Journal of Nonlinear and Convex Analysis
Volume 15, Number 6, 2014, 1181–1191

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

REMARKS ON MULTIVALUED QUASI-NONEXPANSIVE MAPPINGS IN ℝ-TREES

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Abstract. It is shown that, in an ℝ-tree, the notion of proximally quasi-nonexpansive mappings introduced by Song and Cho [25] is weaker than the notion of multivalued quasi-nonexpansive mappings. We also obtain the strong convergence of an Ishikawa iteration for proximally quasi-nonexpansive mappings without assuming the endpoint condition.

1. Introduction

Let E be a nonempty closed convex subset of a Banach space X. A single-valued mapping $t : E \to E$ is said to be *nonexpansive* [12] if

$$\|t(x) - t(y)\| \le \|x - y\|, \text{ for all } x, y \in E.$$

A point $x \in E$ is called a *fixed point* of t if $x = t(x)$. We shall denote by $\mathrm{Fix}(t)$ the set of all fixed points of t. If $\mathrm{Fix}(t) \neq \emptyset$ and $\|t(x) - t(p)\| \le \|x - p\|$ for all $x \in E$ and $p \in \mathrm{Fix}(t)$, then t is called a *quasi-nonexpansive mapping.*

In 1974, Ishikawa [9] introduced an iteration process for approximating fixed points of a mapping t on a Hilbert space H by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n t\left((1 - \beta_n)x_n + \beta_n t(x_n)\right), \ n \ge 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying some certain restrictions. For more details and literature on the convergence of the Ishikawa iteration for single-valued mappings, see e.g., [2, 3, 7, 8, 10, 11, 14, 16, 18, 20, 27, 29].

Fixed Point Theory for multivalued mappings (that is, mappings such that the image of a point is a set) has many useful applications in Applied Sciences, in particular, in Game Theory and Optimization Theory. Thus, it is natural to study the extension of the known fixed point results for single-valued mappings to the setting of multivalued mappings.

The first result concerning to the convergence of an Ishikawa iteration for multivalued nonexpansive mappings was proved by Sastry and Babu [22] in a Hilbert space. Panyanak [17] extended the result of Sastry and Babu to a uniformly convex Banach space. Since then the strong convergence theorems of the Ishikawa iterations for multivalued nonexpansive and quasi-nonexpansive mappings have been rapidly developed and many of papers have appeared (see e.g., [4, 13, 19, 23, 24, 26]). But, all of them assumed the endpoint condition (i.e., $T(x) = \{x\}$ for each $x \in \mathrm{Fix}(T)$)

2010 *Mathematics Subject Classification.* 47H09, 47H10 .

Key words and phrases. Fixed point, multivalued mapping, strong convergence, Ishikawa iteration, ℝ-tree.

The first author was supported by the Thai Government Scholarships in the Area of Science and Technology and the Graduate School of Chiang Mai University. The second author was supported by Chiang Mai University.

which is a strong one. Among other things, Shahzad and Zegeye [23] defined an Ishikawa iteration for multivalued mappings and obtained the interesting results as follows.

Let E be a nonempty closed convex subset of a Banach space X, $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$, and $T : E \to 2^E$ be a multivalued mapping.
(A): The sequence of Ishikawa iterates [23] is defined by $x_1 \in E$,

$$y_n = \beta_n z_n + (1-\beta_n)x_n, \;\; n \geq 1,$$

where $z_n \in T(x_n)$, and

$$x_{n+1} = \alpha_n z_n' + (1-\alpha_n)x_n, \;\; n \geq 1,$$

where $z_n' \in T(y_n)$.

Theorem 1.1 ([23, Theorem 2.3])**.** Let X be a uniformly convex Banach space, E be a nonempty closed convex subset of X and $T : E \to 2^E$ be a quasi-nonexpansive mapping whose values are nonempty closed bounded subsets of E. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by (A) with $\alpha_n, \beta_n \in [a,b] \subset (0,1)$. If T satisfies condition (I) and the endpoint condition, then $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 1.2 ([23, Theorem 2.5])**.** Let X be a uniformly convex Banach space, E be a nonempty closed convex subset of X and $T : E \to 2^E$ be a quasi-nonexpansive mapping whose values are nonempty closed bounded subsets of E. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by (A) with $0 \leq \alpha_n, \beta_n < 1$, $\beta_n \to 0$ and $\sum \alpha_n \beta_n = \infty$. If T is hemicompact and continuous and satisfies the endpoint condition, then $\{x_n\}$ converges strongly to a fixed point of T.

Moreover, to avoid the endpoint condition, Shahzad and Zegeye [23] constructed a modified Ishikawa iteration for proximally nonexpansive mappings and proved strong convergence theorems of the proposed iteration as the following results.

Let E be a nonempty closed convex subset of a Banach space X, $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$, and $T : E \to 2^E$ be a multivalued mapping whose values are nonempty proximinal subsets of E. For each $x \in E$, let $P_T(x) = \{u \in T(x) : \|x-u\| = \mathrm{dist}(x, T(x))\}$.
(B): The sequence of Ishikawa iterates [23] is defined by $x_1 \in E$,

$$y_n = \beta_n z_n + (1-\beta_n)x_n, \;\; n \geq 1,$$

where $z_n \in P_T(x_n)$, and

$$x_{n+1} = \alpha_n z_n' + (1-\alpha_n)x_n, \;\; n \geq 1,$$

where $z_n' \in P_T(y_n)$.

Theorem 1.3 ([23, Theorem 2.7])**.** Let X be a uniformly convex Banach space, E be a nonempty closed convex subset of X and $T : E \to 2^E$ be a proximally nonexpansive mapping whose values are nonempty proximinal subsets of E with $\mathrm{Fix}(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by (B) with $\alpha_n, \beta_n \in [a,b] \subset (0,1)$. If T satisfies condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 1.4 ([23, Theorem 2.8])**.** Let X be a uniformly convex Banach space, E be a nonempty closed convex subset of X and $T : E \to 2^E$ be a proximally nonexpansive mapping whose values are nonempty proximinal subsets of E with $\mathrm{Fix}(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by (B) with $0 \leq \alpha_n, \beta_n < 1$, $\beta_n \to 0$ and $\sum \alpha_n \beta_n = \infty$. If T is hemicompact, then $\{x_n\}$ converges strongly to a fixed point of T.

In 2011, Song and Cho [25] introduced the notion of proximally quasi-nonexpansive mappings and extended Theorems 1.3 and 1.4 to the case of proximally quasi-nonexpansive mappings. However, in general Banach spaces, the class of such mappings is different from the class of multivalued quasi-nonexpansive mappings (see Examples 4.2 and 4.3). Therefore, the convergence of an Ishikawa iteration for multivalued quasi-nonexpansive mappings without the endpoint condition is still unknown.

On the other hand, the present authors [21] proved the convergence of an Ishikawa iteration for multivalued quasi-nonexpansive mappings in an $\mathbb{R}$-tree by assuming the gate condition which is weaker than the endpoint condition. Summary: there is no any result in Banach or metric spaces concerning the convergence of an Ishikawa iteration for multivalued quasi-nonexpansive mappings which completely removes the endpoint condition. Therefore, the following question can be of interest:

Question 1.5. *Let T be a multivalued quasi-nonexpansive mapping defined on a complete $\mathbb{R}$-tree X and let $\{x_n\}$ be a sequence of the Ishikawa iterates defined from T. Assume that T satisfies condition (I) or T is hemicompact and continuous. Does $\{x_n\}$ converge to a fixed point of T?*

In this paper, we show that every multivalued quasi-nonexpansive mapping defined on a subset of a complete $\mathbb{R}$-tree is proximally quasi-nonexpansive. Moreover, we prove the strong convergence of the Ishikawa iteration defined by (B) for proximally quasi-nonexpansive mappings without assuming the endpoint condition. This gives an affirmative answer to the question mentioned above.

2. Preliminaries and lemmas

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a *geodesic segment* joining x and y. When it is unique this geodesic is denoted by $[x, y]$. For $x, y \in X$ and $\alpha \in [0, 1]$, we denote the point $z \in [x, y]$ such that $d(x, z) = \alpha d(x, y)$ by $(1 - \alpha)x \oplus \alpha y$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset E of X is said to be *convex* if E includes every geodesic segment joining any two of its points. If $x \in X$ and $E \subset X$, then the distance from x to E is defined by

$$\mathrm{dist}(x, E) = \inf\{d(x, y) : y \in E\}.$$

The set E is called *proximinal* if for each $x \in X$, there exists an element $y \in E$ such that $d(x, y) = \mathrm{dist}(x, E)$, and E is said to be *gated* if for any point $x \notin E$ there is a unique point y_x such that for any $z \in E$,

$$d(x,z) = d(x,y_x) + d(y_x,z).$$

The point y_x is called the *gate* of x in E. We shall denote by 2^E the family of nonempty subsets of E, by $\mathcal{P}(E)$ the family of nonempty proximinal subsets of E, by $\mathcal{C}(E)$ the family of nonempty closed subsets of E and by $\mathcal{CC}(E)$ the family of nonempty closed convex subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on 2^E, i.e.,

$$H(A,B) = \max\left\{\sup_{a\in A} \operatorname{dist}(a,B), \sup_{b\in B} \operatorname{dist}(b,A)\right\}, \quad A,B \in 2^E.$$

Let $T : E \to 2^E$ be a multivalued mapping. For each $x \in E$, we let

$$P_{T(x)}(x) = \{u \in T(x) : d(x,u) = \operatorname{dist}(x,T(x))\}.$$

In the case of $P_{T(x)}(x)$ is a singleton we will assume, without loss of generality, that $P_{T(x)}(x)$ is a point in E. A point $x \in E$ is called a *fixed point* of T if $x \in T(x)$.

Definition 2.1. Let E be a nonempty subset of a metric space (X,d) and $T : E \to 2^E$. Then T is said to be

(i) *nonexpansive* if $H(T(x),T(y)) \le d(x,y)$ for all $x,y \in E$;

(ii) *quasi-nonexpansive* if $Fix(T) \neq \emptyset$ and

$$H(T(x),T(p)) \le d(x,p) \text{ for all } x \in E \text{ and } p \in Fix(T).$$

(iii) *proximally nonexpansive* if the map $P_T : E \to 2^E$ defined by $x \mapsto P_{T(x)}(x)$ is nonexpansive;

(iv) *proximally quasi-nonexpansive* if the map $P_T : E \to 2^E$ defined by $x \mapsto P_{T(x)}(x)$ is quasi-nonexpansive.

The mapping T is said to satisfy *condition (I)* if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$ such that

$$\operatorname{dist}(x,T(x)) \ge f(\operatorname{dist}(x,\operatorname{Fix}(T))) \text{ for all } x \in E.$$

The mapping T is called *hemicompact* if for any sequence $\{x_n\}$ in E such that

$$\lim_{n\to\infty} \operatorname{dist}(x_n,T(x_n)) = 0,$$

there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q \in E$ such that $\lim_{k\to\infty} x_{n_k} = q$.

Definition 2.2. An $\mathbb{R}$-*tree* (sometimes called *metric tree*) is a geodesic metric space X such that:

(i) there is a unique geodesic segment $[x,y]$ joining each pair of points $x,y \in X$;

(ii) if $[y,x] \cap [x,z] = \{x\}$, then $[y,x] \cup [x,z] = [y,z]$.

By (i) and (ii) we have

(iii) if $u,v,w \in X$, then $[u,v] \cap [u,w] = [u,z]$ for some $z \in X$.

An $\mathbb{R}$-tree is a special case of a CAT(0) space. For a thorough discussion of these spaces and their applications, see [1]. We now collect some basic properties of $\mathbb{R}$-trees.

Lemma 2.3. Let X be a complete $\mathbb{R}$-tree and E be a nonempty subset of X. Then the following statements hold:

(i) [6, page 1048] the gate subsets of X are precisely its closed and convex subsets;
(ii) [1, page 176] if E is closed and convex, then for each $x \in X$, there exists a unique point $P_E(x) \in E$ such that
$$d(x, P_E(x)) = \operatorname{dist}(x, E).$$
That is, every nonempty closed convex subset of a complete $\mathbb{R}$-tree is proximinal.
(iii) [1, page 176] if E is closed convex and if x' belong to $[x, P_E(x)]$, then $P_E(x') = P_E(x)$;
(iv) [15, Lemma 3.1] if A and B are closed convex subsets of X, then, for any $u \in X$,
$$d(P_A(u), P_B(u)) \le H(A, B);$$
(v) [5, Lemma 2.5] if $x, y, z \in X$ and $\alpha \in [0, 1]$, then
$$d^2((1-\alpha)x \oplus \alpha y, z) \le (1-\alpha)d^2(x, z) + \alpha d^2(y, z) - \alpha(1-\alpha)d^2(x, y);$$
(vi) [21, Lemma 3.1] if E is closed convex and there exist $u \in X$ and $v \in X - E$ such that $v \in [P_E(v), u]$, then $P_E(v) = P_E(u)$.

The following lemma can be found in [25]. We observe that the boundedness of the images of T is superfluous.

Lemma 2.4. Let E be a nonempty subset of an $\mathbb{R}$-tree X and $T : E \to \mathcal{P}(E)$ be a multivalued mapping. Then
(i) $\operatorname{dist}(x, T(x)) = \operatorname{dist}(x, P_{T(x)}(x))$ for all $x \in E$;
(ii) $x \in \operatorname{Fix}(T) \iff x \in \operatorname{Fix}(P_T) \iff P_{T(x)}(x) = \{x\}$;
(iii) $\operatorname{Fix}(T) = \operatorname{Fix}(P_T)$.

The following lemma is also needed.

Lemma 2.5 ([17])**.** Let $\{\alpha_n\}, \{\beta_n\}$ be two real sequences such that
(i) $0 \le \alpha_n, \beta_n < 1$;
(ii) $\beta_n \to 0$ as $n \to \infty$;
(iii) $\sum \alpha_n \beta_n = \infty$.
Let $\{\gamma_n\}$ be a nonnegative real sequence such that $\sum \alpha_n \beta_n (1 - \beta_n)\gamma_n$ is bounded. Then $\{\gamma_n\}$ has a subsequence which converges to zero.

3. Main results

Proposition 3.1. Let E be a nonempty subset of a complete $\mathbb{R}$-tree X. If $T : E \to \mathcal{CC}(E)$ is quasi-nonexpansive, then T is proximally quasi-nonexpansive.

Proof. Let $x \in E$ and $p \in \operatorname{Fix}(P_T)$. By Lemma 2.4, $P_{T(p)}(p) = \{p\}$. In the case of $P_{T(x)}(x) \in [x, p]$, we have $d(P_{T(x)}(x), P_{T(p)}(p)) \le d(x, p)$ and the conclusion follows. But, if $P_{T(x)}(x) \notin [x, p]$, we will show that $P_{T(x)}(x) = P_{T(x)}(p)$. Let v be the gate of $P_{T(x)}(x)$ in $[x, p]$. Then $v \ne P_{T(x)}(x)$. Since $v \in [x, P_{T(x)}(x)]$, then by Lemma 2.3(iii) we have $P_{T(x)}(v) = P_{T(x)}(x)$. This implies that $v \in [P_{T(x)}(v), p]$. Since $v \notin T(x)$, by Lemma 2.3(vi) we have
$$P_{T(x)}(x) = P_{T(x)}(v) = P_{T(x)}(p).$$

This, together with Lemma 2.3(iv), we obtain that

$$\begin{aligned} d(P_{T(x)}(x), P_{T(p)}(p)) &= d(P_{T(x)}(p), P_{T(p)}(p)) \\ &\leq H(T(x), T(p)) \\ &\leq d(x, p). \end{aligned}$$

Therefore, the proof is complete. □

If E is a nonempty subset of a complete $\mathbb{R}$-tree X and $T : E \to \mathcal{CC}(E)$ a multivalued mapping with $\text{Fix}(T) \neq \emptyset$, then by using Lemma 2.4, Proposition 3.1 and [21, Proposition 3.2] we can obtain the following implications. Examples 4.1 and 4.2 show that the converses do not hold.

$$\begin{array}{ccc} T \text{ is nonexpansive} & \Rightarrow & T \text{ is quasi-nonexpansive} \\ \Downarrow & & \Downarrow \\ T \text{ is proximally nonexpansive} & \Rightarrow & T \text{ is proximally quasi-nonexpansive} \end{array}$$

The following theorem is a consequence of Theorem 3.2 in [19].

Theorem 3.2. Let X be a complete $\mathbb{R}$-tree, E be a nonempty closed convex subset of X and $T : E \to \mathcal{C}(E)$ be a quasi-nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in \text{Fix}(T)$. Let $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$ and $\{x_n\}$ be the sequence of Ishikawa iterates defined by (A) (replacing $+$ with $\oplus$). If T satisfies condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.

Now, we are ready to prove our main theorems.

Theorem 3.3. Let X be a complete $\mathbb{R}$-tree, E be a nonempty closed convex subset of X and $T : E \to \mathcal{P}(E)$ be a proximally quasi-nonexpansive mapping. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by (B) (replacing $+$ with $\oplus$). Assume that T satisfies condition (I) and $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$. Then $\{x_n\}$ converges to a fixed point of T.

Proof. It follows from Lemmas 2.4 that $\text{dist}(x, P_{T(x)}(x)) = \text{dist}(x, T(x))$ for all $x \in E$,

$$\text{Fix}(P_T) = \text{Fix}(T) \text{ and } P_T(p) = \{p\} \text{ for all } p \in \text{Fix}(P_T).$$

Since T satisfies condition (I), for each $x \in E$ we have

$$\text{dist}(x, P_{T(x)}(x)) = \text{dist}(x, T(x)) \geq f(\text{dist}(x, \text{Fix}(T))) = f(\text{dist}(x, \text{Fix}(P_T))).$$

That is, P_T satisfies condition (I). Next, we show that $P_{T(x)}(x)$ is closed for any $x \in E$. Let $\{y_n\} \subset P_{T(x)}(x)$ and $\lim_{n\to\infty} y_n = y$ for some $y \in E$. Then

$$d(x, y_n) = \text{dist}(x, T(x)) \text{ and } \lim_{n\to\infty} d(x, y_n) = d(x, y).$$

It follows that $d(x, y) = \text{dist}(x, T(x))$ and this implies $y \in P_{T(x)}(x)$. Applying Theorem 3.2 to the map P_T, we can conclude that the sequence $\{x_n\}$ defined by (B) converges to a point $x_0 \in \text{Fix}(P_T) = \text{Fix}(T)$. This completes the proof. □

As a consequence of Proposition 3.1 and Theorem 3.3, we obtain

Corollary 3.4. Let X be a complete $\mathbb{R}$-tree, E be a nonempty closed convex subset of X and $T : E \to \mathcal{CC}(E)$ be a quasi-nonexpansive mapping. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by (B) (replacing $+$ with $\oplus$). Assume that T satisfies condition (I) and $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$. Then $\{x_n\}$ converges to a fixed point of T.

Theorem 3.5. Let X be a complete $\mathbb{R}$-tree, E be a nonempty closed convex subset of X and $T : E \to \mathcal{P}(E)$ be a proximally quasi-nonexpansive mapping. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by (B) (replacing $+$ with $\oplus$). Assume that T is hemicompact and continuous and (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$ and (iii) $\sum \alpha_n \beta_n = \infty$. Then $\{x_n\}$ converges to a fixed point of T.

Proof. Let $p \in \mathrm{Fix}(T) = \mathrm{Fix}(P_T)$. Then $P_{T(p)}(p) = \{p\}$. For each $n \geq 1$, we have

$$\begin{aligned} d(y_n, p) &= d(\beta_n z_n \oplus (1-\beta_n)x_n, p) \\ &\leq \beta_n d(z_n, p) + (1-\beta_n)d(x_n, p) \\ &= \beta_n d(z_n, P_{T(p)}(p)) + (1-\beta_n)d(x_n, p) \\ &\leq \beta_n H(P_{T(x_n)}(x_n), P_{T(p)}(p)) + (1-\beta_n)d(x_n, p) \\ &\leq \beta_n d(x_n, p) + (1-\beta_n)d(x_n, p) \\ &\leq d(x_n, p) \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n z_n' \oplus (1-\alpha_n)x_n, p) \\ &\leq \alpha_n d(z_n', p) + (1-\alpha_n)d(x_n, p) \\ &= \alpha_n d(z_n', P_{T(p)}(p)) + (1-\alpha_n)d(x_n, p) \\ &\leq \alpha_n H(P_{T(y_n)}(y_n), P_{T(p)}(p)) + (1-\alpha_n)d(x_n, p) \\ &\leq \alpha_n d(y_n, p) + (1-\alpha_n)d(x_n, p) \\ &\leq d(x_n, p). \end{aligned}$$

This shows that the sequence $\{d(x_n, p)\}$ is decreasing and bounded below. Thus $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in Fix(T)$. Applying Lemma 2.3(v), for each $p \in \mathrm{Fix}(T)$ we have

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(\alpha_n z_n' \oplus (1-\alpha_n)x_n, p) \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(z_n', p) - \alpha_n(1-\alpha_n)d^2(x_n, z_n') \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n H^2(P_{T(y_n)}(y_n), P_{T(p)}(p)) \\ &\quad -\alpha_n(1-\alpha_n)d^2(x_n, z_n') \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(y_n, p) - \alpha_n(1-\alpha_n)d^2(x_n, z_n') \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(y_n, p). \qquad (3.1) \end{aligned}$$

and

$$\begin{aligned} d^2(y_n, p) &= d^2(\beta_n z_n \oplus (1-\beta_n)x_n, p) \\ &\leq (1-\beta_n)d^2(x_n, p) + \beta_n d^2(z_n, p) - \beta_n(1-\beta_n)d^2(x_n, z_n) \\ &\leq (1-\beta_n)d^2(x_n, p) + \beta_n H^2(P_{T(x_n)}(x_n), P_{T(p)}(p)) \end{aligned}$$

$$
\begin{aligned}
&\quad -\beta_n(1-\beta_n)d^2(x_n,z_n)\\
&\leq (1-\beta_n)d^2(x_n,p)+\beta_n d^2(x_n,p)-\beta_n(1-\beta_n)d^2(x_n,z_n)\\
&\leq d^2(x_n,p)-\beta_n(1-\beta_n)d^2(x_n,z_n). \qquad (3.2)
\end{aligned}
$$

By (3.1) and (3.2), we have

$$d^2(x_{n+1},p) \leq (1-\alpha_n)d^2(x_n,p)+\alpha_n d^2(x_n,p)-\alpha_n\beta_n(1-\beta_n)d^2(x_n,z_n).$$

Therefore,

$$\alpha_n\beta_n(1-\beta_n)d^2(x_n,z_n) \leq d^2(x_n,p)-d^2(x_{n+1},p).$$

This implies

$$\sum_{n=1}^{\infty}\alpha_n\beta_n(1-\beta_n)d^2(x_n,z_n) < \infty.$$

Thus, by Lemma 2.5, there exists subsequences $\{x_{n_k}\}$ and $\{z_{n_k}\}$ of $\{x_n\}$ and $\{z_n\}$ respectively, such that $\lim_{k\to\infty} d(x_{n_k},z_{n_k})=0$. Hence

$$\lim_{k\to\infty}\operatorname{dist}(x_{n_k},T(x_{n_k}))=\lim_{k\to\infty}\operatorname{dist}(x_{n_k},P_{T(x_{n_k})}(x_{n_k}))\leq\lim_{k\to\infty}d(x_{n_k},z_{n_k})=0.$$

Since T is hemicompact, by passing through a subsequence, we may assume that $x_{n_k}\to q$ for some $q\in E$. Since T is continuous,

$$\operatorname{dist}(q,T(q))\leq d(q,x_{n_k})+\operatorname{dist}(x_{n_k},T(x_{n_k}))+H(T(x_{n_k}),T(q))\to 0 \text{ as } k\to\infty.$$

This implies that $q\in \mathrm{Fix}(T)$. Thus $\lim_{n\to\infty} d(x_n,q)$ exists and hence q is the limit of $\{x_n\}$ itself. □

The following corollary can also be obtained.

Corollary 3.6. Let X be a complete $\mathbb{R}$-tree, E be a nonempty closed convex subset of X and $T: E\to \mathcal{CC}(E)$ be a quasi-nonexpansive mapping. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by (B) (replacing $+$ with $\oplus$). Assume that T is hemicompact and continuous and (i) $0\leq\alpha_n,\beta_n<1$; (ii) $\beta_n\to 0$ and (iii) $\sum\alpha_n\beta_n=\infty$. Then $\{x_n\}$ converges to a fixed point of T.

4. Examples

Example 4.1 (see [23] (A quasi-nonexpansive mapping which is not nonexpansive and a proximally quasi-nonexpansive mapping which is not proximally nonexpansive.))**.** Let $E=[0,\infty)$ and $T: E\to\mathcal{CC}(E)$ be defined by

$$T(x)=\begin{cases}\{0\} & \text{if } 0\leq x\leq 1;\\ \left[x-\frac{3}{4},x-\frac{1}{2}\right] & \text{if } 1<x<10;\\ \left[x-\frac{3}{4},x-\frac{1}{10}\right] & \text{if } x\geq 10.\end{cases}$$

Then $\mathrm{Fix}(T)=\{0\}$. It is easy to see that $H(T(x),T(0))\leq d(x,0)$ for each $x\in E$. This implies that T is quasi-nonexpansive and hence T is proximally quasi-nonexpansive by Proposition 3.1. However, for $x=10$ and $y=2$, we have

$$d\left(P_{T(10)}(10),P_{T(2)}(2)\right)=d\left(10-\frac{1}{10},2-\frac{1}{2}\right)=\frac{84}{10}>d(10,2).$$

This implies that T is not proximally nonexpansive. Hence it is not nonexpansive.

Example 4.2 (see [28] (A proximally nonexpansive mapping which is not nonexpansive and a proximally quasi-nonexpansive mapping which is not quasi-nonexpansive.)**.** Let $E = [0, \infty)$ and $T : E \to \mathcal{CC}(E)$ be defined by

$$T(x) = [x, 2x] \text{ for all } x \in E.$$

By Lemma 2.4, $\text{Fix}(T) = \text{Fix}(P_T) = [0, \infty)$ and $P_{T(x)}(x) = \{x\}$ for every $x \in E$. For each $x, y \in E$ we have

$$d(P_{T(x)}(x), P_{T(y)}(y)) \leq d(x, y).$$

This implies that T is proximally nonexpansive and hence proximally quasi-nonexpansive. However, for $x = 1$ and $y = 0$, we have

$$H(T(1), T(0)) = H([1, 2], 0) > d(1, 0).$$

This shows that T is not quasi-nonexpansive and hence it is not nonexpansive.

Example 4.3 (see [28] (A quasi-nonexpansive mapping which is not proximally quasi-nonexpansive))**.** Let E be the triangle in the Euclidean plane with vertexes $O(0, 0), A(1, 0), B(0, 1)$. Let $T : E \to \mathcal{CC}(E)$ be given by

$$T(x, y) = \text{ the segment joining } (0, 1) \text{ and } (x, 0).$$

Then $P_T(x, y)$ is the point in $T(x, y)$ which is nearest to (x, y) as shown in Figure 1. By Lemma 2.4, $\text{Fix}(T) = \text{Fix}(P_T) = \{(x, 0) : x \in [0, 1]\} \cup \{(0, y) : y \in [0, 1]\}$. For each $(x_1, y_1), (x_2, y_2) \in E$, we have

$$H(T(x_1, y_1), T(x_2, y_2)) = |x_1 - x_2| \leq d((x_1, y_1), (x_2, y_2)).$$

That is, T is nonexpansive and hence quasi-nonexpansive. But, for $(x, y) \in E$ with $0 < x, y < 1$ we have

$$d(P_T(x, y), P_T(1, 0)) = d(P_T(x, y), (1, 0)) > d\left((x, y), (1, 0)\right).$$

This implies that P_T is not quasi-nonexpansive.

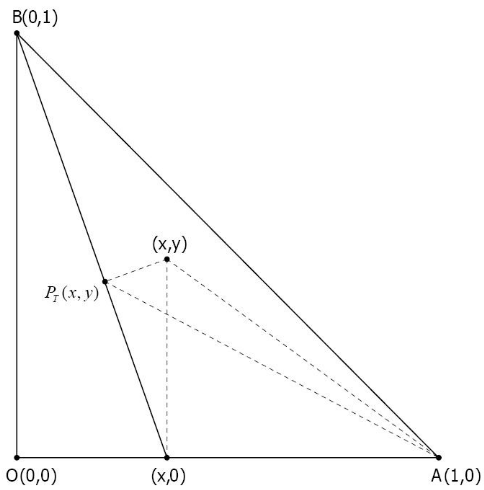

FIGURE 1. $P_T(x, y)$ is the point in $T(x, y)$ which is nearest to (x, y).

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Manuscript received February 27, 2013
revised August 26, 2013

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