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FIXED POINTS OF CORRESPONDENCES IN VECTOR VALUED METRIC SPACES AND APPLICATIONS IN INTEGRAL EQUATIONS

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Abstract. In the present paper, we give some fixed point theorems for correspondences defined on vector valued metric spaces. Our main results reformulate the fixed point theorems given by Assad, Kirk, Zamfirescu, Kannan, Chatterjea and Ćirić in vector valued metric spaces. We also apply the main results in solving some integral equations.

1. Introduction and preliminaries

The motivation of vector valued metric spaces was initiated in [1] by the first author and then formulated in [2] to give a vector version of Caristi's fixed point theorem and Kirk's problem.

The purpose of this work is to investigate the fixed points of correspondences satisfying special contractive conditions in vector valued metric spaces. We show how the results in this paper reformulate the Assad- Kirk [3], Zamfirescu [8], Kannan [7], Chatterjea [4], and Ćirić theorems to non-self correspondences on vector valued metric spaces. Also we give applications of the main results for integral equations.

We first recall some notions which will be used in the sequel.

Let $(\mathcal{E}, \preceq)$ be an ordered Banach space with (positive) cone $\mathcal{E}_+ = \{c \in \mathcal{E} : c \succeq \theta\}$, where θ is the null vector.

Throughout this paper the notation $x \prec y$ indicates that $x \preceq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in \text{int}\mathcal{E}_+$, where $\text{int}\mathcal{E}_+$ denotes the interior of $\mathcal{E}_+$.

Definition 1.1 ([2]). Let X be a nonempty set and $(\mathcal{E}, \preceq)$ be an ordered Banach space. If a mapping $d : X \times X \to \mathcal{E}$ satisfies the following conditions:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \preceq d(x, z) + d(y, z)$ for all $x, y, z \in X$;

then (X, d) is called a vector valued metric space (vvms for short).

Let $\{x_n\}$ be a sequence in vvms X, $x \in X$, and $c \gg \theta$. If there is a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be *convergent* to x, denoted by $x_n \to x$. If there is a positive integer N such that $d(x_n, x_m) \ll c$ for all $n, m > N$, then $\{x_n\}$ is called a *Cauchy sequence* in X. If every Cauchy sequence is convergent to a point of X, then X is called a *complete* vvms.

A subset $C \subset X$ is said to be *closed* if it contains the limit of all its convergent sequences. The closure of C denoted by $\overline{C}$ is defined as the set of all points of X

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that are the limit point of some sequence in C. For a subset C of X, the set of all $x \in \overline{C} \cap \overline{C^c}$ is called the *boundary* of C and it is denoted by ∂C.

2. Main results

A correspondence T from a set C to a set X assigns to each x in X a nonempty subset $T(x)$ of X. For any subset C of X and a correspondence $T : C \twoheadrightarrow X$, an element $x \in C$ is said to be a fixed point if $x \in T(x)$. Also, $T(C) := \cup_{c \in C} T(c)$.

In the sequel, it is assumed that X is a vvms over an ordered Banach space $(\mathcal{E}, \preceq)$ which its positive cone has nonempty interior, C a closed subset of X with nonempty boundary, and for any $x \in C$, $y \in X \setminus C$, there exists a point $z \in \partial C$ such that $d(x,z) + d(z,y) = d(x,y)$. It is also supposed that $T : C \twoheadrightarrow X$ is a correspondence with closed values and $T(x) \subseteq C$ for each $x \in \partial C$.

Theorem 2.1. *Let $T : C \twoheadrightarrow X$ be a correspondence that for each $x, y \in C$, and $p \in T(x)$ there exists $q \in T(y)$ such that*

$$d(p,q) \preceq k u(x,y,p,q),$$

where $k \in (0,1)$ and

$$u(x,y,p,q) \in \left\{ \frac{d(x,y)}{2}, \frac{d(x,p)+d(y,q)}{3}, \frac{d(x,q)+d(y,p)}{3}, \frac{d(x,p)}{2}, \frac{d(y,q)}{2} \right\}. \tag{2.1}$$

Then T has a fixed point.

Proof. We first construct a sequence $\{p_n\}$ in C as follows. Choose $p_0 \in C$ and $p_1' \in T(p_0)$. If $p_1' \in C$, set $p_1 = p_1'$. Otherwise, select a point $p_1 \in \partial C \subseteq C$ such that $d(p_0, p_1) + d(p_1, p_1') = d(p_0, p_1')$. We choose $p_2' \in T(p_1)$ such that

$$d(p_1', p_2') \preceq k u(p_0, p_1, p_1', p_2').$$

Now, if $p_2' \in C$, set $p_2' = p_2$, otherwise again choose $p_2 \in \partial C$ such that $d(p_1, p_2) + d(p_2, p_2') = d(p_1, p_2')$. Continuing in this manner, we obtain sequences $\{p_n\}$ and $\{p_n'\}$ such that for $n \in \mathbb{N} \cup \{0\}$,

- $p_{n+1}' \in T(p_n)$,
- $d(p_{n+1}', p_n') \preceq k u(p_{n-1}, p_n, p_n', p_{n+1}')$,
- if $p_{n+1}' \in C$, then $p_{n+1}' = p_{n+1}$; otherwise $p_{n+1} \in \partial C$ such that $d(p_n, p_{n+1}) + d(p_{n+1}, p_{n+1}') = d(p_n, p_{n+1}')$.

Now, set

$$P := \{p_i \in \{p_n\} : p_i = p_i', i \in \mathbb{N} \cup \{0\}\},$$
$$Q := \{p_i \in \{p_n\} : p_i \neq p_i', i \in \mathbb{N} \cup \{0\}\}.$$

Choose $n \geq 2$. We have the following three cases:

Case 1. Suppose that $p_n, p_{n+1} \in P$.
Then, if

$$u(p_{n-1}, p_n, p_n', p_{n+1}') = \frac{d(p_n, p_{n-1})}{2},$$

we have

$$\begin{aligned} d(p_n, p_{n+1}) &= d(p_n', p_{n+1}') \\ &\preceq \tfrac{k}{2} d(p_n, p_{n-1}). \end{aligned}$$

If $u(p_{n-1}, p_n, p'_n, p'_{n+1}) = \frac{1}{3}(d(p_{n-1}, p'_n) + d(p'_{n+1}, p_n))$, we have

$$\begin{aligned} d(p_n, p_{n+1}) &= d(p'_n, p'_{n+1}) \\ &\preceq \tfrac{k}{3}(d(p_n, p_{n-1}) + d(p_{n+1}, p_n)), \end{aligned}$$

therefore

$$d(p_n, p_{n+1}) \preceq \frac{k}{3-k} d(p_n, p_{n-1}).$$

If $u(p_{n-1}, p_n, p'_n, p'_{n+1}) = \frac{1}{3}(d(p_n, p'_n) + d(p'_{n+1}, p_{n-1}))$, we have

$$\begin{aligned} d(p_n, p_{n+1}) &= d(p'_n, p'_{n+1}) \\ &\preceq \tfrac{k}{3}(d(p_{n+1}, p_{n-1}) + d(p'_n, p_n)) \\ &\preceq \tfrac{k}{3}(d(p_{n+1}, p_n) + d(p_n, p_{n-1})), \end{aligned}$$

so

$$d(p_n, p_{n+1}) \preceq \frac{k}{3-k} d(p_n, p_{n-1}).$$

If $u(p_{n-1}, p_n, p'_n, p'_{n+1}) = \frac{1}{2} d(p_{n-1}, p'_n)$, we have

$$\begin{aligned} d(p_n, p_{n+1}) &= d(p'_n, p'_{n+1}) \\ &\preceq \tfrac{k}{2} d(p_n, p_{n-1}). \end{aligned}$$

If $u(p_{n-1}, p_n, p'_n, p'_{n+1}) = \frac{1}{2} d(p_n, p'_{n+1})$, we have

$$\begin{aligned} d(p_n, p_{n+1}) &= d(p'_n, p'_{n+1}) \\ &\preceq \tfrac{k}{2} d(p_n, p_{n+1}), \end{aligned}$$

so $d(p_n, p_{n+1}) = 0$.

Thus, for $p_n, p_{n+1} \in P$ we have

$$d(p_n, p_{n+1}) \preceq k d(p_n, p_{n-1}).$$

Case 2. Suppose that $p_n \in P, p_{n+1} \in Q$.

If $u(p_{n-1}, p_n, p'_n, p'_{n+1}) = \frac{1}{2} d(p_{n-1}, p_n)$, we have

$$\begin{aligned} d(p_n, p'_{n+1}) &= d(p'_n, p'_{n+1}) \\ &\preceq \tfrac{k}{2} d(p_n, p_{n-1}). \end{aligned}$$

If $u(p_{n-1}, p_n, p'_n, p'_{n+1}) = \frac{1}{3}(d(p_{n-1}, p'_n) + d(p'_{n+1}, p_n))$, we have

$$\begin{aligned} d(p_n, p'_{n+1}) &= d(p'_n, p'_{n+1}) \\ &\preceq \tfrac{k}{3}(d(p'_n, p_{n-1}) + d(p'_{n+1}, p_n)). \end{aligned}$$

so

$$d(p_n, p'_{n+1}) \preceq \frac{k}{3-k} d(p_n, p_{n-1}).$$

If $u(p_{n-1}, p_n, p'_n, p'_{n+1}) = \frac{1}{3}(d(p_n, p'_n) + d(p'_{n+1}, p_{n-1}))$, we have

$$\begin{aligned} d(p_n, p'_{n+1}) &= d(p'_n, p'_{n+1}) \\ &\preceq \tfrac{k}{3}(d(p'_{n+1}, p_{n-1}) + d(p'_n, p_n)) \\ &\preceq \tfrac{k}{3}(d(p'_{n+1}, p_n) + d(p_n, p_{n-1})), \end{aligned}$$

so

$$d(p_n, p'_{n+1}) \preceq \frac{k}{3-k} d(p_n, p_{n-1}).$$

If $u(p_{n-1}, p_n, p'_n, p'_{n+1}) = \frac{1}{2}d(p_{n-1}, p'_n)$, we have

$$\begin{aligned} d(p_n, p'_{n+1}) &= d(p'_n, p'_{n+1}) \\ &\preceq \tfrac{k}{2} d(p_n, p_{n-1}). \end{aligned}$$

If $u(p_{n-1}, p_n, p'_n, p'_{n+1}) = \frac{1}{2}d(p_n, p'_{n+1})$, we have

$$\begin{aligned} d(p_n, p'_{n+1}) &= d(p'_n, p'_{n+1}) \\ &\preceq \tfrac{k}{2} d(p_n, p_{n+1}), \end{aligned}$$

and therefore $d(p_n, p_{n+1}) = 0$ since $d(p_n, p_{n+1}) + d(p_{n+1}, p'_{n+1}) = d(p_n, p'_{n+1})$ implies $d(p_n, p_{n+1}) \preceq d(p_n, p'_{n+1})$. Observe that if $p_n \in P, p_{n+1} \in Q$, by the relations above we get

$$d(p_n, p_{n+1}) \preceq \frac{k}{2} d(p_n, p_{n-1}). \tag{2.2}$$

Next suppose $p_n \in Q$. Note $p_n \neq p'_n$, i.e. $p'_n \notin C$ and so $p_n \in \partial C$. Since $T(\partial C) \subset C$ and $p'_{n+1} \in T(p_n)$, therefore $p'_{n+1} \in C$ and consequently $p'_{n+1} = p_{n+1}$, i.e., $p_{n+1} \in P$. This implies that the case $p_n \in Q$ and $p_{n+1} \in Q$ is impossible. Thus, it remains to consider the following case.

Case 3. Suppose $p_n \in Q$ and $p_{n+1} \in P$.
We note that by the last argument $p_{n-1} \in P$. If $u(p_{n-1}, p_n, p'_n, p'_{n+1}) = \frac{1}{2}d(p_{n-1}, p_n)$, we have

$$\begin{aligned} d(p_n, p_{n+1}) &\preceq d(p_n, p'_n) + d(p'_n, p_{n+1}) \\ &\preceq d(p'_n, p_n) + \tfrac{k}{2} d(p_n, p_{n-1}) \\ &= d(p_{n-1}, p'_n) \\ &= d(p'_{n-1}, p'_n) \\ &\preceq \tfrac{k}{2} d(p_{n-2}, p_{n-1}). \end{aligned}$$

If $u(p_{n-1}, p_n, p'_n, p'_{n+1}) = \frac{1}{3}(d(p_{n-1}, p'_n) + d(p'_{n+1}, p_n))$, we have

$$\begin{aligned}
d(p_n, p_{n+1}) &= d(p_n, p'_{n+1}) \\
&\preceq d(p'_n, p_n) + d(p'_n, p'_{n+1}) \\
&\preceq d(p_{n-1}, p'_n) + d(p'_n, p'_{n+1}) \\
&\preceq \tfrac{k}{2} d(p_{n-2}, p_{n-1}) + \tfrac{k}{3}(d(p'_n, p_{n-1}) + d(p_n, p_{n+1})) \\
&\preceq \tfrac{k}{3}(\tfrac{k}{2} d(p_{n-2}, p_{n-1}) + d(p_n, p_{n+1})) + \tfrac{k}{2} d(p_{n-2}, p_{n-1}) \\
&= \tfrac{k^2}{6} d(p_{n-2}, p_{n-1}) + \tfrac{k}{3} d(p_n, p_{n+1})) + \tfrac{k}{2} d(p_{n-2}, p_{n-1}).
\end{aligned}$$

Therefore

$$d(p_n, p_{n+1}) \preceq \frac{k^2 + 3k}{6 - 2k} d(p_{n-2}, p_{n-1}).$$

If $u(p_{n-1}, p_n, p'_n, p'_{n+1}) = \frac{1}{3}(d(p_n, p'_n) + d(p'_{n+1}, p_{n-1}))$, we have

$$\begin{aligned}
d(p_n, p_{n+1} &= d(p_n, p'_{n+1}) \\
&\preceq d(p'_n, p_n) + d(p'_n, p'_{n+1}) \\
&\preceq d(p'_{n-1}, p'_n) + d(p'_n, p'_{n+1}) \\
&\preceq \frac{k}{2} d(p_{n-2}, p_{n-1}) + \frac{k}{3}(d(p'_n, p_n) + d(p_{n-1}, p_{n+1})) \\
&\preceq \frac{k}{2} d(p_{n-2}, p_{n-1}) + \frac{k}{3}(d(p'_n, p'_{n-1}) - d(p_{n-1}, p_n) + d(p_{n-1}, p_{n+1})) \\
&\preceq \frac{k}{2} d(p_{n-2}, p_{n-1}) \\
&\quad + \frac{k}{3}(d(p'_n, p'_{n-1}) - d(p_{n-1}, p_n) + d(p_{n-1}, p_n) + d(p_n, p_{n+1})) \\
&\preceq \frac{k}{2} d(p_{n-2}, p_{n-1}) + \frac{k}{3}(\frac{k}{2} d(p_{n-2}, p_{n-1}) + d(p_n, p_{n+1})),
\end{aligned}$$

and so

$$d(p_n, p_{n+1}) \preceq \frac{k^2 + 3k}{6 - 2k} d(p_{n-2}, p_{n-1}).$$

If $u(p_{n-1}, p_n, p'_n, p'_{n+1}) = \frac{1}{2} d(p_{n-1}, p'_n)$, we have

$$\begin{aligned}
d(p_n, p_{n+1}) &\preceq d(p_n, p'_n) + d(p'_n, p_{n+1}) \\
&\preceq d(p'_n, p_n) + \tfrac{k}{2} d(p'_n, p_{n-1}) \\
&\preceq d(p_{n-1}, p'_n) + \tfrac{k}{2} d(p'_n, p_{n-1}) \\
&\preceq \tfrac{k^2 + 2k}{4} d(p_{n-2}, p_{n-1})
\end{aligned}$$

If $u(p_{n-1}, p_n, p'_n, p'_{n+1}) = \frac{1}{2} d(p_n, p'_{n+1})$, we have

$$\begin{aligned} d(p_n, p_{n+1}) &\preceq d(p_n, p'_n) + d(p'_n, p_{n+1}) \\ &\preceq d(p'_n, p_n) + \tfrac{k}{2} d(p_n, p_{n+1}) \\ &\preceq d(p_{n-1}, p'_n) + \tfrac{k}{2} d(p_n, p_{n+1}) \\ &\preceq \tfrac{k}{2} d(p_{n-2}, p_{n-1}) + \tfrac{k}{2} d(p_n, p_{n+1}), \end{aligned}$$

so

$$d(p_n, p_{n+1}) \preceq \frac{k}{2-k} d(p_{n-2}, p_{n-1}).$$

Combining all cases, for $n \geq 2$, we have

$$d(p_n, p_{n+1}) \preceq k d(p_n, p_{n-1}),$$

or

$$d(p_n, p_{n+1}) \preceq k d(p_{n-2}, p_{n-1}).$$

Let $M \in \{d(p_0, p_1), d(p_1, p_2)\}$. We have

$$d(p_n, p_m) \preceq \sum_{i=n}^{m-1} d(p_i, p_{i+1}) \preceq \frac{k^n}{1-k} M.$$

Let $c \gg \theta$ be given. Choose $\delta > 0$ such that $c + N_\delta(0) \subseteq \mathcal{P}$, where

$$N_\delta(0) = \{y \in X : \|y\| < \delta\}.$$

Then $\frac{k^n}{1-k} M \in N_\delta(0)$, for all sufficiently large n. Thus for all sufficiently large m and n where $m > n$, we have

$$d(p_n, p_m) \preceq \frac{k^n}{1-k} M \ll c.$$

That is $\{p_n\}$ is a Cauchy sequence in (X, d). Now, by the completeness of (X, d), there exists $p \in X$ such that $p_n \to p$. By our choice of $\{p_n\}$, there exists a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ such that $p_{n_i} \in P$, i.e., $p_{n_i} = p'_{n_i}$, $i = 1, 2, \ldots$. Note $p'_{n_i} \in T(p_{n_i - 1})$ for $i \in \mathbb{N}$. Let $c \gg \theta$ be given. For $p'_{n_i} \in T(p_{n_i-1})$ there exists $q_{n_i} \in T(p)$ such that

$$d(q_{n_i}, p_{n_i}) \preceq k u(p_{n_i-1}, p, p_{n_i}, q_{n_i}).$$

If $u(p_{n_i-1}, p, p_{n_i}, q_{n_i}) = \frac{1}{2} d(p_{n_i-1}, p)$, we have

$$\begin{aligned} d(q_{n_i}, p) &\preceq d(q_{n_i}, p_{n_i}) + d(p_{n_i}, p) \\ &\preceq d(p_{n_i}, p) + \tfrac{k}{2} d(p_{n_i-1}, p). \end{aligned}$$

If $u(p_{n_i-1}, p, p_{n_i}, q_{n_i}) = \frac{1}{3}(d(p_{n_i-1}, p_{n_i}) + d(p, q_{n_i}))$, then

$$\begin{aligned} d(q_{n_i}, p) &\preceq d(p, p'_{n_i}) + d(p'_{n_i}, q_{n_i}) \\ &\preceq \tfrac{k}{3}(d(p_{n_i}, p_{n_i-1}) + d(q_{n_i}, p)) + d(p_{n_i}, p), \end{aligned}$$

so $(1-\frac{k}{3})d(q_{n_i},p) \preceq \frac{k}{3}d(p_{n_i},p_{n_i-1}) + d(p_{n_i},p)$, and hence

$$d(q_{n_i},p) \preceq \frac{k}{3-k}d(p_{n_i},p_{n_i-1}) + \frac{3}{3-k}d(p_{n_i},p).$$

If $u(p_{n_i-1},p,p_{n_i},q_{n_i}) = \frac{1}{3}(d(p_{n_i},p) + d(q_{n_i},p_{n_i-1}))$, we have

$$\begin{aligned} d(q_{n_i},p) &\preceq d(q_{n_i},p_{n_i}) + d(p_{n_i},p) \\ &\preceq \tfrac{k}{3}(d(p_{n_i-1},q_{n_i}) + d(p'_{n_i},p)) + d(p_{n_i},p) \\ &\preceq \tfrac{k}{3}(d(p_{n_i-1},p) + d(p,q_{n_i}) + d(p'_{n_i},p)) + d(p_{n_i},p). \end{aligned}$$

This implies that

$$d(q_{n_i},p) \preceq \frac{k}{3-k}(d(p_{n_i-1},p) + d(p'_{n_i},p)) + \frac{3}{3-k}d(p_{n_i},p).$$

If $u(p_{n_i-1},p,p_{n_i},q_{n_i}) = \frac{1}{2}d(p_{n_i-1},p_{n_i})$, we have

$$\begin{aligned} d(q_{n_i},p) &\preceq d(q_{n_i},p_{n_i}) + d(p_{n_i},p) \\ &\preceq d(p_{n_i},p) + \tfrac{k}{2}d(p_{n_i-1},p_{n_i}). \end{aligned}$$

If $u(p_{n_i-1},p,p_{n_i},q_{n_i}) = \frac{1}{2}d(p,q_{n_i})$, we have

$$\begin{aligned} d(q_{n_i},p) &\preceq d(q_{n_i},p_{n_i}) + d(p_{n_i},p) \\ &\preceq d(p_{n_i},p) + \tfrac{k}{2}d(p,q_{n_i}). \end{aligned}$$

Thus, $d(q_{n_i},p) \ll c$ for sufficiently large n. Since $T(p)$ is closed, $p \in T(p)$. □

Arguing similarly as in the theorem above, we also have the next result which generalizes Ćirić's Theorem [5] for the quasi-contraction mappings when $k \in (0,\frac{1}{2})$. We omit its proof.

Theorem 2.2. *Let $T : C \twoheadrightarrow X$ be a correspondence that for each $x, y \in C$, $p \in T(x)$ there exists $q \in T(y)$ such that*

$$d(p,q) \preceq ku(x,y,p,q),$$

where $k \in (0,\frac{1}{2})$ and

$$u(x,y,p,q) \in \{d(x,y), d(x,q), d(y,p), d(x,p), d(y,q)\}.$$

Then T has a fixed point.

Now, we list some corollaries of Theorem 2.1.

Corollary 2.3. *Let $T : C \twoheadrightarrow X$ be a correspondence that for each $x, y \in C$, $p \in T(x)$ there exists $q \in T(y)$ such that*

$$d(p,q) \preceq ku(x,y,p,q),$$

where

$$\text{(2.3)} \quad u(x,y,p,q) \in \left\{\frac{d(x,y)}{2}, \frac{d(x,p)}{2}, \frac{d(y,q)}{2}, \frac{d(x,p)+d(y,q)}{\xi}, \frac{d(x,q)+d(y,p)}{\xi}\right\},$$

and $k \in (0,1), \xi \geq 1+2k$. Then T has a fixed point.

Proof. It is easy to see that (2.3) implies that

$$d(p,q) \preceq k_1 u(x,y,p,q),$$

where

$$u(x,y,p,q) \in \Big\{\frac{d(x,y)}{2}, \frac{d(x,p)}{2}, \frac{d(y,q)}{2}, \frac{d(x,p)+d(y,q)}{3}, \frac{d(x,q)+d(y,p)}{3}\Big\},$$

and $k_1 = \frac{3k}{1+2k}$. Thus, by Theorem 2.1, T has a fixed point. □

The following corollary is a generalization of Zamfirescu's Theorem [8] when $0 < \alpha < \frac{1}{2}$, $0 < \beta < \frac{1}{3}$ and $0 < \gamma < \frac{1}{3}$.

Corollary 2.4. *Let α, β, η be positive real numbers with $\alpha < \frac{1}{2}$, $\beta < \frac{1}{3}$, $\gamma < \frac{1}{3}$, and $T : C \twoheadrightarrow X$ such that for each $x, y \in C$ and $p \in T(x)$ there exists $q \in T(y)$ which at least one of the following conditions holds:*

a) $d(p,q) \preceq \alpha d(x,y)$,
b) $d(p,q) \preceq \beta(d(x,p) + d(q,y))$,
c) $d(p,q) \preceq \gamma(d(y,p) + d(q,x))$.

Then T has a fixed point.

Proof. Let $k := \max\{2\alpha, 3\beta, 3\gamma\}$. Then, we have $d(p,q) \preceq ku(x,y,p,q)$, where

$$u(x,y,p,q) \in \Big\{\frac{d(x,y)}{2}, \frac{d(x,p)+d(q,y)}{3}, \frac{d(y,p)+d(q,x)}{3}\Big\}.$$

Now, Theorem 2.1 guarantees that T has a fixed point. □

Corollary 2.5. *Let $T : C \twoheadrightarrow X$ be a correspondence that for each $x, y \in C$, $p \in T(x)$ there exists $q \in T(y)$ such that*

$$d(p,q) \preceq kd(x,y), \tag{2.4}$$

where $k \in (0, \frac{1}{2})$. Then T has a fixed point.

Remark 2.6. By an argument similar to that in Theorem 2.1, we can show that Corollary 2.5 is also valid for $k \in (0,1)$. For the reader's convenience, we give the details: Let a sequence $\{p_n\}$ in C be constructed as the proof of Theorem 2.1. To estimate the distance $d(p_n, p_{n+1})$ for $n \geq 2$, we consider the following three cases:

First, if $p_n, p_{n+1} \in P$, we have

$$\begin{aligned} d(p_n, p_{n+1}) &= d(p'_n, p'_{n+1}) \\ &\preceq kd(p_n, p_{n-1}). \end{aligned}$$

Second, if $p_n \in P, p_{n+1} \in Q$, we have

$$\begin{aligned} d(p_n, p_{n+1}) &\preceq d(p_n, p'_{n+1}) \\ &\preceq kd(p_n, p_{n-1}). \end{aligned}$$

Third, if $p_n \in Q, p_{n+1} \in P$, we get

$$\begin{aligned} d(p_n, p_{n+1}) &\preceq d(p_n, p'_n) + d(p'_n, p_{n+1}) \\ &\preceq d(p'_n, p_n) + kd(p_n, p_{n-1}) \\ &= d(p_{n-1}, p'_n) \\ &= d(p'_{n-1}, p'_n) \\ &\preceq kd(p_{n-2}, p_{n-1}). \end{aligned}$$

Thus, for $n \geq 2$, we have

$$d(p_n, p_{n+1}) \preceq kd(p_n, p_{n-1})$$

or

$$d(p_n, p_{n+1}) \preceq kd(p_{n-2}, p_{n-1}).$$

Let $M \in \{d(p_0, p_1), d(p_1, p_2)\}$ and $m > n$. We have

$$d(p_n, p_m) \preceq \sum_{i=n}^{m-1} d(p_i, p_{i+1}) \preceq \frac{k^n}{1-k} M.$$

This implies that $\{p_n\}$ is a Cauchy sequence in (X, d) and therefore is convergent. Let $p_n \to p$. By our choice of $\{p_n\}$, there exists a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ such that $p_{n_i} \in P$, i.e., $p_{n_i} = p'_{n_i}$, $i = 1, 2, \ldots$. Note that $p'_{n_i} \in T(p_{n_i - 1})$ for $i \in \mathbb{N}$. Now for $p'_{n_i} \in T(p_{n_i-1})$ there exists $q_{n_i} \in T(p)$ such that $d(q_{n_i}, p'_{n_i}) \preceq kd(p_{n_i-1}, p)$. We have

$$\begin{aligned} d(q_{n_i}, p) &\preceq d(q_{n_i}, p_{n_i}) + d(p_{n_i}, p) \\ &\preceq d(p_{n_i}, p) + kd(p_{n_i-1}, p). \end{aligned}$$

Now the closedness of $T(p)$ implies that $p \in T(p)$.

Remark 2.7. We recall that a metric space (Y, d) is metrically convex if for any $x, y \in Y$ with $x \neq y$, there exists $z \in Y$ such that $x \neq y \neq z$ and $d(x, z) + d(z, y) = d(x, y)$. This fact immediately yields that if B is a nonempty closed subset of a complete and metrically convex space Y, then for any $x \in B$, $y \in Y \setminus B$, there exists a point $z \in \partial B$ such that $d(x, z) + d(z, y) = d(x, y)$. Now, if Y is a complete metrically convex space, B a closed subset of Y and $T : B \twoheadrightarrow Y$ satisfies (2.4) and also $T(\partial B) \subseteq B$, then since for each $x \in B$, $y \in Y \setminus B$, there exists $z \in \partial B$ such that $d(x, z) + d(z, y) = d(x, y)$, by Remark 2.6, T has a fixed point. Hence, Remark 2.6 simultaneously generalizes a type of Assad- Kirk's Theorem for correspondences on vector valued metric spaces [3] in terms of a given contraction.

The next corollary is a type of Kannan contraction [7] for the correspondences that generalizes it to the setting of vvms when $k \in (0, \frac{1}{3})$.

Corollary 2.8. *Let $T : C \twoheadrightarrow X$ be a correspondence that for each $x, y \in C$, $p \in T(x)$ there exists $q \in T(y)$ such that*

$$d(p, q) \preceq k(d(x, p) + d(q, y)), \tag{2.5}$$

where $k \in (0, \frac{1}{3})$. Then T has a fixed point.

In the following corollary, we see a type of Chatterjea Theorem [4] for $k \in (0, \frac{1}{3})$.

Corollary 2.9. *Let $T : C \twoheadrightarrow X$ be a correspondence that for each $x, y \in C$, $p \in T(x)$ there exists $q \in T(y)$ such that*

$$d(p,q) \preceq k(d(y,p) + d(q,x)), \tag{2.6}$$

where $k \in (0, \frac{1}{3})$. Then T has a fixed point.

Theorem 2.10. *Let $T : C \twoheadrightarrow X$ be a correspondence that for each $x, y \in C$, $p \in T(x)$ there exists $q \in T(y)$ such that*

$$d(p,q) \preceq kd(y,x) + ld(y,p), \tag{2.7}$$

where $k + l < 1$. Then T has a fixed point.

Proof. Let $\{p_n\}$ be a sequence in C constructed as in the proof of Theorem 2.1. To estimate the distance $d(p_n, p_{n+1})$ for $n \geq 2$, we consider the following three steps:
Step 1. If $p_n, p_{n+1} \in P$, then

$$\begin{aligned} d(p_n, p_{n+1}) &= d(p'_n, p'_{n+1}) \\ &\preceq kd(p_n, p_{n-1}). \end{aligned}$$

Step 2. If $p_n \in P, p_{n+1} \in Q$, then

$$\begin{aligned} d(p_n, p_{n+1}) &\preceq d(p_n, p'_{n+1}) \\ &= d(p'_n, p'_{n+1}) \\ &\preceq kd(p_n, p_{n-1}). \end{aligned}$$

Step 3. If $p_n \in Q, p_{n+1} \in P$, and $\rho = d(p_n, p_{n+1})$, then

$$\begin{aligned} \rho &= d(p_n, p'_{n+1}) \\ &\preceq d(p'_n, p_n) + d(p'_n, p'_{n+1}) \\ &\preceq d(p_n, p'_n) + kd(p_{n-1}, p_n) + ld(p_n, p'_n) \\ &\preceq d(p_{n-1}, p'_n) - d(p_{n-1}, p_n) + kd(p_{n-1}, p_n) + l(d(p_{n-1}), p'_n) - d(p_{n-1}, p_n)) \\ &= (k + kl)d(p_{n-2}, p_{n-1}) \end{aligned}$$

This implies that, as in the proof of Theorem 2.1, $\{p_n\}$ is a Cauchy sequence. Because C is closed, $\{p_n\}$ converges to a point $p \in C$. By our choice of $\{p_n\}$, there exists a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ such that $p_{n_i} \in P$, i.e., $p_{n_i} = p'_{n_i}$, $i = 1, 2, \ldots$. Note that $p'_{n_i} \in T(p_{n_i-1})$ for $i \in \mathbb{N}$. Now for $p'_{n_i} \in T(p_{n_i-1})$ there exists $q_{n_i} \in T(p)$ such that $d(q_{n_i}, p'_{n_i}) \preceq kd(p_{n_i-1}, p) + ld(p'_{n_i}, q_{n_i})$. This implies $d(q_{n_i}, p_{n_i}) \to 0$ and therefore $q_{n_i} \to p$. Since $T(p)$ is closed and $q_{n_i} \in T(p)$, $p \in T(p)$. □

3. Applications in integral equations

In this section, we denote min and max for minimum and maximum functions, respectively, and $C^1_{\mathbb{R}}[0,1]$ the space of all real continuously differentiable functions on $[0,1]$.

Theorem 3.1. *Consider the integral equation*

$$x(t) = \int_0^1 k(s, x(t), \min\{x(s) : s \in [0,1]\}, \max\{x(s) : s \in [0,1]\})ds + g(t), \tag{3.1}$$

where

1) $k : \mathbb{R}^4 \to \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$ *are continuous mappings,*
2) *there exist* $a, b \in (0,1)$ *and* $c \in \mathbb{R}$ *such that*

$$k(c, b, a, b) < -g(1) \quad or \quad k(c, a, a, b) > 1 - g(0)$$

and

$$-g(t) \le k(s, \gamma, 0, \eta) \le 1 - g(t) \quad and \quad -g(t) \le k(s, \gamma, \xi, 1) \le 1 - g(t),$$

for every $t, \gamma \in \mathbb{R}$ *and* $s, \xi, \eta \in [0,1]$,
3) *there exists* $k_0 \in (0,1)$ *such that*

$$|k(s, \gamma_1, \xi, \eta) - k(s, \gamma_2, \xi, \eta)| \le k_0 U,$$

where

$$U \in \{P, I, G, H, J\},$$

and

$$\begin{aligned}
P &= \frac{1}{2}|\gamma_1 - \gamma_2|,\\
I &= \frac{1}{2}|\gamma_1 - \int_0^1 k(s, \gamma_1, \xi, \eta)ds|,\\
G &= \frac{1}{3}(|\gamma_1 - \int_0^1 k(s, \gamma_1, \xi, \eta)ds| + |\gamma_2 - \int_0^1 k(s, \gamma_2, \xi, \eta)ds|),\\
H &= \frac{1}{3}(|\gamma_2 - \int_0^1 k(s, \gamma_1, \xi, \eta)ds| + |\gamma_1 - \int_0^1 k(s, \gamma_2, \xi, \eta)ds|),\\
J &= \frac{1}{2}|\gamma_2 - \int_0^1 k(s, \gamma_2, \xi, \eta)ds|, \quad \text{for all} \quad \gamma_1, \gamma_2 \in \mathbb{R} \quad \text{and} \quad s, \xi, \eta \in [0,1].
\end{aligned}$$

Then the integral equation (3.1) has a solution.

Proof. Consider the Banach space $E = \{(x, y) : x, y \in C^1_{\mathbb{R}}[0,1]\}$ equipped with the norm

$$\|(x, y)\| = \sqrt{\|x\|_1^2 + \|y\|_1^2},$$

where $\|f\|_1 = \|f\|_\infty + \|f'\|_\infty$, $f \in C^1_{\mathbb{R}}[0,1]$. Let

$$\mathcal{P} = \{(x, y) : x, y \in C^1_{\mathbb{R}}[0,1], x(t) \ge 0, y(t) \ge 0\}.$$

We show that $\partial\mathcal{P} \subseteq \mathcal{P}$. By contradiction, suppose that there exists $(x_0, y_0) \in \partial\mathcal{P} \backslash \mathcal{P}$. Without loss of generality, we suppose there exists $t_0 \in [0,1]$ such that $x_0(t_0) < 0$. Now the inequality

$$\|(x, y) - (x_0, y_0)\| > \frac{|x_0(t_0)|}{2} \qquad ((x, y) \in \mathcal{P}),$$

gives a contradiction. Hence $\mathcal{P}$ is a closed subset of E. Now we prove that

$$\partial\mathcal{P} = \{(x, y) \in \mathcal{P} : \min_{t\in[0,1]} x(t) = 0 \ \text{ or } \ \min_{t\in[0,1]} y(t) = 0\}. \tag{3.2}$$

Let (x, y) be an element of the right hand side of (3.2) and $\min_{t\in[0,1]} x(t) = 0$. Then for each $\epsilon > 0$ we have

$$\Big(x - \frac{\epsilon}{2}, y\Big) \in C^1_{\mathbb{R}}[0,1]\backslash\mathcal{P}, \qquad \Big\|(x - \frac{\epsilon}{2}, y) - (x, y)\Big\| < \epsilon.$$

Therefore

$$\{(x, y) \in \mathcal{P} : \min_{t\in[0,1]} x(t) = 0 \vee \min_{t\in[0,1]} y(t) = 0\} \subseteq \partial\mathcal{P}.$$

For the converse, let $(x_1, y_1) \in \partial\mathcal{P}$ be such that

$$\min_{t\in[0,1]} x_1(t) > 0 \quad \text{and} \quad \min_{t\in[0,1]} y_1(t) > 0.$$

Letting

$$\epsilon = \min\{\min_{t\in[0,1]} y_1(t) > 0, \min_{t\in[0,1]} x_1(t) > 0\}$$

and $(x, y) \in \mathcal{P}^c$ we may suppose that, without loss of generality, there exists $t_1 \in [0, 1]$ such that $x(t_1) < 0$. Then, we clearly get $\|(x, y) - (x_1, y_1)\| > \epsilon$ which is a contradiction. On other hand

$$\text{int}\mathcal{P} = \{(x, y) \in \mathcal{P} : x(t) > 0, y(t) > 0, \text{for each } t \in [0, 1]\}.$$

Let $X = C[0, 1]$ be the Banach space consisting of all real continuous functions on $[0, 1]$ equipped with the sup norm. Suppose that mapping $d : X \times X \to \mathcal{P}$ is defined as

$$d(x, y) = (\|x - y\|_\infty, \varphi\|x - y\|_\infty),$$

where $\varphi \in C^1_{\mathbb{R}}[0, 1]$ and $\varphi(t) \geq 0$ for $t \in [0, 1]$. Also, suppose that $C = \{x \in X : 0 \leq x(t) \leq 1, t \in [0, 1]\}$. Now, the vvms X is complete. Indeed, let $\{x_n\}$ be a Cauchy sequence in X and $a', b' \in \mathbb{R}^{>0}$. We have $d(x_n, x_m) \ll (a', b')$, for sufficiently large m, n, hence $\|x_n - x_m\|_\infty < a'$. Since $(C[0, 1], \|\cdot\|_\infty)$ is a Banach space, there exists $x \in X$ such that $\|x_n - x\|_\infty \to 0$. Now, if $(f, g) \in \text{int}\mathcal{P}$ there exists $a', b' \in \mathbb{R}^{>0}$ such that $d(x_n, x) = (\|x_n - x\|_\infty, \varphi\|x_n - x\|_\infty) \ll (a', b') \preceq (f, g)$, for sufficiently large n. This implies that (X, d) is a complete vvms.

A similar argument given for $\mathcal{P}$ shows that

$$\partial C = \{x \in C : \min_{t\in[0,1]} x(t) = 0 \vee \max_{t\in[0,1]} x(t) = 1\} \subseteq C.$$

It is not hard to see that for each $x \in C$ and $y \in C[0, 1]\backslash C$ there exists $z \in \partial C$ such that $d(x, z) + d(z, y) = d(x, y)$. Now define the mapping $A : C \to X$ by

$$A(x(t)) = \int_0^1 k(s, x(t), \min\{x(s) : s \in [0, 1]\}, \max\{x(s) : s \in [0, 1]\})ds + g(t).$$

Let a, b be as given in the assumption and

$$x_1(t) = a + (b - a)t \qquad t \in [0, 1].$$

By assumption (2), because $x_1 \in C$ and $A(x_1) \notin C$, A is not a self-mapping. Again, by using (2), we have $A(\partial C) \subseteq C$. For each $x, y \in X$, we have for $t \in [0, 1]$,

$$(|A(x(t)) - A(y(t))|, \varphi|A(x(t)) - A(y(t))|) =$$

$$\Big(\Big|\int_0^1 k(s, x(t), \min_{s\in[0,1]} x(s), \max_{s\in[0,1]} x(s)) - k(s, y(t), \min_{s\in[0,1]} y(s), \max_{s\in[0,1]} y(s))ds\Big|,$$

$$\varphi\Big|\int_0^1 k(s,x(t),\min_{s\in[0,1]}x(s),\max_{s\in[0,1]}x(s))-k(s,y(t),\min_{s\in[0,1]}y(s),\max_{s\in[0,1]}y(s))ds\Big|\Big)$$
$$\preceq k_0(1,\varphi)U\preceq k_0u(x,y,A(x),A(y)).$$

Since A satisfies the assumptions in Theorem 2.1, it has a fixed point. □

Theorem 3.2. *Consider the integral equation*

$$x(t)=\int_0^1 k(x(t)-s,x(s))f(s)ds\vee x(t)=\int_0^1 h(x(t)-s,x(s))g(s)ds,$$

where

1) $k,h:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^{\geq 0}$ *and* $f,g:\mathbb{R}\to\mathbb{R}^{\geq 0}$ *are continuous mappings,*
2) $\int_0^1 k(-s,0)f(s)ds=0$, $\int_0^1 h(-s,0)g(s)ds=0$ *and there exists* $a,b\in(0,1)$ *such that* $k(-a,b)f(b)>0$ *or* $h(-a,b)g(b)>0$,
3) $(k(\gamma_1-s,\gamma_1')+k(\gamma_2-s,\gamma_2'))f(s)\leq k_0(\gamma_1+\gamma_2)+l_0(\gamma_2+k(\gamma_1-s,\gamma_1')f(s))$ *and* $(h(\gamma_1-s,\gamma_1')+h(\gamma_2-s,\gamma_2'))f(s)\leq k_0'(\gamma_1+\gamma_2)+l_0'(\gamma_2+h(\gamma_1-s,\gamma_1')f(s))$, *where* $\gamma_1,\gamma_1',\gamma_2,\gamma_2'\in\mathbb{R}$ *and* $k_0+l_0\leq 1$, $k_0'+l_0'\leq 1$.

Then the given integral equation has a solution.

Proof. Let $E=C^1_{\mathbb{R}}[0,1]$ be equipped with the norm $\|x\|=\|x\|_\infty+\|x'\|_\infty$ and $\mathcal{P}=\{x\in E: x(t)\geq 0\}$. The cone $\mathcal{P}$ has nonempty interior (e.g., $x(t)=\frac{1}{2}, t\in[0,1]$ is an interior point of $\mathcal{P}$). Suppose that $X=\mathcal{P}$ and $C=\{x\in X: x(0)=0\}$. Next, we define the vector valued metric d on X by

$$d(x,y)=\begin{cases} x+y & x\neq y\\ 0 & x=y.\end{cases}$$

Since the only Cauchy sequence of X is $\{0\}$, (X,d) is a complete vvms and $C\subset X$ is closed. It is easy to see that for each $x\in C$ and $y\notin C$ there exists $z=0\in\partial C$ such that $d(x,y)=d(x,0)+d(0,y)$. Define

$$A(x(t))=\Big\{\int_0^1 k(x(t)-s,x(s))f(s)ds,\int_0^1 h(x(t)-s,x(s))g(s)ds\Big\}.$$

Note A is a non-self correspondence on C with $A(\partial C)\subseteq C$ ($x_1(t)=\frac{b}{a}t, t\in[0,1]$ belongs to C, but $A(x_1)\notin C$). Now, we show that A satisfies (2.7). Let $x,y\in C$ and $p\in A(x)$. If $p=\int_0^1 k(x(t)-s,x(s))f(s)ds$, choose $q=\int_0^1 k(y(t)-s,y(s))f(s)ds$ and note

$\int_0^1 k(x(t)-s,x(s))f(s)ds+\int_0^1 k(y(t)-s,y(s))f(s)ds\preceq\int_0^1(k(x(t)-s,x(s))+k(y(t)-s,y(s)))f(s)ds\preceq\int_0^1 k_0(x(t)+y(t))ds+l_0\int_0^1(y(t)+k(x(t)-s,x(s)))ds\preceq k_0(x(t)+y(t))+l_0(y(t)+\int_0^1 k(x(t)-s,x(s))ds)$.

Therefore

$$d(p,q)\preceq k_0d(x,y)+l_0d(y,p). \tag{3.3}$$

A similar argument with

$$p=\int_0^1 h(x(t)-s,x(s))g(s)ds$$

and

$$q = \int_0^1 h(y(t) - s, y(s))g(s)ds$$

gives again (3.3) with k_0 replaced by k'_0 and l_0 replaced by l'_0. Thus, by Theorem 2.10, A has a fixed point. □

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