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COINCIDENCE AND COMMON FIXED POINT THEOREMS IN NORMED BOOLEAN VECTOR SPACES

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Abstract. In this paper we obtain some coincidence and common fixed point theorems in normed Boolean vector spaces without using continuity. Our approach is entirely algebraic.

1. Introduction and preliminaries

Fixed point theory of Boolean functions is an active area of research (see, for instance, Ghilezan [1], Rudeanu [2] and references thereof). Fixed points of Boolean functions have numerous applications in the theory of error-correcting codes, applications to switching theory and to the relationship between the consistency of a Boolean equation, cryptography, convergence of some recursive parallel array processes in Boolean arrays, memory-efficient solution techniques in computer science etc.

Subrahmanyam [3], introduced the notion of Boolean and normed Boolean vector spaces and studied the basis and convergence in a normed Boolean vector space over a σ-complete Boolean algebra (see also [4]). Further he proved that a normed Boolean vector space over a σ-complete Boolean algebra with finite basis is topologically complete.

In this paper we obtain some coincidence and common fixed point theorems on finite dimensional normed Boolean vector spaces.

Throughout this paper $\mathfrak{V} = (\mathfrak{V}, +)$ denotes an additive abelian group and $\mathscr{B} = (\mathscr{B}, +, \cdot, ')$ a Boolean algebra. For the sake of completeness we recall the following from [3].

Definition 1.1. An additive abelian group $\mathfrak{V}$ with two operations namely 'addition' and 'scalar multiplication' is said to be a Boolean vector space over $\mathscr{B}$ (or simply, a $\mathscr{B}$-vector space) if for all $x, y \in \mathfrak{V}$ and $a, b \in \mathscr{B}$,

(i) $a(x + y) = ax + ay$;
(ii) $(ab)x = a(bx) = b(ax)$;
(iii) $1x = x$; and
(iv) if $ab = 0$, then $(a + b)x = ax + bx$.

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The elements of $\mathfrak{V}$ and $\mathscr{B}$ will be denoted respectively, by x, y, z and a, b, c (with or without indices); the zero of $\mathfrak{V}$ and also null-element of $\mathscr{B}$ will both be denoted by 0, while the universal element $(= 0^{'})$ of $\mathscr{B}$ will be denoted by 1.

Example 1.2. Let $\mathscr{B}$ be any Boolean algebra and $\mathfrak{V}$ be the additive group of the corresponding Boolean ring; then $\mathfrak{V}$ is a $\mathscr{B}$-vector space if we define: For $a \in \mathscr{B}$ and $x \in \mathfrak{V}$, $ax =$the (*Boolean*) product of a and x in $\mathscr{B}$.

Example 1.3. Let R be any Ring with unity element 1 and let $\mathscr{B}$ denote the set of all the central idempotents of R ; then it is known that $(\mathscr{B}, \cup, \cap, \prime)$ is a Boolean algebra, where, by definition, $a \cup b = a + b - ab$, $a \cap b = ab$ and $a' = 1 - a$. If $\mathfrak{V}$ is the additive group of the ring R, and for $a \in \mathscr{B}$ and $x \in \mathfrak{V}$, $ax =$ the product of a and x in R, then $\mathfrak{V}$ is a Boolean vector space over $(\mathscr{B}, \cup, \cap, \prime)$.

Definition 1.4. A Boolean vector space $\mathfrak{V}$ over a Boolean algebra $\mathscr{B}$ is said to be $\mathscr{B}$-normed (or simply, normed) if and only if there exists a mapping $\|.\|$ (called norm): $\mathfrak{V} \to \mathscr{B}$ such that

(i) $\|x\| = 0$ if and only if $x = 0$, and
(ii) $\|ax\| = a\|x\|$ for all $a \in \mathscr{B}$ and $x \in \mathfrak{V}$.

In view of [3, Cor. 3.2], we note the following.

Let $\mathfrak{V}$ be a $\mathscr{B}$-normed vector space and $d^* : \mathfrak{V} \times \mathfrak{V} \to \mathscr{B}$ then $d^*(x, y) = \|x - y\|$ defines a Boolean metric on $\mathfrak{V}$, i.e.

(i) $d^*(x, y) = 0$ if and only if $x = y$;
(ii) $d^*(x, y) = d(y, x)$ and
(iii) $d^*(x, z) < d^*(x, y) + d^*(y, z)$.

Definition 1.5. Let $\mathscr{B}$ be a σ-complete (= countably complete) Boolean algebra. If $\{a_n\}$ is a sequence of elements of $\mathscr{B}$, we define :

$$\liminf a_n = \cup_{k\geq 1} \cap_{n\geq k} a_n; \text{ and } \limsup a_n = \cap_{k\geq 1} \cup_{n\geq k} a_n;$$

and if

$$\liminf a_n = a = \limsup a_n,$$

then we say that a_n converges to a, and will be written as $a_n \to a$. A sequence $\{a_n\}$ in $\mathscr{B}$ is a Cauchy sequence if and only if $d(a_n, a_m) \to 0$, where d is the Boolean metric on $\mathscr{B}$ defined by $d(a, b) = a^{'}b + ab^{'}$.

Definition 1.6. If $\{x_n\}$ is a sequence of elements of $\mathfrak{V}$, we say that $x_n \to x$ $(x \in \mathfrak{V})$ if and only if $\|x_n - x\| \to 0$; and a sequence $\{x_n\}$ in $\mathfrak{V}$ is a Cauchy sequence if and only if $\|x_n - x_m\| \to 0$.

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[1]We shall use $\cup$ and $\cap$ instead of $+$ and . whenever confusion is possible; however, for the sake of comparison with the usual concept of a vector space (over a field), we generally prefer to use $+$ and $\cdot$, which is not without a precedent.

2. Coincidence and common fixed point theorems

Let $\mathfrak{V}$ be a normed Boolean vector space and $T : \mathfrak{V} \to \mathfrak{V}$ a mapping. A point $z \in \mathfrak{V}$ is called fixed point of T if $Tz = z$. The point z is a called a coincidence point of $f, T : \mathfrak{V} \to \mathfrak{V}$ if $fz = Tz$ and a common fixed point if $z = fz = Tz$.

Let Φ denotes the class of all functions $\varphi : \mathscr{B} \to \mathscr{B}$ satisfying:

(i) φ is monotonically increasing and continuous;
(ii) $\varphi(0) = 0$;
(iii) $\lim_{n\to\infty} \varphi^n(a) = 0$ for all $a \in \mathscr{B}$.

Example 2.1. Let A be a set and $\mathcal{B}$ the class of all subsets of A with three set operation $\cup$, $\cap$, $'$ (union, intersection and complement). Then $\mathcal{B}$ defines a Boolean algebra. For all $X, Y \in \mathcal{B}$ we define a function $\varphi : \mathcal{B} \to \mathcal{B}$ by

$$\varphi(X) = \begin{cases} \emptyset & \text{if } X = \emptyset, \\ Y & \text{if } X \neq \emptyset, \text{ whenever } Y \subset X \text{ and } Y \neq X. \end{cases}$$

Then it can be easily verified that $\lim_{n\to\infty} \varphi^n(X) = \emptyset$ for all $X \in \mathcal{B}$.

To prove our main result we need the following lemma due to Rao and Pant [5].

Lemma 2.2. Let $\mathfrak{V}$ be a finite dimensional normed Boolean vector space with a finite basis $G = \{g_1, g_2, \ldots, g_t\}$. If $\{y_n\}$ is a sequence in $\mathfrak{V}$ such that for $n \to \infty$, $\|y_n - y_{n+1}\| \to 0$ then

$$\lim_{n\to\infty} d(a_{n+1j}, a_{nj}) = 0,$$

where for each n, $y_n = \sum_{j=1}^{t} a_{nj} g_j$, $1 \leq j \leq t$ with $a_{nj}.a_{nk} = 0$ for $j \neq k$ and d is the Boolean metric on $\mathscr{B}$ defined by $d(a, b) = a'b + ab'$.

Now, we obtain a coincidence and common fixed point theorem for a pair of self mapping on a finite dimensional normed Boolean vector space.

Theorem 2.3. *Let T, f be self mappings of a finite dimensional normed Boolean vector space $\mathfrak{V}$ over a σ-Complete Boolean algebra $\mathscr{B}$ such that*

(A) $T\mathfrak{V} \subseteq f\mathfrak{V}$;
(B) $\|Tx - Ty\| < \varphi(\|fx - fy\|)$ *for all* $x, y \in \mathfrak{V}$, *where* $\varphi \in \Phi$.

Then f and T have a coincidence in $\mathfrak{V}$.

Further, f and T have a common fixed point provided that $ffu = fu$ and f and T commute at the coincidence point.

Proof. Let $x_0 \in \mathfrak{V}$. Define sequences $\{x_n\}$ and $\{y_n\}$ in $\mathfrak{V}$ by $y_{n+1} = Tx_n = fx_{n+1}$, $n = 0, 1, 2, \ldots$. This can be done since the range of f contains the range of T. We show that $\{y_n\}$ is a Cauchy sequence.

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|Tx_n - Tx_{n-1}\| \\ &< \varphi(\|fx_n - fx_{n-1}\|) < \varphi^2(\|fx_{n-1} - fx_{n-2}\|) \ldots \\ &< \varphi^n(\|fx_1 - fx_0\|). \end{aligned}$$

Since $\varphi \in \Phi$, we have

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| < \lim_{n\to\infty} \varphi^n(\|fx_1 - fx_0\|) = 0. \tag{2.1}$$

Since $\mathfrak{V}$ is finite dimensional, let $G = \{g_1, g_2, \dots, g_t\}$ be the basis of $\mathfrak{V}$. Then for each n, $y_n = \sum_{j=1}^{t} a_{nj} g_j$, $1 \le j \le t$ with $a_{nj}.a_{nk} = 0$ for $j \ne k$. Hence from (2.1) and Lemma 2.2 it follows that $d(a_{n+1j}, a_{nj}) \to 0$.

Now we show that $\{a_{nj}\}$ is a Cauchy sequence. Observe that for any $m, n \in \mathbb{N}$

$$\limsup d(a_{mj}, a_{nj}) = (\limsup a_{nj})(\liminf a_{nj})^{'}.$$

For $m, n \in \mathbb{N}$

$$\begin{aligned} \limsup d(a_{mj}, a_{nj}) &= \limsup (a_{mj} a_{nj}^{'} + a_{mj}^{'} a_{nj}) \\ &= \cap_{k \ge 1} \cup_{m,n \ge k} (a_{mj} a_{nj}^{'} + a_{mj}^{'} a_{nj}) \\ &= \cap_{k \ge 1} \cup_{m,n \ge k} (a_{mj} a_{nj}^{'}) \\ &= \cap_{k \ge 1} \{\cup_{m \ge k} a_{mj} \cup_{n \ge k} a_{nj}^{'}\} \\ &= \limsup a_{nj} \cdot \limsup a_{nj}^{'} \\ &= (\limsup a_{nj})(\liminf a_{nj})^{'}. \end{aligned} \tag{2.2}$$

For $m = n + 1$, the above equation reduces to

$$\limsup d(a_{n+1j}, a_{nj}) = (\limsup a_{nj})(\liminf a_{nj})^{'}.$$

Since $d(a_{n+1j}, a_{nj}) \to 0$ so $\limsup d(a_{n+1j}, a_{nj}) \to 0$. Therefore

$$(\limsup a_{nj})(\liminf a_{nj})^{'} = \limsup d(a_{nj}, a_{n+1j}) = 0. \tag{2.3}$$

By (2.2) and (2.3)

$$\limsup d(a_{mj}, a_{nj}) = (\limsup a_{nj})(\liminf a_{nj})^{'} = 0.$$

Thus $d(a_{mj}, a_{nj}) \to 0$ which yields that $\{a_{nj}\}$ is a Cauchy sequence in $\mathscr{B}$. In view of [3, Lem. 18], the sequence $\{y_n\}$ is Cauchy in $\mathfrak{V}$. Since $\mathfrak{V}$ is a normed Boolean-vector space over a σ-complete Boolean algebra $\mathscr{B}$ with a finite basis, by [3, Th. 12], $\mathfrak{V}$ is a complete (topologically). Hence $\{y_n\}$ has a limit in $\mathfrak{V}$. Call it z, then $fu = z$ for some $u \in \mathfrak{V}$. Using (B), we get

$$\|y_{n+1} - Tu\| = \|Tx_n - Tu\| < \varphi(\|fx_n - fu\|).$$

Making $n \to \infty$, we obtain $fu = Tu$ and u is a coincidence point of f and T. Further, if $ffu = fu$, and the mappings f and T commute at their coincidence point u then $fu = fTu = Tfu$ and fu is a common fixed point of f and T. □

Now we obtain the following result which ensure the uniqueness of common fixed point.

Theorem 2.4. *Let T, f be self mappings of a finite dimensional normed Boolean vector space $\mathfrak{V}$ over a σ-Complete Boolean algebra $\mathscr{B}$ such that*

(A) $T\mathfrak{V} \subseteq f\mathfrak{V}$;

(B) $\|Tx - Ty\| < \varphi(\|fx - fy\|)$ *for all $x, y \in \mathfrak{V}$, where $\varphi \in \Phi$.*

Then f and T have a coincidence in $\mathfrak{V}$.

Further, in addition to (A) *and* (B) *suppose that the following condition holds*

(C) $\|Tx - Ty\| = \psi(\|fx - fy\|)$ *for all* $x, y \in \mathfrak{V}$, *where* $\psi : \mathscr{B} \to \mathscr{B}$ *is a continuous function such that* $\psi(a) < a'$.

Then f *and* T *have a unique common fixed point provided that* $ffu = fu$ *and* T *and* f *commute at the coincidence point.*

Proof. By Theorem 2.3, we know that f and T have a common fixed point. Suppose z_1, z_2 are two common fixed points of f and T. Then $Tz_1 = fz_1 = z_1$ and $Tz_2 = fz_2 = z_2$. Using the condition (C)

$$\|z_1 - z_2\| = \|Tz_1 - Tz_2\| = \psi(\|fz_1 - fz_2\|) < (\|z_1 - z_2\|)'.$$

Which follows that $z_1 = z_2$. □

Corollary 2.5. *Let* $\mathfrak{V}$ *be a finite dimensional normed Boolean vector space over a* σ*-complete Boolean algebra* $\mathscr{B}$ *and* $T : \mathfrak{V} \to \mathfrak{V}$ *such that*

$$\|Tx - Ty\| < \varphi(\|x - y\|) \tag{2.4}$$

for all $x, y \in \mathfrak{V}$, *where* $\varphi \in \Phi$. *Then* T *has a fixed point in* $\mathfrak{V}$.

Proof. This comes from Theorem 2.3, when f is an identity mapping on $\mathfrak{V}$.

Corollary 2.6. *Let* $\mathfrak{V}$ *be a finite dimensional normed Boolean vector space over a* σ*-complete Boolean algebra* $\mathscr{B}$ *and* $T : \mathfrak{V} \to \mathfrak{V}$ *such that*

(i) $\|Tx - Ty\| < \varphi(\|x - y\|)$ *for all* $x, y \in \mathfrak{V}$, *where* $\varphi \in \Phi$;

(ii) $\|Tx - Ty\| = \psi(\|x - y\|)$ *for all* $x, y \in \mathfrak{V}$, *where* ψ *is as in* Theorem 2.4. *Then* T *has a unique fixed point in* $\mathfrak{V}$.

Proof. This comes from Theorem 2.4, when f is an identity mapping on $\mathfrak{V}$.

Now we present an example to illustrate our results.

Example 2.7. Let A be a non-empty finite set and $\mathscr{B}$ the class of all subsets of A. Then the class $\mathscr{B}$ with three set operation $+$, $\cdot$, $'$ (union, intersection, and complement) defines a Boolean algebra. Further, this class $\mathscr{B}$ with the set operation "exclusive-or addition" $\oplus$ (symmetric difference of sets) defines a Boolean ring. Let $\mathfrak{V} = (\mathfrak{V}, \oplus)$ be the additive abelian group of this Boolean ring. For a in $\mathscr{B}$ and x in $\mathfrak{V}$, we define $ax = a \cdot x$ (the Boolean) product of a and x in $\mathscr{B}$. Then $\mathfrak{V}$ is a Boolean vector space over $\mathscr{B}$.

Let $T, f : \mathfrak{V} \to \mathfrak{V}$ be self-mappings defined by

$$fx = x \text{ and } Tx = \xi \text{ for all } x \in \mathfrak{V} \ (\xi \text{ is some element in } \mathfrak{V}).$$

Let $\varphi : \mathscr{B} \to \mathscr{B}$ defined as in Example 2.1 and $\psi : \mathscr{B} \to \mathscr{B}$ defined by $\psi(a) = a - 1$ for all $a \in \mathscr{B}$, where '1' is the universal element of $\mathscr{B}$.

Then $T\mathfrak{V} \subset f\mathfrak{V}$ and for all $x, y \in \mathfrak{V}$

$$\|Tx - Ty\| < \varphi(\|fx - fy\|)$$

where $\|.\|$ is any norm defined on $\mathfrak{V}$. Thus all the hypotheses of Theorem 2.4 are satisfied and ξ is a unique common fixed point of f and T.

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