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HOPFIELD NETWORKS WITH ASYMMETRIC COUPLING ARCHITECTURE

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Abstract. In 1982, the biophysicist John J. Hopfield proposed a symmetric network architecture and showed that the network dynamics can tend toward a fixed point when the retrieval operation is performed asynchronously. Hopfield networks are remarkable in that (i) they include Hebbian synaptic plasticity to construct symmetric coupling matrices; (ii) single units in Hopfield networks asynchronously adjust their states over time; (iii) equilibria (fixed points) are states of local energy minimum. Here we wish to propose an asymmetric network architecture that extends the convergence theorem of Hopfield networks. Our result shows that (i) the coupling matrices can be asymmetric; (ii) ensemble units in networks can assemblingly adjust their states over time; (iii) equilibria (fixed points) are not necessarily states of local energy minimum.

1. Introduction

Hopfield in 1982 proposed a symmetric network architecture and used a global energy function to show that network dynamics can tend toward a fixed point when the retrieval operation is performed asynchronously [2]. To state that, we consider a dynamical system of n coupled units modeled by the equation [3–5]:

$$x(t+1) = H_A(x(t), s(t)), \quad t = 0, 1, \dots, \tag{1.1}$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t)) \in \{0,1\}^n$ is the vector of unit states at time t, $A = (a_{ij}) \in M_n(\mathbb{R})$ is the coupling matrix of n coupled units, $s(t) \subset \{1, 2, \dots, n\}$ denotes the units that adjust their states at time t, and $H_A(\cdot, s(t))$ is a function whose ith component is defined by

$$[H_A(x, s(t))]_i = \mathbb{1}\left(\sum_{j=1}^{n} a_{ij}x_j - b_i\right) \quad \text{if } i \in s(t),$$

otherwise $[H_A(x, s(t))]_i = x_i$, where $b_i \in \mathbb{R}$ is the threshold of unit i and the function $\mathbb{1}$ is the Heaviside function: $\mathbb{1}(u) = 1$ for $u \geq 0$, otherwise 0, which describes an instantaneous unit pulse. The dynamical system generates the vector of unit states according to (1.1), resulting in the phase flow $x(t)$, $t = 0, 1, \dots$. We say that the phase flow $x(t)$ converges to a fixed point ξ if there exists $T \geq 0$ such that $H_A(x(t), s(t)) = \xi$ for each $t \geq T$. The convergence theorem of Hopfield networks can be rewritten as follows.

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Theorem 1.1 (Hopfield [2]). *If $A \in M_n(\mathbb{R})$ is symmetric with nonnegative diagonal entries, then each phase flow $x(t)$ of the network modeled by* (1.1)*, with each unit adjusting randomly and asynchronously (i.e., $\sharp s(t) = 1$ for each $t = 0, 1, \ldots$), will converge to a fixed point.*

The purpose of this paper is to propose an asymmetric network architecture that extends the convergence theorem of Hopfield networks. The extension is nontrivial not only because we construct an asymmetric network architecture, but because we do not use a global energy function to control network dynamics. This implies that equilibria (fixed points) are not necessarily states of local energy minimum, in accordance with related lines of research in switched linear networked systems showing that networked systems are asymptotically stable, but no common quadratic energy function exists [1].

To state that, let $\{0,1\}^n$ denote the binary code consisting of all 01-strings of fixed length n. For each 01-string $x = x_1 x_2 \cdots x_n$, let

$$\mathbf{1}(x) = \{i;\ x_i = 1,\ 1 \le i \le n\},$$
$$\mathbf{0}(x) = \{i;\ x_i = 0,\ 1 \le i \le n\}.$$

Let us recall that the symmetric difference of two sets U and V is the set $U \triangle V$, each of whose elements belongs to U but not to V, or belongs to V but not to U. Define $\Omega = [x^0, x^1, \ldots, x^p]$ a loop of states in $\{0,1\}^n$, meaning that $p > 1$, $x^0, x^1, \ldots, x^p \in \{0,1\}^n$, $x^0 = x^p$, and $x^i \ne x^j$ for some $i, j \in \{1, 2, \ldots, p\}$. For every $i = 1, 2, \ldots, n$, let

$$M_i(\Omega) = \{m;\ i \in \mathbf{1}(x^{m-1}) \triangle \mathbf{1}(x^m), m = 1, 2, \ldots, p\},$$
$$M_i(\Omega)^+ = \{m;\ i \in \mathbf{1}(x^m) \setminus \mathbf{1}(x^{m-1}), m = 1, 2, \ldots, p\},$$
$$M_i(\Omega)^- = \{m;\ i \in \mathbf{1}(x^{m-1}) \setminus \mathbf{1}(x^m), m = 1, 2, \ldots, p\}.$$

For every loop Ω and for every coupling (i, j), let

$$\alpha_{ij}(\Omega) = \sharp(M_i(\Omega)^+ \cap M_j(\Omega)^+) + \sharp(M_i(\Omega)^- \cap M_j(\Omega)^-) - \sharp(M_i(\Omega)^+ \cap M_j(\Omega)^-) - \sharp(M_i(\Omega)^- \cap M_j(\Omega)^+)$$

and

$$\beta_{ij}(\Omega) = \min\{\sharp M_i(\Omega), \sharp M_j(\Omega)\}.$$

Our principal result is the following:

Theorem 1.2. *Let $\delta \ge 1$. If the coupling matrix A (whether symmetric or asymmetric) satisfies*

$$\sum_{1 \le i,j \le n} \alpha_{ij}(\Omega)\,(a_{ij} + a_{ji}) \ge \sum_{1 \le i,j \le n} \beta_{ij}(\Omega)|a_{ij} - a_{ji}| \tag{1.2}$$

for each loop $\Omega = [x^0, x^1, \ldots, x^p]$ with $\sharp((\mathbf{1}(x^0) \triangle \mathbf{1}(x^1)) \cup (\mathbf{1}(x^1) \triangle \mathbf{1}(x^2)) \cup \cdots \cup (\mathbf{1}(x^{p-1}) \triangle \mathbf{1}(x^p))) \ge 1$, and $\sharp(\mathbf{1}(x^{m-1}) \triangle \mathbf{1}(x^m)) \le \delta$ for each $m = 1, 2, \ldots, p$, then for every initial unit state $x(0)$ and for every updating $s(t)$ with $\sharp s(t) \le \delta$ for each $t = 0, 1, \ldots$, the resulting phase flow $x(t)$ tends toward a fixed point. (Here the retrieval updating is asynchronous if δ is selected to be one such that $\sharp s(t) \le 1$ for each $t = 0, 1, \ldots$; otherwise the retrieval updating is assembling.)

2. Construction of symmetric or asymmetric Hopfield networks

The coupling architecture (1.2) given in Theorem 1.2 can help to construct three classes of symmetric or asymmetric Hopfield networks.

(I) If the coupling matrix A is symmetric with diagonal entries $a_{ii} \geq 0$ for each $i = 1, 2, \dots, n$, then

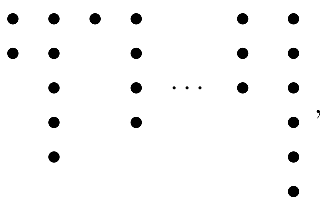
$$\sum_{1\leq i,j\leq n} \alpha_{ij}(\Omega)\,(a_{ij} + a_{ji}) = \sum_{1\leq i\leq n} 2\alpha_{ii}(\Omega) a_{ii} \geq 0 = \sum_{1\leq i,j\leq n} \beta_{ij}(\Omega)|a_{ij} - a_{ji}|$$

for every loop $\Omega = [x^0, x^1, \dots, x^p]$ with $\sharp((\mathbf{1}(x^0)\triangle\mathbf{1}(x^1)) \cup (\mathbf{1}(x^1)\triangle\mathbf{1}(x^2)) \cup \cdots \cup (\mathbf{1}(x^{p-1})\triangle\mathbf{1}(x^p))) \geq 1$, and $\sharp(\mathbf{1}(x^{m-1})\triangle\mathbf{1}(x^m)) \leq \delta$ for each $m = 1, 2, \dots, p$. Applying Theorem 1.2 to $\delta = 1$, we obtain that the phase flow $x(t)$ tends toward a fixed point underlying asynchronous retrieval updating. (Here δ is selected to be one so that the retrieval updating is asynchronous.) This reduces to the convergence theorem of classical Hopfield networks [2].

(II) Suppose that the coupling matrix A (whether symmetric or asymmetric) satisfies

$$\sum_{i\in I} a_{ii} \geq \sum_{i,j\in I} \frac{1}{2}|a_{ij} - a_{ji}| \quad \text{for each} \quad I \subset \{1, 2, \dots, n\}. \tag{2.1}$$

Let $\delta = 1$ and associate to every loop $\Omega = [x^0, x^1, \dots, x^p]$ with $\sharp((\mathbf{1}(x^0)\triangle\mathbf{1}(x^1)) \cup (\mathbf{1}(x^1)\triangle\mathbf{1}(x^2)) \cup \cdots \cup (\mathbf{1}(x^{p-1})\triangle\mathbf{1}(x^p))) \geq 1$, and $\sharp(\mathbf{1}(x^{m-1})\triangle\mathbf{1}(x^m)) \leq \delta$ for each $m = 1, 2, \dots, p$, the array of dots constructed as follows:



where the dots in the ith column are arranged consecutively from the top and the number of dots in the ith column is equal to $\sharp M_i(\Omega)$, we conclude from (2.1) that

$$\begin{aligned}
\sum_{1\leq i,j\leq n} \alpha_{ij}(\Omega)\,(a_{ij} + a_{ji}) &= \sum_{1\leq i\leq n} 2\alpha_{ii}(\Omega) a_{ii} \\
&= \sum_{1\leq i\leq n} 2 \cdot \sharp M_i(\Omega) \cdot a_{ii} \\
&= 2 \sum_{1\leq k\leq r} \sum_{i\in V_k} a_{ii} \\
&\geq \sum_{1\leq k\leq r} \sum_{i,j\in V_k} |a_{ij} - a_{ji}| \\
&= \sum_{1\leq i,j\leq n} \min\{\sharp M_i(\Omega), \sharp M_j(\Omega)\} |a_{ij} - a_{ji}| \\
&= \sum_{1\leq i,j\leq n} \beta_{ij}(\Omega)|a_{ij} - a_{ji}|,
\end{aligned}$$

where r denotes the number of rows in the array and V_k denotes the set

$$\{i;\ \text{the } (k,i)\text{-entry of the array is equipped with } \bullet\ ,\ i = 1, 2, \ldots, n\}$$

for every $k = 1, 2, \ldots, r$. Thus we conclude from Theorem 1.2 that the phase flow $x(t)$ can tend toward a fixed point underlying asynchronous retrieval updating.

(III) Let $\delta \geq 1$ and let A be a symmetric coupling matrix satisfying

$$\sum_{i\in I} a_{ii} \geq \sum_{i,j\in I, i\neq j} |a_{ij}| \text{ for each } I \subset \{1, 2, \ldots, n\} \text{ with } \sharp\, I \leq \delta. \tag{2.2}$$

We say that a pair of sequences $\{J_1^+, J_2^+, \ldots, J_p^+\}$, $\{J_1^-, J_2^-, \ldots, J_p^-\}$ in $2^{\{1,2,\ldots,n\}}$ is complementary covering if they have the common covering region $\bigcup_{1\leq m\leq p} J_m^+ = \bigcup_{1\leq m\leq p} J_m^-$ and they are complementary in the sense that $J_m^+ \cap J_m^- = \emptyset$ for each $m = 1, 2, \ldots, p$ and $J_m^+ \cap J_{m+1}^+ = \emptyset$, $J_m^- \cap J_{m+1}^- = \emptyset$ for each $m = 1, 2, \ldots, p-1$. For each pair of complementary covering sequences $\{J_1^+, J_2^+, \ldots, J_p^+\}$, $\{J_1^-, J_2^-, \ldots, J_p^-\}$ in $2^{\{1,2,\ldots,n\}}$ with $\sharp(J_m^+ \cup J_m^-) \leq \delta$ for each $m = 1, 2, \ldots, p$, by (2.2), we have

$$\begin{aligned}
&\sum_{1\leq m\leq p}\left(\sum_{i,j\in J_m^+} a_{ij} + \sum_{i,j\in J_m^-} a_{ij}\right)\\
&= \sum_{1\leq m\leq p}\left(\sum_{i\in J_m^+\cup J_m^-} a_{ii} + \sum_{i,j\in J_m^+, i\neq j} a_{ij} + \sum_{i,j\in J_m^-, i\neq j} a_{ij}\right)\\
&\geq \sum_{1\leq m\leq p}\left(\sum_{i,j\in J_m^+\cup J_m^-, i\neq j} |a_{ij}| + \sum_{i,j\in J_m^+, i\neq j} a_{ij} + \sum_{i,j\in J_m^-, i\neq j} a_{ij}\right) \qquad (2.3)\\
&= \sum_{1\leq m\leq p}\left(\sum_{i\in J_m^+, j\in J_m^-} |a_{ij}| + \sum_{i\in J_m^-, j\in J_m^+} |a_{ij}|\right.\\
&\qquad \left. + \sum_{i,j\in J_m^+, i\neq j} (|a_{ij}| + a_{ij}) + \sum_{i,j\in J_m^-, i\neq j} (|a_{ij}| + a_{ij})\right)\\
&\geq \sum_{1\leq m\leq p}\left(\sum_{i\in J_m^+, j\in J_m^-} a_{ij} + \sum_{i\in J_m^-, j\in J_m^+} a_{ij}\right).
\end{aligned}$$

Let $\Omega = [x^0, x^1, \ldots, x^p]$ be a loop with $\sharp((\mathbf{1}(x^0)\triangle\mathbf{1}(x^1)) \cup (\mathbf{1}(x^1)\triangle\mathbf{1}(x^2)) \cup \cdots \cup (\mathbf{1}(x^{p-1})\triangle\mathbf{1}(x^p))) \geq 1$, and $\sharp(\mathbf{1}(x^{m-1})\triangle\mathbf{1}(x^m)) \leq \delta$ for each $m = 1, 2, \ldots, p$. Then the pair of sequences $\{\mathbf{1}(x^1) \setminus \mathbf{1}(x^0), \mathbf{1}(x^2) \setminus \mathbf{1}(x^1), \ldots, \mathbf{1}(x^p) \setminus \mathbf{1}(x^{p-1})\}$, $\{\mathbf{1}(x^0) \setminus \mathbf{1}(x^1), \mathbf{1}(x^1) \setminus \mathbf{1}(x^2), \ldots, \mathbf{1}(x^{p-1}) \setminus \mathbf{1}(x^p)\}$ is complementary covering. Since A is symmetric, it follows from (2.3) that

$$\begin{aligned}
&\sum_{1\leq i,j\leq n} \alpha_{ij}(\Omega)\,(a_{ij} + a_{ji})\\
&= 2\sum_{1\leq i,j\leq n} \big(\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^+) + \sharp(M_i(\Omega)^- \cap M_j(\Omega)^-)
\end{aligned}$$

$$
\begin{aligned}
&-\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^-) - \sharp(M_i(\Omega)^- \cap M_j(\Omega)^+)\big)\, a_{ij} \\
= \;& 2 \sum_{1\le m\le p} \Bigg(\sum_{i,j\in \mathbf{1}(x^m)\setminus \mathbf{1}(x^{m-1})} a_{ij} + \sum_{i,j\in \mathbf{1}(x^{m-1})\setminus \mathbf{1}(x^m)} a_{ij} \\
&- \sum_{i\in \mathbf{1}(x^m)\setminus \mathbf{1}(x^{m-1}),\, j\in \mathbf{1}(x^{m-1})\setminus \mathbf{1}(x^m)} a_{ij} - \sum_{i\in \mathbf{1}(x^{m-1})\setminus \mathbf{1}(x^m),\, j\in \mathbf{1}(x^m)\setminus \mathbf{1}(x^{m-1})} a_{ij} \Bigg) \\
\ge \;& 0 = \sum_{1\le i,j\le n} \beta_{ij}(\Omega)|a_{ij}-a_{ji}|.
\end{aligned}
$$

Thus, by Theorem 1.2, the phase flow $x(t)$ tends to an equilibrium underlying asynchronous retrieval updating if δ is selected to be one; otherwise the phase flow $x(t)$ tends toward a fixed point underlying assembling retrieval updating.

3. Proof of Theorem 1.2

Let $A \in M_n(\mathbb{R})$ be a coupling matrix and let $\Omega = [x^0, x^1, \ldots, x^p]$ be a loop satisfying the conditions given in Theorem 1.2. For every $i, j = 1, 2, \ldots, n$, we assign an integer, denoted by $c_{ij}(\Omega)$, according to the rule:

$$c_{ij}(\Omega) = x_j^0(x_i^0 - x_i^1) + x_j^1(x_i^1 - x_i^2) + \cdots + x_j^{p-1}(x_i^{p-1} - x_i^p).$$

Let $C(\Omega) = (c_{ij}(\Omega))$. Let $H(A) = \frac{1}{2}\left(A + A^T\right)$ and $S(A) = \frac{1}{2}\left(A - A^T\right)$ be the symmetric part and the skew-symmetric part of the coupling matrix A, respectively. Let $H(C(\Omega)) = \frac{1}{2}\left(C(\Omega) + C(\Omega)^T\right)$ and $S(C(\Omega)) = \frac{1}{2}\left(C(\Omega) - C(\Omega)^T\right)$ be the symmetric part and the skew-symmetric part of $C(\Omega)$, respectively. The symmetry of $H(A)$, $H(C(\Omega))$ and the skew-symmetry of $S(A)$, $S(C(\Omega))$ together imply that

$$\langle A, C(\Omega)\rangle = \langle H(A), H(C(\Omega))\rangle + \langle S(A), S(C(\Omega))\rangle. \tag{3.1}$$

According to [6, Proof of Theorem 1], we have

$$
\begin{aligned}
&\langle H(A), H(C(\Omega))\rangle \\
= \;& \sum_{1\le i\le n} H(A)_{ii}\cdot \frac{1}{2}\left(\sharp(M_i(\Omega)^+ \cap M_i(\Omega)^+) + \sharp(M_i(\Omega)^- \cap M_i(\Omega)^-)\right) \\
&+ \sum_{1\le i,j\le n, i\ne j} H(A)_{ij}\cdot \frac{1}{2}\left(\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^+) + \sharp(M_i(\Omega)^- \cap M_j(\Omega)^-)\right) \\
&- \sum_{1\le i,j\le n, i\ne j} H(A)_{ij}\cdot \frac{1}{2}\left(\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^-) + \sharp(M_i(\Omega)^- \cap M_j(\Omega)^+)\right)
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
\langle S(A), S(C(\Omega))\rangle \;\ge\; & -\sum_{1\le i,j\le n} \frac{1}{4}\cdot \min\{\sharp M_i(\Omega), \sharp M_j(\Omega)\}\cdot |a_{ij}-a_{ji}| \\
&+ \sum_{1\le i,j\le n, S(A)_{ij}>0} \frac{1}{2}\cdot \sharp(M_i(\Omega)\cap M_j(\Omega))\cdot (S(A)_{ij} - S(A)_{ji}).
\end{aligned}
\tag{3.3}
$$

Consider the term

$$\begin{aligned}
&\sum_{1\le i,j\le n, i\ne j} H(A)_{ij}\cdot\frac{1}{2}\left(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\right)\\
-&\sum_{1\le i,j\le n, i\ne j} H(A)_{ij}\cdot\frac{1}{2}\left(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^+)\right)\\
+&\sum_{1\le i,j\le n, S(A)_{ij}>0} \frac{1}{2}\cdot\sharp(M_i(\Omega)\cap M_j(\Omega))\cdot(S(A)_{ij}-S(A)_{ji})
\end{aligned}$$

deriving from the partial sum of (3.2) and (3.3). Since

$$\begin{aligned}
&\sum_{1\le i,j\le n, S(A)_{ij}\ne 0} H(A)_{ij}\cdot\frac{1}{2}\left(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\right)\\
=&\sum_{1\le i,j\le n, S(A)_{ij}>0} H(A)_{ij}\cdot\frac{1}{2}\left(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\right)\\
&+\sum_{1\le i,j\le n, S(A)_{ij}>0} H(A)_{ji}\cdot\frac{1}{2}\left(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\right)
\end{aligned}$$

and since

$$\begin{aligned}
&\sum_{1\le i,j\le n, S(A)_{ij}\ne 0} H(A)_{ij}\cdot\frac{1}{2}\left(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^+)\right)\\
=&\sum_{1\le i,j\le n, S(A)_{ij}>0} H(A)_{ij}\cdot\frac{1}{2}\left(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^+)\right)\\
&+\sum_{1\le i,j\le n, S(A)_{ij}>0} H(A)_{ji}\cdot\frac{1}{2}\left(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^+)\right),
\end{aligned}$$

it follows from the equality

$$\begin{aligned}
\sharp(M_i(\Omega)\cap M_j(\Omega)) &= \sharp\left((M_i(\Omega)^+\cup M_i(\Omega)^-)\cap(M_j(\Omega)^+\cup M_j(\Omega)^-)\right)\\
&= \sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\\
&\quad+\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^+)
\end{aligned}$$

that

$$\begin{aligned}
&\sum_{1\le i,j\le n, i\ne j} H(A)_{ij}\cdot\frac{1}{2}\left(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\right)\\
&-\sum_{1\le i,j\le n, i\ne j} H(A)_{ij}\cdot\frac{1}{2}\left(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^+)\right)\\
&+\sum_{1\le i,j\le n, S(A)_{ij}>0} \frac{1}{2}\cdot\sharp(M_i(\Omega)\cap M_j(\Omega))\cdot(S(A)_{ij}-S(A)_{ji})\\
=&\sum_{1\le i,j\le n, S(A)_{ij}\ne 0} \frac{1}{2}\cdot H(A)_{ij}\cdot\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{1\le i,j\le n, S(A)_{ij}\neq 0} \frac{1}{2}\cdot H(A)_{ij}\cdot \sharp(M_i(\Omega)^- \cap M_j(\Omega)^-) \\
&- \sum_{1\le i,j\le n, S(A)_{ij}\neq 0} \frac{1}{2}\cdot H(A)_{ij}\cdot \sharp(M_i(\Omega)^+ \cap M_j(\Omega)^-) \\
&- \sum_{1\le i,j\le n, S(A)_{ij}\neq 0} \frac{1}{2}\cdot H(A)_{ij}\cdot \sharp(M_i(\Omega)^- \cap M_j(\Omega)^+) \\
&+ \sum_{1\le i,j\le n, S(A)_{ij}> 0} \frac{1}{2}\cdot \sharp(M_i(\Omega) \cap M_j(\Omega))\cdot (S(A)_{ij}-S(A)_{ji}) \\
&+ \sum_{1\le i,j\le n, i\neq j, S(A)_{ij}= 0} \frac{1}{2}\cdot H(A)_{ij}\cdot \sharp(M_i(\Omega)^+ \cap M_j(\Omega)^+) \\
&+ \sum_{1\le i,j\le n, i\neq j, S(A)_{ij}= 0} \frac{1}{2}\cdot H(A)_{ij}\cdot \sharp(M_i(\Omega)^- \cap M_j(\Omega)^-) \\
&- \sum_{1\le i,j\le n, i\neq j, S(A)_{ij}= 0} \frac{1}{2}\cdot H(A)_{ij}\cdot \sharp(M_i(\Omega)^+ \cap M_j(\Omega)^-) \\
&- \sum_{1\le i,j\le n, i\neq j, S(A)_{ij}= 0} \frac{1}{2}\cdot H(A)_{ij}\cdot \sharp(M_i(\Omega)^- \cap M_j(\Omega)^+) \\
= &\sum_{1\le i,j\le n, S(A)_{ij}> 0} \frac{1}{2}\cdot (H(A)_{ij}+S(A)_{ij})\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^+) \\
&+ \sum_{1\le i,j\le n, S(A)_{ij}> 0} \frac{1}{2}\cdot (H(A)_{ij}+S(A)_{ij})\sharp(M_i(\Omega)^- \cap M_j(\Omega)^-) \qquad (3.4)\\
&- \sum_{1\le i,j\le n, S(A)_{ij}> 0} \frac{1}{2}\cdot (H(A)_{ij}-S(A)_{ij})\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^-) \\
&- \sum_{1\le i,j\le n, S(A)_{ij}> 0} \frac{1}{2}\cdot (H(A)_{ij}-S(A)_{ij})\sharp(M_i(\Omega)^- \cap M_j(\Omega)^+) \\
&+ \sum_{1\le i,j\le n, S(A)_{ij}> 0} \frac{1}{2}\cdot (H(A)_{ji}-S(A)_{ji})\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^+) \\
&+ \sum_{1\le i,j\le n, S(A)_{ij}> 0} \frac{1}{2}\cdot (H(A)_{ji}-S(A)_{ji})\sharp(M_i(\Omega)^- \cap M_j(\Omega)^-) \\
&- \sum_{1\le i,j\le n, S(A)_{ij}> 0} \frac{1}{2}\cdot (H(A)_{ji}+S(A)_{ji})\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^-) \\
&- \sum_{1\le i,j\le n, S(A)_{ij}> 0} \frac{1}{2}\cdot (H(A)_{ji}+S(A)_{ji})\sharp(M_i(\Omega)^- \cap M_j(\Omega)^+)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{1\le i,j\le n, i\ne j, S(A)_{ij}=0} \frac{1}{2}\cdot H(A)_{ij}\cdot \sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)\\
&+\sum_{1\le i,j\le n, i\ne j, S(A)_{ij}=0} \frac{1}{2}\cdot H(A)_{ij}\cdot \sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\\
&-\sum_{1\le i,j\le n, i\ne j, S(A)_{ij}=0} \frac{1}{2}\cdot H(A)_{ij}\cdot \sharp(M_i(\Omega)^+\cap M_j(\Omega)^-)\\
&-\sum_{1\le i,j\le n, i\ne j, S(A)_{ij}=0} \frac{1}{2}\cdot H(A)_{ij}\cdot \sharp(M_i(\Omega)^-\cap M_j(\Omega)^+).
\end{aligned}
$$

Note that for each $i, j = 1, 2, \ldots, n$,

$$
\begin{aligned}
H(A)_{ij}+S(A)_{ij} &= \frac{1}{2}(a_{ij}+a_{ji})+\frac{1}{2}(a_{ij}-a_{ji}) = a_{ij},\\
H(A)_{ij}-S(A)_{ij} &= \frac{1}{2}(a_{ij}+a_{ji})-\frac{1}{2}(a_{ij}-a_{ji}) = a_{ji},\\
H(A)_{ji}-S(A)_{ji} &= \frac{1}{2}(a_{ji}+a_{ij})-\frac{1}{2}(a_{ji}-a_{ij}) = a_{ij},\\
H(A)_{ji}+S(A)_{ji} &= \frac{1}{2}(a_{ji}+a_{ij})+\frac{1}{2}(a_{ji}-a_{ij}) = a_{ji},
\end{aligned}
$$

and that

$$S(A)_{ij} = 0 \text{ if and only if } a_{ij} = a_{ji},$$

we conclude from (3.4) that

$$
\begin{aligned}
&\sum_{1\le i,j\le n, i\ne j} H(A)_{ij}\cdot\frac{1}{2}\left(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\right)\\
&-\sum_{1\le i,j\le n, i\ne j} H(A)_{ij}\cdot\frac{1}{2}\left(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^+)\right)\\
&+\sum_{1\le i,j\le n, S(A)_{ij}>0} \frac{1}{2}\cdot\sharp(M_i(\Omega)\cap M_j(\Omega))\cdot(S(A)_{ij}-S(A)_{ji}) \qquad (3.5)\\
=&\sum_{1\le i,j\le n, a_{ij}>a_{ji}} a_{ij}\cdot\left(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\right)\\
&-\sum_{1\le i,j\le n, a_{ij}>a_{ji}} a_{ji}\cdot\left(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^+)\right)\\
&+\sum_{1\le i,j\le n, i\ne j, a_{ij}=a_{ji}} \frac{1}{2}\cdot a_{ij}\cdot\left(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\right)\\
&-\sum_{1\le i,j\le n, i\ne j, a_{ij}=a_{ji}} \frac{1}{2}\cdot a_{ji}\cdot\left(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^+)\right).
\end{aligned}
$$

Since

$$\begin{aligned}
&\sum_{1\le i,j\le n, a_{ij}>a_{ji}} \frac{1}{2}\cdot|a_{ij}-a_{ji}|\cdot\big(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\big)\\
=\ &\sum_{1\le i,j\le n} \frac{1}{4}\cdot|a_{ij}-a_{ji}|\cdot\big(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\big)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{1\le i,j\le n, a_{ij}>a_{ji}} \frac{1}{2}\cdot|a_{ij}-a_{ji}|\cdot\big(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^+)\big)\\
=\ &\sum_{1\le i,j\le n} \frac{1}{4}\cdot|a_{ij}-a_{ji}|\cdot\big(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^+)\big)\,,
\end{aligned}$$

we have

$$\begin{aligned}
&\sum_{1\le i,j\le n, a_{ij}>a_{ji}} a_{ij}\cdot\big(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\big)\\
&\quad-\sum_{1\le i,j\le n, a_{ij}>a_{ji}} a_{ji}\cdot\big(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^+)\big)\\
=\ &\sum_{1\le i,j\le n, a_{ij}>a_{ji}} \left(a_{ij}-\frac{1}{2}\cdot|a_{ij}-a_{ji}|\right)\cdot\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)\\
&\quad+\sum_{1\le i,j\le n, a_{ij}>a_{ji}} \left(a_{ij}-\frac{1}{2}\cdot|a_{ij}-a_{ji}|\right)\cdot\sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\\
&\quad-\sum_{1\le i,j\le n, a_{ij}>a_{ji}} \left(a_{ji}+\frac{1}{2}\cdot|a_{ij}-a_{ji}|\right)\cdot\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-)\\
&\quad-\sum_{1\le i,j\le n, a_{ij}>a_{ji}} \left(a_{ji}+\frac{1}{2}\cdot|a_{ij}-a_{ji}|\right)\cdot\sharp(M_i(\Omega)^-\cap M_j(\Omega)^+)\\
&\quad+\sum_{1\le i,j\le n} \frac{1}{4}\cdot|a_{ij}-a_{ji}|\cdot\big(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\big)\\
&\quad+\sum_{1\le i,j\le n} \frac{1}{4}\cdot|a_{ij}-a_{ji}|\cdot\big(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-)+\sharp(M_i(\Omega)^-\cap M_j(\Omega)^+)\big)\\
=\ &\sum_{1\le i,j\le n, a_{ij}>a_{ji}} \frac{1}{2}\cdot(a_{ij}+a_{ji})\cdot\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)\\
&\quad+\sum_{1\le i,j\le n, a_{ij}>a_{ji}} \frac{1}{2}\cdot(a_{ij}+a_{ji})\cdot\sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\\
&\quad-\sum_{1\le i,j\le n, a_{ij}>a_{ji}} \frac{1}{2}\cdot(a_{ij}+a_{ji})\cdot\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-)
\end{aligned} \tag{3.6}$$

$$
\begin{aligned}
&- \sum_{1\le i,j\le n, a_{ij}>a_{ji}} \frac{1}{2}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^- \cap M_j(\Omega)^+) \\
&+ \sum_{1\le i,j\le n} \frac{1}{4}\cdot |a_{ij}-a_{ji}| \cdot \left(\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^+) + \sharp(M_i(\Omega)^- \cap M_j(\Omega)^-)\right) \\
&+ \sum_{1\le i,j\le n} \frac{1}{4}\cdot |a_{ij}-a_{ji}| \cdot \left(\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^-) + \sharp(M_i(\Omega)^- \cap M_j(\Omega)^+)\right).
\end{aligned}
$$

Combining (3.5) and (3.6) shows that

$$
\begin{aligned}
& \sum_{1\le i,j\le n, i\ne j} H(A)_{ij}\cdot \frac{1}{2}\left(\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^+) + \sharp(M_i(\Omega)^- \cap M_j(\Omega)^-)\right) \\
&- \sum_{1\le i,j\le n, i\ne j} H(A)_{ij}\cdot \frac{1}{2}\left(\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^-) + \sharp(M_i(\Omega)^- \cap M_j(\Omega)^+)\right) \\
&+ \sum_{1\le i,j\le n, S(A)_{ij}>0} \frac{1}{2}\cdot \sharp(M_i(\Omega)\cap M_j(\Omega))\cdot (S(A)_{ij}-S(A)_{ji}) \\
= & \sum_{1\le i,j\le n, a_{ij}>a_{ji}} \frac{1}{2}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^+ \cap M_j(\Omega)^+) \\
&+ \sum_{1\le i,j\le n, a_{ij}>a_{ji}} \frac{1}{2}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^- \cap M_j(\Omega)^-) \\
&- \sum_{1\le i,j\le n, a_{ij}>a_{ji}} \frac{1}{2}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^+ \cap M_j(\Omega)^-) \\
&- \sum_{1\le i,j\le n, a_{ij}>a_{ji}} \frac{1}{2}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^- \cap M_j(\Omega)^+) \\
&+ \sum_{1\le i,j\le n} \frac{1}{4}\cdot |a_{ij}-a_{ji}| \cdot \left(\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^+) + \sharp(M_i(\Omega)^- \cap M_j(\Omega)^-)\right) \\
&+ \sum_{1\le i,j\le n} \frac{1}{4}\cdot |a_{ij}-a_{ji}| \cdot \left(\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^-) + \sharp(M_i(\Omega)^- \cap M_j(\Omega)^+)\right) \\
&+ \sum_{1\le i,j\le n, i\ne j, a_{ij}=a_{ji}} \frac{1}{2}\cdot a_{ij}\cdot \left(\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^+) + \sharp(M_i(\Omega)^- \cap M_j(\Omega)^-)\right) \qquad (3.7) \\
&- \sum_{1\le i,j\le n, i\ne j, a_{ij}=a_{ji}} \frac{1}{2}\cdot a_{ji}\cdot \left(\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^-) + \sharp(M_i(\Omega)^- \cap M_j(\Omega)^+)\right) \\
= & \sum_{1\le i,j\le n, a_{ij}\ne a_{ji}} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^+ \cap M_j(\Omega)^+) \\
&+ \sum_{1\le i,j\le n, a_{ij}\ne a_{ji}} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^- \cap M_j(\Omega)^-)
\end{aligned}
$$

$$
\begin{aligned}
& - \sum_{1\le i,j\le n, a_{ij}\neq a_{ji}} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^+\cap M_j(\Omega)^-) \\
& - \sum_{1\le i,j\le n, a_{ij}\neq a_{ji}} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^-\cap M_j(\Omega)^+) \\
& + \sum_{1\le i,j\le n} \frac{1}{4}\cdot |a_{ij}-a_{ji}|\cdot \big(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+) + \sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\big) \\
& + \sum_{1\le i,j\le n} \frac{1}{4}\cdot |a_{ij}-a_{ji}|\cdot \big(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-) + \sharp(M_i(\Omega)^-\cap M_j(\Omega)^+)\big) \\
& + \sum_{1\le i,j\le n, i\neq j, a_{ij}=a_{ji}} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^+\cap M_j(\Omega)^+) \\
& + \sum_{1\le i,j\le n, i\neq j, a_{ij}=a_{ji}} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^-\cap M_j(\Omega)^-) \\
& - \sum_{1\le i,j\le n, i\neq j, a_{ij}=a_{ji}} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^+\cap M_j(\Omega)^-) \\
& - \sum_{1\le i,j\le n, i\neq j, a_{ij}=a_{ji}} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^-\cap M_j(\Omega)^+) \\
= & \sum_{1\le i,j\le n, i\neq j} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \big(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+) + \sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\big) \\
& - \sum_{1\le i,j\le n, i\neq j} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \big(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-) + \sharp(M_i(\Omega)^-\cap M_j(\Omega)^+)\big) \\
& + \sum_{1\le i,j\le n} \frac{1}{4}\cdot |a_{ij}-a_{ji}|\cdot \big(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^+) + \sharp(M_i(\Omega)^-\cap M_j(\Omega)^-)\big) \\
& + \sum_{1\le i,j\le n} \frac{1}{4}\cdot |a_{ij}-a_{ji}|\cdot \big(\sharp(M_i(\Omega)^+\cap M_j(\Omega)^-) + \sharp(M_i(\Omega)^-\cap M_j(\Omega)^+)\big).
\end{aligned}
$$

Composing the a priori estimates in (3.1), (3.2), (3.3), and (3.7), we conclude that

$$
\begin{aligned}
& \langle A, C(\Omega)\rangle \\
= & \langle H(A), H(C(\Omega))\rangle + \langle S(A), S(C(\Omega))\rangle \\
\geq & \sum_{1\le i\le n} H(A)_{ii}\cdot \frac{1}{2}\big(\sharp(M_i(\Omega)^+\cap M_i(\Omega)^+) + \sharp(M_i(\Omega)^-\cap M_i(\Omega)^-)\big) \\
& - \sum_{1\le i,j\le n} \frac{1}{4}\cdot \min\{\sharp M_i(\Omega), \sharp M_j(\Omega)\}\cdot |a_{ij}-a_{ji}| \\
& + \sum_{1\le i,j\le n, i\neq j} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^+\cap M_j(\Omega)^+)
\end{aligned}
$$

$$
\begin{aligned}
& + \sum_{1\le i,j\le n, i\ne j} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^- \cap M_j(\Omega)^-) \\
& - \sum_{1\le i,j\le n, i\ne j} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^+ \cap M_j(\Omega)^-) \\
& - \sum_{1\le i,j\le n, i\ne j} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^- \cap M_j(\Omega)^+) \qquad (3.8)\\
& + \sum_{1\le i,j\le n} \frac{1}{4}\cdot |a_{ij}-a_{ji}|\cdot \big(\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^+) + \sharp(M_i(\Omega)^- \cap M_j(\Omega)^-)\big) \\
& + \sum_{1\le i,j\le n} \frac{1}{4}\cdot |a_{ij}-a_{ji}|\cdot \big(\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^-) + \sharp(M_i(\Omega)^- \cap M_j(\Omega)^+)\big) \\
\ge & \sum_{1\le i\le n} \frac{1}{4}\cdot (a_{ii}+a_{ii})\cdot \big(\sharp(M_i(\Omega)^+ \cap M_i(\Omega)^+) + \sharp(M_i(\Omega)^- \cap M_i(\Omega)^-)\big) \\
& + \sum_{1\le i,j\le n, i\ne j} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^+ \cap M_j(\Omega)^+) \\
& + \sum_{1\le i,j\le n, i\ne j} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^- \cap M_j(\Omega)^-) \\
& - \sum_{1\le i,j\le n, i\ne j} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^+ \cap M_j(\Omega)^-) \\
& - \sum_{1\le i,j\le n, i\ne j} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \sharp(M_i(\Omega)^- \cap M_j(\Omega)^+) \\
& - \sum_{1\le i,j\le n} \frac{1}{4}\cdot \min\{\sharp M_i(\Omega), \sharp M_j(\Omega)\}\cdot |a_{ij}-a_{ji}|.
\end{aligned}
$$

Since for each $i = 1, 2, \ldots, n$,

$$\sharp(M_i(\Omega)^+ \cap M_i(\Omega)^-) + \sharp(M_i(\Omega)^- \cap M_i(\Omega)^+) = 0,$$

it follows from (1.2) and (3.8) that

$$
\begin{aligned}
& \langle A, C(\Omega)\rangle \\
\ge & \sum_{1\le i,j\le n} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \big(\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^+) + \sharp(M_i(\Omega)^- \cap M_j(\Omega)^-)\big) \\
& - \sum_{1\le i,j\le n} \frac{1}{4}\cdot (a_{ij}+a_{ji})\cdot \big(\sharp(M_i(\Omega)^+ \cap M_j(\Omega)^-) + \sharp(M_i(\Omega)^- \cap M_j(\Omega)^+)\big) \qquad (3.9)\\
& - \sum_{1\le i,j\le n} \frac{1}{4}\cdot \min\{\sharp M_i(\Omega), \sharp M_j(\Omega)\}\cdot |a_{ij}-a_{ji}| \ge 0.
\end{aligned}
$$

Now let $x(t)$ be the phase flow generated by (1.1). According to [5, Theorem 2], we conclude that the phase flow $x(t)$ cannot behave in

$$x(T) = x^0, x(T+1) = x^1, \ldots, x(T+p) = x^p$$

for each $T = 0, 1, \ldots$. We claim that there exists $T \geq 0$ such that

$$[x(T), x(T+1), \ldots, x(T')] = [x(T), x(T), \ldots, x(T)] \text{ for each } T' > T. \tag{3.10}$$

If (3.10) is not valid, there exist $T_1, T_2, \ldots$ with $T_1 < T_2 < \cdots$ such that

$$x(T_k) \neq x(T_{k+1}) \text{ for each } k = 1, 2, \ldots. \tag{3.11}$$

From (3.11), we can find positive integers k_1, k_2 with $k_1 + 1 < k_2$ such that $x(T_{k_1}) = x(T_{k_2})$ and the loop

$$\Omega' = [x(T_{k_1}), x(T_{k_1} + 1), \ldots, x(T_{k_1+1}), \ldots, x(T_{k_2})]$$

satisfies $\langle A, C(\Omega') \rangle \geq 0$ by (3.9), contradicting [5, Theorem 2], and that completes the proof.

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