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SECOND-ORDER IMPLICIT DIFFERENTIAL INCLUSIONS WITH DISCONTINUOUS RIGHT-HAND SIDE

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Abstract. Given a multifunction $F : [a,b] \times \mathbb{R} \to 2^{\mathbb{R}}$, we consider the implicit multivalued boundary value problem

$$\begin{cases} h(u''(t)) \in F(t,u(t)) & \text{a.e. in } [a,b] \\ u(a) = u(b) = 0. \end{cases}$$

We prove an existence theorem for solutions $u \in W^{2,p}([a,b])$, where for each $t \in [a,b]$ the multifunction $F(t,\cdot)$ can fail to be lower semicontinuous even at all points $x \in \mathbb{R}$. In particular, our assumptions are satisfied, for instance, if there exist a neglegible set $E \subseteq \mathbb{R}$ and a multifunction $G : [a,b] \times \mathbb{R} \to 2^{\mathbb{R}}$ such that for a.a. $t \in [a,b]$ one has

$$\{x \in \mathbb{R} : G(t,\cdot) \text{ is not l.s.c. at } x\} \cup \{x \in \mathbb{R} : G(t,x) \neq F(t,x)\} \subseteq E.$$

No monotonicity assumption is required for h or F. Our result extends Theorem 3 of [5], in which the explicit case is considered.

1. Introduction

Recently, in [16], J. Saint Raymond proved the following deep result concerning the existence of Riemann-measurable selections (that is, selections that are continuous at almost each point of their domain) of a given multifunction.

Theorem 1.1 (Theorem 3 of [16]). *Let X be a Polish space (that is, a complete separable metric space) equipped with a σ-finite regular Borel measure, E a metric space and $F : X \to 2^E$ a multifunction with nonempty complete values. If F is lower semicontinuous at almost every point of X, then F admits a selection which is continuous at almost every point of X.*

Then, in the paper [5], the following parametrized version of Theorem 1.1 was established (where $\mathcal{T}_\mu$ denotes the completion of the Borel σ-algebra $\mathcal{B}(T)$ of T with respect to the measure μ).

Theorem 1.2 (Theorem 2 of [5]). *Let T, X be two Polish spaces and let μ, ψ be two positive regular Borel measures on T and X, respectively, with μ finite and ψ σ-finite. Let S be a separable metric space, $F : T \times X \to 2^S$ a multifunction with nonempty complete values, and let $E \subseteq X$ be a given set. Assume that:*

(i) *F is $\mathcal{T}_\mu \otimes \mathcal{B}(X)$-measurable;*

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(ii) *for a.a.* $t \in T$, *one has*

$$\{x \in X : F(t, \cdot) \text{ is not lower semicontinuous at } x\} \subseteq E.$$

Then, there exist a selection $\phi : T \times X \to S$ *of* F *and a set* $V \in \mathcal{B}(X)$, *with* $\psi(V) = 0$, *such that:*

(a) $\phi(\cdot, x)$ *is* $\mathcal{T}_\mu$*-measurable for each* $x \in X \setminus (E \cup V)$;

(b) *for a.a.* $t \in T$, *one has*

$$\{x \in X : \phi(t, \cdot) \text{ is not continuous at } x\} \subseteq E \cup V.$$

As an application of Theorem 1.2, the following existence result for solutions of second-order (scalar) differential inclusions was proved in [5] (as usual, $W^{2,p}([a,b])$ denotes the space of all functions $u \in C^1([a,b])$ such that u' is absolutely continuous in $[a,b]$ and $u'' \in L^p([a,b])$; moreover, $\mathcal{L}([a,b])$ will denote the family of all Lebesgue-measurable subsets of $[a,b]$).

Theorem 1.3 (Theorem 3 of [5]). *Let* $[a,b]$ *be a closed interval,* $F : [a,b] \times I\!R \to 2^{I\!R}$ *a multifunction,* $p \in [1, +\infty[$.

Assume that there exists a multifunction $G : [a,b] \times I\!R \to 2^{I\!R}$, *with nonempty closed values, and two functions* $\alpha : [a,b] \to]0, +\infty[$ *and* $\beta \in L^p([a,b])$ *such that:*

(i) G *is* $\mathcal{L}([a,b]) \otimes \mathcal{B}(I\!R)$ *-measurable;*

(ii) *there exists* $E \subseteq I\!R$, *with* $m(E) = 0$, *such that for a.a.* $t \in [a,b]$ *one has*

$$\big\{x \in I\!R : G(t, \cdot\,) \text{ is not l.s.c. at } x\big\} \cup \big\{x \in I\!R : G(t,x) \not\subseteq F(t,x)\big\} \subseteq E;$$

(iii) *for a.a.* $t \in [a,b]$ *and for all* $x \in I\!R$, *one has*

$$G(t,x) \subseteq [\alpha(t), \beta(t)].$$

Then, there exists $u \in W^{2,p}([a,b])$ *such that*

$$\begin{cases} u''(t) \in F(t, u(t)) & \text{for a.a. } t \in [a,b] \\ u(a) = u(b) = 0. \end{cases}$$

The aim of this note is simply to extend Theorem 1.3 to the implicit case. Namely, our aim is to prove the following result.

Theorem 1.4. *Let* $[a,b]$, F, p, G, α *and* β *be as in Theorem 1.3. Let* $A \subseteq]0, +\infty[$ *be another interval, and* $h : A \to I\!R$ *a continuous function. Assume that assumptions* (i) *and* (ii) *of Theorem 1.3 are satisfied. Moreover, assume that:*

(iii)′ $\mathrm{int}(h^{-1}(z)) = \emptyset$ *for all* $z \in \mathrm{int}(h(A))$;

(iv)′ *for a.a.* $t \in [a,b]$ *and for all* $x \in I\!R$, *one has*

$$G(t,x) \subseteq h(A) \qquad \text{and} \qquad h^{-1}(G(t,x)) \subseteq [\alpha(t), \beta(t)] \subseteq A.$$

Then, there exists $u \in W^{2,p}([a,b])$ *such that*

$$\begin{cases} h(u''(t)) \in F(t, u(t)) & \text{for a.a. } t \in [a,b] \\ u(a) = u(b) = 0. \end{cases} \tag{1.1}$$

Of course, the main peculiarity of Theorems 1.3 and 1.4 resides in the kind of discontinuity that is allowed for F. Indeed, assumption (ii) of Theorem 1.3 is satisfied, for instance, if (taking $F = G$) there exists a null-measure set $E \subseteq \mathbb{R}$ such that for a.a. $t \in [a, b]$ one has

$$\{x \in \mathbb{R} : F(t, \cdot\,) \text{ is not lower semicontinuous at } x\big\} \subseteq E.$$

As a matter of fact, Theorems 1.3 and 1.4 allow much more discontinuity for F. Assumption (ii) is satisfied, for instance, if for a.a. $t \in [a, b]$ one has

$$\big\{x \in \mathbb{R} : G(t, \cdot\,) \text{ is not l.s.c. at } x\big\} \cup \big\{x \in \mathbb{R} : G(t, x) \neq F(t, x)\big\} \subseteq E,$$

where $G : [a, b] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multifunction and $E \subseteq \mathbb{R}$ has null Lebesgue measure. In this case, for a.a. $t \in [a, b]$, the multifunction $F(t, \cdot)$ is required to be a.e. equal to a multifunction $G(t, \cdot)$ which, in turn, is a.e. lower semicontinuous in $\mathbb{R}$. As can be easily seen, such a multifunction F can fail to be lower semicontinuous even at all points $x \in \mathbb{R}$. Therefore, assumption (ii) of Theorem 1.3 seems to be much less restrictive than the usual lower semicontinuity, Lipschitz, Caratheodory, almost lower semicontinuity conditions usually required in the literature (see, for instance, [1–3, 6–8, 12, 13] and the references therein).

In particular, when F is single-valued, assumption (ii) is satisfied if there exist a $\mathcal{L}([a, b]) \otimes \mathcal{B}(\mathbb{R})$-measurable single-valued function $g : [a, b] \times \mathbb{R} \to \mathbb{R}$ and a null-measure set $E \subseteq \mathbb{R}$ such that for a.e. $t \in [a, b]$ one has

$$\big\{x \in \mathbb{R} : g(t, \cdot\,) \text{ is discontinuous at } x\big\} \cup \big\{x \in \mathbb{R} : g(t, x) \neq F(t, x)\big\} \subseteq E.$$

It is easy to see that such a function F can be discontinuous in the second variable *even at all points* $x \in \mathbb{R}$. For instance, one can take (in the autonomous case) $F(x)$ as the usual Dirichlet function, which is discontinuous at each point $x \in \mathbb{R}$ and it is a.e. equal to a constant function. In this connection, it is useful to compare Theorem 1.4 with Theorem 2.2 of [12], where the implicit (single-valued) boundary value problem 1.1 is studied by assuming the continuity of F (under the same hypotheses on h).

The proof of Theorem 1.4 will be given in Section 2. The main tools will be the notion of inductively open function and a related result by B. Ricceri (Theorem 2.4 of [15]), together with Theorem 1.2 and an existence result for explicit and graph-closed differential inclusions (Theorem 3 of [14]).

It is worth noticing that in Theorem 1.4 no convexity assumption is required on the values of F. For the basic definitions and facts on multifunctions, the reader is referred to [11].

2. Proof of Theorem 1.4

First, we observe that by assumption (iii)$'$ and Theorem 2.4 of [15] the function h is inductively open. That is, there exists a set $Y \in \mathcal{B}(A)$ such that the function

$$h|_Y : Y \to h(A)$$

is open and $h(Y) = h(A)$. It follows that the multifunction $T : h(A) \to 2^Y$ defined by putting, for each $s \in h(A)$,

$$T(s) = h^{-1}(s) \cap Y$$

is lower semicontinuous in $h(A)$ with nonempty values. To see this, fix any set $\Omega_Y \subseteq Y$, with Ω_Y open in the relative topology of Y. We get

$$\begin{aligned} T^-(\Omega_Y) : &= \{s \in h(A) : \ T(s) \cap \Omega_Y \neq \emptyset\} \\ &= \{s \in h(A) : \ h^{-1}(s) \cap Y \cap \Omega_Y \neq \emptyset\} \\ &= \{s \in h(A) : \ h^{-1}(s) \cap \Omega_Y \neq \emptyset\} \\ &= h(\Omega_Y). \end{aligned}$$

Since the function $h|_Y : Y \to h(A)$ is open, the set $h(\Omega_Y)$ is open in $h(A)$. It follows that the set $T^-(\Omega_Y)$ is open in $h(A)$, hence T is lower semicontinuous in $h(A)$, as claimed.

Without loss of generality we can assume that assumptions (ii) and (iv)$'$ are satisfied for all $t \in [a,b]$. Let $\Psi : [a,b] \times \mathbb{R} \to 2^Y$ be defined by

$$\Psi(t,x) := T(G(t,x)) = h^{-1}(G(t,x)) \cap Y$$

(note that Ψ is well-defined since $G(t,x) \subseteq h(A)$ for all $(t,x) \in [a,b] \times \mathbb{R}$). We observe the following facts:

(a) the multifunction Ψ is $\mathcal{L}([a,b]) \otimes \mathcal{B}(\mathbb{R})$ - weakly measurable. That is, for each set $\Omega \subseteq Y$, with Ω open in the relative topology of Y, the set

$$\Psi^-(\Omega) = \{(t,x) \in [a,b] \times \mathbb{R} : \ \Psi(t,x) \cap \Omega \neq \emptyset\}$$

belongs to $\mathcal{L}([a,b]) \otimes \mathcal{B}(\mathbb{R})$ (this follows from Proposition 13.2.1 of [11], since G is $\mathcal{L}([a,b]) \otimes \mathcal{B}(\mathbb{R})$-measurable and T is lower semicontinuous);

(b) Ψ has nonempty values and for all $t \in [a,b]$ one has

$$\big\{x \in \mathbb{R} : \Psi(t,\cdot\,) \text{ is not lower semicontinuous at } x\big\} \subseteq E.$$

Indeed, if $t \in [a,b]$ and $\overline{x} \in \mathbb{R} \setminus E$ are fixed, then the multifunction $G(t,\cdot)$ is lower semicontinuous at $\overline{x}$, hence (by the lower semicontinuity of T) the multifunction

$$x \in \mathbb{R} \to \Psi(t,x) = T(G(t,x))$$

is lower semicontinuous at $\overline{x}$, as claimed.

Now, let $\overline{\Psi} : [a,b] \times \mathbb{R} \to 2^{\mathbb{R}}$ be defined by

$$\overline{\Psi}(t,x) = \overline{\Psi(t,x)}.$$

It follows by assumption (iv)$'$ and by construction that

$$\overline{\Psi}(t,x) \subseteq [\alpha(t), \beta(t)] \quad \text{for all} \quad (t,x) \in [a,b] \times \mathbb{R}. \tag{2.1}$$

By Proposition 2.6 and Theorem 3.5 of [10], the multifunction $\overline{\Psi}$ is $\mathcal{L}([a,b]) \otimes \mathcal{B}(\mathbb{R})$-measurable with nonempty values. Moreover, for all $t \in [a,b]$ one has

$$\big\{x \in \mathbb{R} : \overline{\Psi}(t,\cdot\,) \text{ is not l.s.c. at } x\big\} \subseteq E.$$

By Theorem 1.2, there exists a selection $\phi : [a,b] \times \mathbb{R} \to \mathbb{R}$ of $\overline{\Psi}$ and two neglegible sets

$$K_1 \subseteq [a,b] \qquad \text{and} \qquad V \subseteq \mathbb{R}$$

such that, if one puts $D := E \cup V$, one has

(a)$'$ for each $x \in \mathbb{R} \setminus D$, the function $\phi(\cdot,x)$ is measurable;

(b)′ for each $t \in [a,b] \setminus K_1$, one has

$$\{x \in \mathbb{R} : \phi(t,\cdot\,) \text{ is not continuous at } x\} \subseteq D.$$

Moreover, by (2.1) we get

$$\phi(t,x) \in [\alpha(t),\beta(t)] \quad \text{for all} \quad (t,x) \in [a,b] \times \mathbb{R}. \tag{2.2}$$

Since $m(D) = 0$ (m denoting the Lebesgue measure on the real line), there exists a numerable set $P \subseteq \mathbb{R} \setminus D$ which is dense in $\mathbb{R}$. Let $H : [a,b] \times \mathbb{R} \to 2^{\mathbb{R}}$ be defined by

$$H(t,x) = \bigcap_{m\in\mathbb{N}} \overline{\mathrm{co}}\,\overline{\Big(\bigcup_{y\in P, |y-x|\le \frac{1}{m}} \{\phi(t,y)\} \Big)}$$

(where "$\overline{\mathrm{co}}$" stands for "closed convex hull") By Proposition 2 of [4], we have that:
(a)″ H has nonempty closed convex values;
(b)″ for all $x \in \mathbb{R}$, the multifunction $H(\cdot,x)$ is measurable;
(c)″ for all $t \in [a,b]$, $H(t,\cdot)$ has closed graph;
(d)″ for all $t \in [a,b] \setminus K_1$, and for all $x \in \mathbb{R} \setminus D$, one has

$$H(t,x) = \{\phi(t,x)\}.$$

Moreover, it follows by (2.2) that

$$H(t,x) \subseteq [\alpha(t),\beta(t)] \quad \text{for all} \quad (t,x) \in [a,b] \times \mathbb{R}.$$

By Theorem 3 of [14] (where we can take $r = \|\beta\|_{L^p([a,b])}$), there exists a function $u \in W^{2,p}([a,b])$ such that

$$\begin{cases} u''(t) \in H(t,u(t)) & \text{for a.a. } t \in [a,b] \\ u(a) = u(b) = 0. \end{cases}$$

Let $K \subseteq [a,b]$, with $K_1 \subseteq K$ and $m(K) = 0$, be such that

$$u''(t) \in H(t,u(t)) \quad \text{for all} \quad t \in [a,b] \setminus K.$$

Since

$$u''(t) \in H(t,u(t)) \subseteq [\alpha(t),\beta(t)] \quad \text{for a.e.} \quad t \in [a,b],$$

we have that $u''(t) > 0$ a.e. in $[a,b]$. It follows easily that u' is strictly increasing in $[a,b]$. Since $u(a) = u(b) = 0$, there exists a point $c \in]a,b[$ such that

$$u'(t) < 0 \quad \text{for all} \quad t \in [a,c[\,,$$
$$u'(t) > 0 \quad \text{for all} \quad t \in]c,b].$$

Let

$$g_1 := u|_{[a,c]}, \qquad g_2 := u|_{[c,b]}.$$

By Theorem 2 of [17], the functions g_1^{-1} and g_2^{-1} are absolutely continuous. Therefore, by Theorem 18.25 of [9], the sets $g_1^{-1}(D)$ and $g_2^{-1}(D)$ have null Lebesgue measure, hence $m(u^{-1}(D)) = 0$. Consequently, the set

$$S := u^{-1}(D) \cup K$$

has null Lebesgue measure. We claim that

$$h(u''(t)) \in F(t,u(t)) \quad \text{for all} \quad t \in [a,b] \setminus S. \tag{2.3}$$

To this aim, fix $t \in [a,b] \setminus S$. Since $u(t) \notin D$ and $t \notin K$, we get

$$u''(t) \in H(t,u(t)) = \{\phi(t,u(t))\},$$

hence

$$u''(t)) \in \overline{\Psi}(t,u(t)) = \overline{h^{-1}(G(t,u(t)) \cap Y} \subseteq \overline{h^{-1}(G(t,u(t))}. \tag{2.4}$$

Since $G(t,u(t))$ is closed and h is continuous, the set $W := h^{-1}(G(t,u(t))$ is closed in A. By (iv)$'$, we have

$$W := h^{-1}(G(t,u(t)) \subseteq [\alpha(t), \beta(t)] \subseteq A,$$

hence W is closed in $\mathbb{R}$. Consequently, by (2.4) we get

$$u''(t) \in \overline{h^{-1}(G(t,u(t))} = h^{-1}(G(t,u(t)),$$

and thus $h(u''(t)) \in G(t,u(t))$. Since $u(t) \notin E$, we get $G(t,u(t)) \subseteq F(t,u(t))$, hence

$$h(u''(t)) \in F(t,u(t))$$

and our claim (2.3) is proved. Consequently, the function u satisfies the conclusion.

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