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COMMON FIXED POINT OF SUBCOMPATIBLE MAPPINGS AND BEST APPROXIMATIONS IN CONVEX METRIC SPACES

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Abstract. In this paper, we first establish a common fixed point theorem for a pair of compatible mappings satisfying a Ciric's type contraction condition in the setting of convex metric spaces which is utilized to derive new common fixed point result for subcompatible maps under suitable assumptions. Subsequently, as applications to new common fixed point result, we derive invariant approximation results. Several related results are also derived as corollaries to our main results besides furnishing illustrative examples to demonstrate the validity of the hypotheses of our results proved herein.

1. Introduction

Metric fixed point theory is a branch of fixed point theory which finds its primary applications in functional analysis. The interplay between the geometry of Banach spaces and the fixed point theory is not only natural but also proving very fruitful. In particular, geometric properties play a key role in metric fixed point problems (see [11] and references cited therein). There exist numerous results banking heavily on geometric properties of Banach spaces which mark the beginning of a new mathematical field wherein the metric fixed point theorems are proved with the aid of geometric properties of Banach spaces. The utility of approximation theory is enormous. By now approximation theory intersects with almost every other branch of analysis and plays a very fruitful role in the Applied Sciences and Engineering. Broadly speaking, approximation theory is concerned with the approximation of a continuous function by a polynomial which is carried out in several concrete ways these days. In fixed point theory also the approximation of fixed points are carried out and one of the most applied results of fixed point approximation is due to Scarf [23].

In the course of last four decades, fixed point theorems have been extensively applied to best approximation theory and consequently several interesting results were established. One may recall that Meinardus [18] was the first man who notice such a possibility by using Schauder Fixed Point Theorem to best approximation theory. Thereafter, Brosowski [5] obtained a celebrated result and generalized the Meinardus's result. Several authors (e.g. [13, 24, 26]) have further improved the results of Brosowski [5] in several ways. In the year 1988, Sahab et al. [22] extended the result of Hicks and Humpheries [13] and Singh [24] by considering a pair of mappings wherein one is linear and the other one is nonexpansive.

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In 1970, Takahashi [27] introduced the notion of convex metric spaces and proved some fixed point theorems for nonexpansive mappings in such spaces. Afterwards, many authors have discussed the existence of fixed point as well as the convergence of iterative processes for nonexpansive mappings in such spaces (e.g. [7, 9, 14]). Recently, Beg et al. [4] employed convex metric spaces to prove results on the existence of common fixed point and utilize the same to prove the existence of best approximant for relatively contractive commuting mappings which also generalize the core result of Sahab et al. [22] that has witnessed intense research activity it in the course of last several years.

Recently, Al-Mezel and Hussain [2], Akbar and Khan [1], Nashine and Khan [20] and Nashine and Imdad [21] proved common fixed point results for subcompatible pair of mappings under Gregus [10] type contraction condition.

In the present paper, the results are divided into three sections. In Section 3, we establish existence result on common fixed point satisfying Ciric's type contraction condition for compatible mappings in the setting of convex metric space. In Section 4, using the results of Section 3, we derive a new common fixed point result for subcompatible mappings. In the last and final section, as applications of common fixed point theorem of Section 4, we prove some results in the theory of invariant approximation. In process, results due to Beg et al. [4], Al-Thagafi [3], Brosowski [5], Meinardus [18], Nashine and Imdad [20], Nashine and Khan [21], Singh [24, 25] and Sahab et al. [22] are also generalized and improved by considering relatively generalized classes of noncommuting mappings satisfying a Ciric [6] type contraction condition in the setting of convex metric spaces. In the support of our results, we also furnish illustrative examples.

2. Preliminaries

For the material to be presented here, the following definitions are required:

Definition 2.1 ([27])**.** Let $(\mathcal{X}, d)$ be a metric space. A mapping $\mathcal{W} : \mathcal{X}\times\mathcal{X}\times[0, 1] \to \mathcal{X}$ is said to be a convex structure on $\mathcal{X}$, if for all $x, y \in \mathcal{X}$ and $\lambda \in [0, 1]$ the following condition is satisfied:

$$d(u, \mathcal{W}(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $u \in \mathcal{X}$ wherein obviously $\mathcal{W}(x, x, \lambda) = x$.

A metric space $\mathcal{X}$ equipped with a convex structure is called a convex metric space. Obviously, Banach space and each of its convex subsets are simple examples of convex metric spaces with $\mathcal{W}(x, y, \lambda) = \lambda x + (1 - \lambda)y$ but a *Fréchet* space need not be a convex metric space. There are many examples of convex metric spaces which can not be embedded in any Banach space. For substantiation, the following two examples can be recalled:

Example 2.2. Let I be the unit interval $[0, 1]$ and $\mathcal{X}$ be the family of closed intervals $[a_i, b_i]$ such that $0 \leq a_i \leq b_i \leq 1$. For $I_i = [a_i, b_i], I_j = [a_j, b_j]$ and $\lambda(0 \leq \lambda \leq 1)$, we define a mapping $\mathcal{W}$ by $\mathcal{W}(I_i, I_j; \lambda) = [\lambda a_i + (1 - \lambda)a_j, \lambda b_i + (1 - \lambda)b_j]$ and define a metric d in $\mathcal{X}$ by the Hausdorff distance, i.e.

$$d(I_i, I_j) = \sup_{a\in I}\{|\inf_{b\in I_i}\{|a - b|\} - \inf_{c\in I_j}\{|a - c|\}|\}.$$

Example 2.3. A linear space $\mathcal{L}$ equipped with the following two properties forms a natural convex metric space:
(i) For $x, y \in \mathcal{L}$, $d(x,y) = d(x-y, 0)$.
(ii) For $x, y \in \mathcal{L}$ and $\lambda (0 \leq \lambda \leq 1)$,

$$d(\lambda x + (1-\lambda)y, 0) \leq \lambda d(x,0) + (1-\lambda)d(y,0).$$

Definition 2.4. A subset $\mathcal{K}$ of a convex metric space $(\mathcal{X}, d)$ is said to be convex, if $\mathcal{W}(x,y,\lambda) \in \mathcal{K}$ for all $x, y \in \mathcal{K}$ and $\lambda \in [0,1]$. The set $\mathcal{K}$ is said to q-starshaped if there exists $q \in \mathcal{K}$ such that $\mathcal{W}(x,q,\lambda) \in \mathcal{K}$ for all $x \in \mathcal{K}$ and $\lambda \in [0,1]$. Clearly q-starshaped subsets of $\mathcal{X}$ contain all convex subsets of $\mathcal{X}$ as a proper subclass. Takahashi [27] has shown that open spheres $\mathcal{B}(x,r) = \{y \in \mathcal{X} : d(y,x) < r\}$ and closed spheres $\mathcal{B}[x,r] = \{y \in \mathcal{X} : d(y,x) \leq r\}$ are convex in a convex metric space $\mathcal{X}$.

Definition 2.5. Let $(\mathcal{X}, d)$ be a convex metric space wherein for $(x, y \in \mathcal{X})$

$$seg[x,y] = \{W(x,y,\lambda) : 0 \leq \lambda \leq 1\} \text{ and } d(seg[x,y], z) = \inf\{d(t,z) : t \in seg[x,y]\}.$$

Definition 2.6. A convex metric space $(\mathcal{X}, d)$ is said to satisfy the property (I), if for all $x, y, z \in \mathcal{X}$ and $\lambda \in [0,1]$,

$$d(\mathcal{W}(x,z,\lambda), \mathcal{W}(y,z,\lambda)) \leq \lambda d(x,y).$$

For motivation and further details in respect of the Property (I), one can be referred to Guay et al. [12] (e.g. Definition 3.2).

Definition 2.7 ([14, 27])**.** A continuous function $\mathcal{S}$ from a closed convex subset $\mathcal{K}$ of a convex metric space $(\mathcal{X}, d)$ into itself is said to be $\mathcal{W}$-affine if $\mathcal{S}(\mathcal{W}(x,y,\lambda)) = \mathcal{W}(\mathcal{S}x, \mathcal{S}y, \lambda)$ whenever $\lambda \in [0,1] \cap \mathcal{Q}$ and $x, y \in \mathcal{K}$, where $\mathcal{Q}$ stands for the set of rational numbers.

Definition 2.8 ([24])**.** Let $\mathcal{K}$ be a closed subset of a metric space $(\mathcal{X}, d)$. Let $x_0 \in \mathcal{X}$. An element $y \in \mathcal{K}$ is called a best approximant to $x_0 \in \mathcal{X}$, if

$$d(x_0, y) = d(x_0, \mathcal{K}) = \inf\{d(x_0, z) : z \in \mathcal{K}\}.$$

We denote by $\mathcal{P}_{\mathcal{K}}(x_0)$, the set of best $\mathcal{K}-$approximants to x_0.

Definition 2.9. Let $\mathcal{T}$ be a self-map defined on a subset $\mathcal{K}$ of a metric space $(\mathcal{X}, d)$. A best approximant y in $\mathcal{K}$ to an element x_0 in $\mathcal{X}$ (with $\mathcal{T}x_0 = x_0$) is an invariant approximation in $\mathcal{X}$ to y if $\mathcal{T}y = y$.

Remark 2.10. Let $\mathcal{K}$ be a closed convex subset of a convex metric space $(\mathcal{X}, d)$. As, $\mathcal{W}(u,v,\lambda) \in \mathcal{K}$ for $(u,v,\lambda) \in \mathcal{K} \times \mathcal{K} \times [0,1]$, so with $u, v \in \mathcal{P}_{\mathcal{K}}(x)$, the definition of convexity structure on $\mathcal{X}$ further implies that $\mathcal{W}(u,v,\lambda) \in \mathcal{P}_{\mathcal{K}}(x)$. Hence $\mathcal{P}_{\mathcal{K}}(x)$ is a convex subset of $\mathcal{X}$. Also, $\mathcal{P}_{\mathcal{K}}(x)$ is a closed subset of $\mathcal{X}$. Moreover, it can also be shown that $\mathcal{P}_{\mathcal{K}}(x) \subset \partial\mathcal{K}$, where $\partial\mathcal{K}$ stands for the boundary of $\mathcal{K}$.

Example 2.11 ([25])**.** Let $\mathcal{X} = \mathbb{R}$ and $\mathcal{K} = [0, \frac{1}{2}]$. Define $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ as follows:

$$\mathcal{T}x = \begin{cases} x-1, & \text{if } x < 0, \\ x, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{x+1}{2}, & \text{if } x > \frac{1}{2}. \end{cases} \tag{2.1}$$

Clearly, $\mathcal{T}(\mathcal{K}) = \mathcal{K}$ and $\mathcal{T}(1) = 1$ (i.e. $x_0 = 1$). Also

$$\mathcal{P}_{\mathcal{K}}(x_0) = \left\{\frac{1}{2}\right\}.$$

Hence, $\mathcal{T}$ has a fixed point in $\mathcal{P}_{\mathcal{K}}(x_0)$ which is a best approximation to x_0 in $\mathcal{K}$. Thus, $\frac{1}{2}$ is an invariant approximation.

Definition 2.12 ([15])**.** A pair $(\mathcal{T}, \mathcal{S})$ of self-mappings of a metric space $\mathcal{X}$ is said to be compatible, if $d(\mathcal{TS}x_n, \mathcal{ST}x_n) \to 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{T}x_n, \mathcal{S}x_n \to t \in \mathcal{X}$.

Every commuting pair of mappings is compatible but the converse implication is not true in general.

Definition 2.13 ([1,17])**.** Suppose that $\mathcal{K}$ is q-starshaped subset of a convex metric space $\mathcal{X}$. For the self maps $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{K}$ with $q \in Fix(\mathcal{S})$, define $\bigwedge_q(\mathcal{S}, \mathcal{T}) = \bigcup\{\bigwedge(\mathcal{S}, \mathcal{T}_k) : 0 \leq k \leq 1\}$ where $\mathcal{T}_k x = seg[\mathcal{T}x, q]$ and $\bigwedge(\mathcal{S}, \mathcal{T}_k) = \{\{x_n\} \subset \mathcal{K} : \lim_n \mathcal{S}x_n = \lim_n \mathcal{T}_k x_n = t \in \mathcal{K}\}$. Then $\mathcal{S}$ and $\mathcal{T}$ are called subcompatible, if $\lim_n d(\mathcal{ST}x_n, \mathcal{TS}x_n) = 0$ for all sequences $x_n \in \bigwedge_q(\mathcal{S}, \mathcal{T})$.

Obviously, subcompatible maps are compatible but not conversely as substantiated by the following example.

Example 2.14 ([1, 17, 20])**.** Let $\mathcal{X} = \mathbb{R}$ with usual metric and $\mathcal{K} = [1, \infty)$. Let $\mathcal{S}(x) = 2x - 1$ and $\mathcal{T}(x) = x^2$, for all $x \in \mathcal{K}$. Let $q = 1$. Then $\mathcal{K}$ is q-starshaped with $\mathcal{S}q = q$. As for sequences in $\mathcal{K}$ converging to 1, $\lim_n d(\mathcal{ST}x_n, \mathcal{TS}x_n) = 0$, therefore $\mathcal{S}$ and $\mathcal{T}$ are compatible. For any sequence $\{x_n\}$ in $\mathcal{K}$ with $\lim_n x_n = 2$, we have, $\lim_n \mathcal{S}x_n = \lim_n \mathcal{T}_{\frac{2}{3}} x_n = 3 \in \mathcal{K}$. However, $\lim_n d(\mathcal{ST}x_n, \mathcal{TS}x_n) \neq 0$. Thus $\mathcal{S}$ and $\mathcal{T}$ are not subcompatible.

Gregus [10] proved the following theorem.

Theorem 2.15. *Let $\mathcal{K}$ be a closed convex subset of a Banach space $\mathcal{X}$ and let the mapping $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ satisfies*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq a\|x - y\| + b\|x - \mathcal{T}x\| + c\|y - \mathcal{T}y\|$$

for all $x, y \in \mathcal{K}$, where $0 < a < 1$, $b, c \geq 0$ and $a + b + c = 1$. Then $\mathcal{T}$ has a unique fixed point.

Fisher and Sessa [8] obtained the following generalization of Gregus fixed point theorem.

Theorem 2.16. *Let $\mathcal{T}$ and $\mathcal{S}$ be weakly commuting self mappings of a closed convex subset $\mathcal{K}$ of a Banach space $\mathcal{X}$ with $\mathcal{T}(\mathcal{K}) \subseteq \mathcal{S}(\mathcal{K})$ and also satisfy the inequality*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq a\|\mathcal{S}x - \mathcal{S}y\| + (1 - a)\max\{\|\mathcal{T}x - \mathcal{S}x\|, \|\mathcal{T}y - \mathcal{S}y\|\}, \tag{2.2}$$

for $x, y \in \mathcal{K}$, where $0 < a < 1$. If $\mathcal{S}$ is linear and nonexpansive in $\mathcal{K}$, then $\mathcal{T}$ and $\mathcal{S}$ have a unique common fixed point in $\mathcal{K}$.

Mukherjee and Verma [19] replaced the linearity of the mapping $\mathcal{S}$ by affineness in Theorem 2.16, whereas Jungck [16] improved and generalized Theorem 2.16 by replacing the nonexpansive property of $\mathcal{S}$ by continuity of $\mathcal{S}$ and weak commutativity by compatibility.

Ciric [6] employed the following more general contractive condition in normed space to improve the Gregus theorem:

Definition 2.17. Let $\mathcal{T}$ and $\mathcal{S}$ be two self mappings of a nonempty subset $\mathcal{K}$ of a normed linear space $\mathcal{X}$. The self mapping $\mathcal{T}$ of $\mathcal{K}$ is said to satisfy Ciric $\mathcal{S}$-contractive type condition, if there exist real numbers a, b, c with $0 < a < 1$, $b \geq 0$, $a + b = 1$, $0 \leq c < \eta$ such that

$$\begin{aligned} \|\mathcal{T}x - \mathcal{T}y\| \quad \leq \quad & a \max\{\|\mathcal{S}x - \mathcal{S}y\|, c[\|\mathcal{S}x - \mathcal{T}y\| + \|\mathcal{S}y - \mathcal{T}x\|]\} \\ & +b \max\{\|\mathcal{T}x - \mathcal{S}x\|, \|\mathcal{T}y - \mathcal{S}y\|\} \end{aligned} \tag{2.3}$$

for $x, y \in \mathcal{K}$, where $\eta = \min\{\frac{2+a}{5+a}, \frac{2-a}{4}, \frac{4}{9+a}\}$.

It can be observed that $\eta < \frac{1}{2}$.

Definition 2.18. A self mapping $\mathcal{T}$ of a nonempty subset $\mathcal{K}$ of a metric space $(\mathcal{X}, d)$ is said to be compact if $\{\mathcal{T}x_n\}$ admits a convergent subsequence $\{\mathcal{T}x_m\}$ in $\mathcal{K}$ for every bounded sequence $\{x_n\}$ in $\mathcal{K}$.

3. A Ciric's type fixed point theorem

In this section, we prove a common fixed point result satisfying Ciric's type contraction condition for compatible mappings in convex metric space.

Theorem 3.1. *Let $(\mathcal{X}, d)$ be a complete convex metric space with a convex structure $\mathcal{W}$ whereas $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{X}$. Let the pair $(\mathcal{T}, \mathcal{S})$ be a compatible pair of self-mappings defined on $\mathcal{K}$ such that there exist real numbers a, b, c with $0 < a < 1$, $b \geq 0$, $a + 2b = 1$, $0 \leq c < \eta$ with*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) \leq & a \max\{d(\mathcal{S}x, \mathcal{S}y), c[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}y, \mathcal{T}x)]\} \\ & +b \max\{d(\mathcal{T}x, \mathcal{S}x), d(\mathcal{T}y, \mathcal{S}y), \tfrac{1}{2}[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}y, \mathcal{T}x)]\} \end{aligned} \tag{3.1}$$

for $x, y \in \mathcal{K}$, where $\eta = \min\{\frac{2+a}{5+a}, \frac{2-a}{4}, \frac{4}{9+a}\}$. If $\mathcal{T}(\mathcal{K}) \subseteq \mathcal{S}(\mathcal{K})$ and $\mathcal{S}$ is $\mathcal{W}$-affine and continuous, then there exists a unique common fixed point u of $\mathcal{T}$ and $\mathcal{S}$. Moreover, $\mathcal{T}$ is continuous at the unique common fixed point u.

Proof. Since $\mathcal{T}(\mathcal{K}) \subseteq \mathcal{S}(\mathcal{K})$ and x_0 be an arbitrary point in $\mathcal{K}$, therefore we can choose points x_1, x_2 and x_3 in $\mathcal{K}$ such that $\mathcal{S}x_1 = \mathcal{T}x_0$, $\mathcal{S}x_2 = \mathcal{T}x_1$ and $\mathcal{S}x_3 = \mathcal{T}x_2$ so that

$$\mathcal{S}x_r = \mathcal{T}x_{r-1}, \ \ \forall \ r = 1, 2, 3, \ldots. \tag{3.2}$$

which can be done as $\mathcal{T}(\mathcal{K}) \subseteq \mathcal{S}(\mathcal{K})$. By (3.1), (3.2) and the triangle inequality, we have

$$\begin{aligned} d(\mathcal{T}x_r, \mathcal{S}x_r) \quad = \quad & d(\mathcal{T}x_r, \mathcal{T}x_{r-1}) \\ \leq \quad & a \max\{d(\mathcal{S}x_r, \mathcal{S}x_{r-1}), c[d(\mathcal{S}x_r, \mathcal{T}x_{r-1}) + d(\mathcal{S}x_{r-1}, \mathcal{T}x_r)]\} \\ & +b \max\{d(\mathcal{S}x_r, \mathcal{T}x_r), d(\mathcal{S}x_{r-1}, \mathcal{T}x_{r-1}), \frac{1}{2}[d(\mathcal{S}x_r, \mathcal{T}x_{r-1}) \\ & +d(\mathcal{S}x_{r-1}, \mathcal{T}x_r)]\} \\ \leq \quad & a \max\{d(\mathcal{T}x_{r-1}, \mathcal{S}x_{r-1}), c[d(\mathcal{S}x_r, \mathcal{S}x_r) + d(\mathcal{S}x_{r-1}, \mathcal{T}x_{r-1}) \end{aligned}$$

$$\begin{aligned}&+d(\mathcal{T}x_{r-1},\mathcal{T}x_r)]\}+b\max\{d(\mathcal{S}x_r,\mathcal{T}x_r),d(\mathcal{S}x_{r-1},\mathcal{T}x_{r-1}),\\&\frac{1}{2}[d(\mathcal{S}x_r,\mathcal{S}x_r)+d(\mathcal{S}x_{r-1},\mathcal{T}x_{r-1})+d(\mathcal{T}x_{r-1},\mathcal{T}x_r)]\}.\end{aligned}$$

If there exists $r \in \mathbb{N}$ such that $d(\mathcal{T}x_{r-1},\mathcal{S}x_{r-1}) < d(\mathcal{T}x_r,\mathcal{S}x_r)$, it follows from the preceding inequality that

$$\begin{aligned}d(\mathcal{T}x_r,\mathcal{S}x_r) &< a\max\{d(\mathcal{T}x_r,\mathcal{S}x_r),2cd(\mathcal{T}x_r,\mathcal{S}x_r)\}+bd(\mathcal{T}x_r,\mathcal{S}x_r)\\&= (a+b)d(\mathcal{T}x_r,\mathcal{S}x_r)<(a+2b)d(\mathcal{T}x_r,\mathcal{S}x_r),\end{aligned}$$

which is a contradiction and therefore, we have

$$d(\mathcal{T}x_r,\mathcal{S}x_r)\le d(\mathcal{T}x_{r-1},\mathcal{S}x_{r-1}) \text{ for } r=1,2,3,\ldots.$$

so that

$$d(\mathcal{T}x_r,\mathcal{S}x_r) \le d(\mathcal{T}x_0,\mathcal{S}x_0) \text{ for } r=1,2,3,\ldots. \tag{3.3}$$

On using (3.1) and (3.3), we have

$$\begin{aligned}d(\mathcal{T}x_2,\mathcal{S}x_1) &= d(\mathcal{T}x_2,\mathcal{T}x_0)\\&\le a\max\{d(\mathcal{S}x_2,\mathcal{S}x_0),c[d(\mathcal{S}x_2,\mathcal{T}x_0)+d(\mathcal{S}x_0,\mathcal{T}x_2)]\}\\&\quad+b\max\{d(\mathcal{S}x_2,\mathcal{T}x_2),d(\mathcal{S}x_0,\mathcal{T}x_0),\frac{1}{2}[d(\mathcal{S}x_2,\mathcal{T}x_0)+d(\mathcal{S}x_0,\mathcal{T}x_2)]\}\\&\le a\max\{d(\mathcal{T}x_1,\mathcal{S}x_1)+d(\mathcal{T}x_0,\mathcal{S}x_0),c[d(\mathcal{T}x_1,\mathcal{S}x_1)+d(\mathcal{T}x_0,\mathcal{S}x_0)\\&\quad+d(\mathcal{S}x_1,\mathcal{T}x_1)+d(\mathcal{S}x_2,\mathcal{T}x_2)]\}+b\max\{d(\mathcal{S}x_2,\mathcal{T}x_2),d(\mathcal{S}x_0,\mathcal{T}x_0),\\&\quad\frac{1}{2}[d(\mathcal{T}x_1,\mathcal{S}x_1)+d(\mathcal{T}x_0,\mathcal{S}x_0)+d(\mathcal{S}x_1,\mathcal{T}x_1)+d(\mathcal{S}x_2,\mathcal{T}x_2)]\}\\&\le a\max\{2d(\mathcal{S}x_0,\mathcal{T}x_0),4cd(\mathcal{S}x_0,\mathcal{T}x_0)\}+b\max\{d(\mathcal{S}x_0,\mathcal{T}x_0),\\&\quad\frac{1}{2}(4d(\mathcal{T}x_0,\mathcal{S}x_0))\}\\&= (1+a)d(\mathcal{T}x_0,\mathcal{S}x_0),\end{aligned}$$

so that

$$d(\mathcal{T}x_2,\mathcal{S}x_1)=d(\mathcal{T}x_2,\mathcal{T}x_0)\le(1+a)d(\mathcal{T}x_0,\mathcal{S}x_0). \tag{3.4}$$

Write $z=\mathcal{W}(x_2,x_3,\frac{1}{2})$. Since $\mathcal{S}$ is $\mathcal{W}$-affine, on using (3.2), we have

$$\mathcal{S}z=\mathcal{W}\Big(\mathcal{S}x_2,\mathcal{S}x_3,\frac{1}{2}\Big)=\mathcal{W}\Big(\mathcal{T}x_1,\mathcal{T}x_2,\frac{1}{2}\Big), \tag{3.5}$$

so that

$$\begin{aligned}d(\mathcal{T}z,\mathcal{S}z) &= d\Big(\mathcal{T}z,\mathcal{W}\Big(\mathcal{T}x_1,\mathcal{T}x_2,\frac{1}{2}\Big)\Big)\\&\le \frac{1}{2}d(\mathcal{T}z,\mathcal{T}x_1)+\frac{1}{2}d(\mathcal{T}z,\mathcal{T}x_2). \qquad (3.6)\end{aligned}$$

Write $\mathcal{M}=\max\{d(\mathcal{S}z,\mathcal{T}z),d(\mathcal{T}x_0,\mathcal{S}x_0)\}$. Now, on using the inequality (3.1), we have

$$\begin{aligned}d(\mathcal{T}z,\mathcal{T}x_1) &\le a\max\{d(\mathcal{S}z,\mathcal{S}x_1),c[d(\mathcal{S}z,\mathcal{T}x_1)+d(\mathcal{S}x_1,\mathcal{T}z)]\}\\&\quad+b\max\{d(\mathcal{S}z,\mathcal{T}z),d(\mathcal{S}x_1,\mathcal{T}x_1),\\&\quad\frac{1}{2}[d(\mathcal{S}z,\mathcal{T}x_1)+d(\mathcal{S}x_1,\mathcal{T}z)]\}.\end{aligned}$$

Thus,

$$
\begin{aligned}
(3.7)\quad d(\mathcal{T}z,\mathcal{T}x_1) \leq\ & a\max\{d(\mathcal{S}z,\mathcal{S}x_1), c[d(\mathcal{S}z,\mathcal{T}x_1)+d(\mathcal{S}x_1,\mathcal{S}z)\\
& +d(\mathcal{S}z,\mathcal{T}z)]\}+b\max\{d(\mathcal{S}z,\mathcal{T}z), d(\mathcal{S}x_1,\mathcal{T}x_1),\\
& \frac{1}{2}[d(\mathcal{S}z,\mathcal{T}x_1)+d(\mathcal{S}x_1,\mathcal{S}z)+d(\mathcal{S}z,\mathcal{T}z)]\}.
\end{aligned}
$$

Owing to (3.3), (3.4) and (3.5), we get

$$
\begin{aligned}
d(\mathcal{S}z,\mathcal{S}x_1) &= d\Big(\mathcal{S}x_1,\mathcal{W}\Big(\mathcal{T}x_1,\mathcal{T}x_2,\frac{1}{2}\Big)\Big)\\
&\leq \frac{1}{2}d(\mathcal{T}x_1,\mathcal{S}x_1)+\frac{1}{2}d(\mathcal{T}x_2,\mathcal{S}x_1)\\
&\leq \frac{1}{2}d(\mathcal{T}x_0,\mathcal{S}x_0)+\frac{1}{2}(1+a)d(\mathcal{T}x_0,\mathcal{S}x_0)\\
&= \Big(1+\frac{a}{2}\Big)d(\mathcal{T}x_0,\mathcal{S}x_0). \qquad (3.8)
\end{aligned}
$$

Now, on using (3.2), (3.3) and (3.5), we have

$$
\begin{aligned}
d(\mathcal{S}z,\mathcal{S}x_2) &= d(\mathcal{S}z,\mathcal{T}x_1)=d\Big(\mathcal{T}x_1,\mathcal{W}\Big(\mathcal{T}x_1,\mathcal{T}x_2,\frac{1}{2}\Big)\Big)\\
&\leq \frac{1}{2}d(\mathcal{T}x_2,\mathcal{T}x_1)=\frac{1}{2}d(\mathcal{T}x_2,\mathcal{S}x_2)\leq\frac{1}{2}d(\mathcal{T}x_0,\mathcal{S}x_0). \qquad (3.9)
\end{aligned}
$$

and

$$
\begin{aligned}
d(\mathcal{S}z,\mathcal{T}x_2) &= d\Big(\mathcal{T}x_2,\mathcal{W}\Big(\mathcal{T}x_1,\mathcal{T}x_2,\frac{1}{2}\Big)\Big)\\
&\leq \frac{1}{2}d(\mathcal{T}x_1,\mathcal{T}x_2)=\frac{1}{2}d(\mathcal{S}x_2,\mathcal{T}x_2)\leq\frac{1}{2}d(\mathcal{T}x_0,\mathcal{S}x_0). \qquad (3.10)
\end{aligned}
$$

Making use of (3.8) and (3.9) in (3.7), we have

$$
\begin{aligned}
d(\mathcal{T}z,\mathcal{T}x_1) &\leq a\max\Big\{\Big(1+\frac{a}{2}\Big)d(\mathcal{T}x_0,\mathcal{S}x_0), c\Big[\frac{1}{2}d(\mathcal{T}x_0,\mathcal{S}x_0)+\Big(1+\frac{a}{2}\Big)d(\mathcal{T}x_0,\mathcal{S}x_0)\\
&\quad +d(\mathcal{S}z,\mathcal{T}z)\Big]\Big\}+b\max\Big\{d(\mathcal{S}z,\mathcal{T}z), d(\mathcal{T}x_0,\mathcal{S}x_0),\\
&\quad \frac{1}{2}\Big[\frac{1}{2}d(\mathcal{T}x_0,\mathcal{S}x_0)+\Big(1+\frac{a}{2}\Big)d(\mathcal{T}x_0,\mathcal{S}x_0)+d(\mathcal{S}z,\mathcal{T}z)\Big]\Big\}\\
&\leq a\max\Big\{\Big(1+\frac{a}{2}\Big)\mathcal{M}, c\Big(\frac{5+a}{2}\Big)\mathcal{M}\Big\}+b\frac{5+a}{4}\mathcal{M}. \qquad (3.11)
\end{aligned}
$$

Also, using the inequality (3.1), we have

$$
\begin{aligned}
d(\mathcal{T}z,\mathcal{T}x_2) \leq\ & a\max\{d(\mathcal{S}z,\mathcal{S}x_2), c[d(\mathcal{S}z,\mathcal{T}x_2)+d(\mathcal{S}x_2,\mathcal{T}z)]\}\\
& +b\max\{d(\mathcal{S}z,\mathcal{T}z), d(\mathcal{S}x_2,\mathcal{T}x_2), \frac{1}{2}[d(\mathcal{S}z,\mathcal{T}x_2)+d(\mathcal{S}x_2,\mathcal{T}z)]\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
d(\mathcal{T}z,\mathcal{T}x_2) \leq\ & a\max\{d(\mathcal{S}z,\mathcal{S}x_2), c[d(\mathcal{S}z,\mathcal{T}x_2)+d(\mathcal{S}x_2,\mathcal{S}z)+d(\mathcal{S}z,\mathcal{T}z)]\}\\
& +b\max\{d(\mathcal{S}z,\mathcal{T}z), d(\mathcal{S}x_2,\mathcal{T}x_2),\\
& \frac{1}{2}[d(\mathcal{S}z,\mathcal{T}x_2)+d(\mathcal{S}x_2,\mathcal{S}z)+d(\mathcal{S}z,\mathcal{T}z)]\}. \qquad (3.12)
\end{aligned}
$$

On making use of (3.9) and (3.10) in (3.12), we get

$$\begin{aligned} d(\mathcal{T}z,\mathcal{T}x_2) &\le a\max\Big\{\frac{1}{2}d(\mathcal{T}x_0,\mathcal{S}x_0), c\Big[\frac{1}{2}d(\mathcal{T}x_0,\mathcal{S}x_0) \\ &\qquad +\frac{1}{2}d(\mathcal{T}x_0,\mathcal{S}x_0)+d(\mathcal{S}z,\mathcal{T}z)\Big]\Big\} \\ &\quad +b\max\Big\{d(\mathcal{S}z,\mathcal{T}z), d(\mathcal{T}x_0,\mathcal{S}x_0), \frac{1}{2}\Big[\frac{1}{2}d(\mathcal{T}x_0,\mathcal{S}x_0) \\ &\qquad +\frac{1}{2}d(\mathcal{T}x_0,\mathcal{S}x_0)+d(\mathcal{S}z,\mathcal{T}z)\Big]\Big\} \\ &\le a\max\{\frac{1}{2}\mathcal{M}, 2c\mathcal{M}\}+b\mathcal{M}, \qquad (3.13) \end{aligned}$$

whereas making use of (3.11) and (3.13) in (3.6), we have

$$\begin{aligned} d(\mathcal{T}z,\mathcal{S}z) &\le \frac{1}{2}\Big[a\max\Big\{\Big(1+\frac{a}{2}\Big)\mathcal{M}, \Big(\frac{5+a}{2}\Big)c\mathcal{M}\Big\}+b\Big(\frac{5+a}{4}\Big)\mathcal{M}\Big] \\ &\quad +\frac{1}{2}\Big[a\max\Big\{\frac{1}{2}\mathcal{M}, 2c\mathcal{M}\Big\}+b\mathcal{M}\Big] \\ &= \frac{a}{2}\Big[\max\Big\{\Big(1+\frac{a}{2}\Big)\mathcal{M}, \Big(\frac{5+a}{2}\Big)c\mathcal{M}\}\Big] \qquad (3.14) \\ &\quad +\frac{a}{2}\Big[\max\Big\{\frac{1}{2}\mathcal{M}, 2c\mathcal{M}\Big\}\Big]+b\Big(\frac{9+a}{8}\Big)\mathcal{M}. \end{aligned}$$

Now the following four possible cases may arise in (3.14).

Case 1. If $\max\{(1+\frac{a}{2})\mathcal{M}, (\frac{5+a}{2})c\mathcal{M}\}=(1+\frac{a}{2})\mathcal{M}$ and $\max\{\frac{1}{2}\mathcal{M}, 2c\mathcal{M}\}=\frac{1}{2}\mathcal{M}$, then (owing to (3.14)) we have

$$\begin{aligned} d(\mathcal{T}z,\mathcal{S}z) &\le \Big[\frac{a}{2}\Big(1+\frac{a}{2}\Big)+\frac{a}{2}.\frac{1}{2}+\Big(\frac{1-a}{2}\Big)\Big(\frac{9+a}{8}\Big)\Big]\mathcal{M} \\ &= \Big[\frac{a(2+a)}{4}+\frac{a}{4}+(1-a)\Big(\frac{9+a}{16}\Big)\Big]\mathcal{M} \\ &= \lambda_1.\mathcal{M}, \qquad (3.15) \end{aligned}$$

where $\lambda_1=\frac{3a^2+4a+9}{16}(<1)$.

Case 2. If $\max\{(1+\frac{a}{2})\mathcal{M}, (\frac{5+a}{2})c\mathcal{M}\}=(1+\frac{a}{2})\mathcal{M}$ and $\max\{\frac{1}{2}\mathcal{M}, 2c\mathcal{M}\}=2c\mathcal{M}$, then (owing to (3.14)) we have

$$\begin{aligned} d(\mathcal{T}z,\mathcal{S}z) &\le \Big[\frac{a}{2}\Big(1+\frac{a}{2}\Big)+\frac{a}{2}2c+\Big(\frac{1-a}{2}\Big)\Big(\frac{9+a}{8}\Big)\Big]\mathcal{M} \\ &= \Big[\frac{a(2+a)}{4}+ac+(1-a)\Big(\frac{9+a}{16}\Big)\Big]\mathcal{M} \\ &= \lambda_2.\mathcal{M}, \qquad (3.16) \end{aligned}$$

where $\lambda_2=\frac{3a^2+16ac+9}{16}(<1)$.

Case 3. If $\max\{(1+\frac{a}{2})\mathcal{M}, (\frac{5+a}{2})c\mathcal{M}\}=(\frac{5+a}{2})c\mathcal{M}$, then we have $\frac{2+a}{5+a}\le c$. As $c<\eta=\min\{\frac{2+a}{5+a}, \frac{2-a}{4}, \frac{4}{9+a}\}$, therefore $\frac{2+a}{5+a}\le c<\eta\le\frac{2+a}{5+a}$, so that $\frac{2+a}{5+a}<\frac{2+a}{5+a}$, which is a contradiction. Therefore the situation $\max\{(1+\frac{a}{2})\mathcal{M}, (\frac{5+a}{2})c\mathcal{M}\}=(\frac{5+a}{2})c\mathcal{M}$, and $\max\{\frac{1}{2}\mathcal{M}, 2c\mathcal{M}\}=2c\mathcal{M}$ can not arise.

Case 4. In view of the explanations furnished in Case 3, the situation $\max\{(1+\frac{a}{2})\mathcal{M}, (\frac{5+a}{2})c\mathcal{M}\} = (\frac{5+a}{2})c\mathcal{M}$,and $\max\{\frac{1}{2}\mathcal{M}, 2c\mathcal{M}\} = \frac{1}{2}\mathcal{M}$ also can not arise.

In view of (3.15), (3.16), we have

$$d(\mathcal{T}z, \mathcal{S}z) \quad \leq \quad \lambda.\mathcal{M},$$

where $\lambda = \max\{\lambda_1, \lambda_2\}$. Now one can have

$$d(\mathcal{T}z, \mathcal{S}z) \leq \lambda \max\{d(\mathcal{S}z, \mathcal{T}z), d(\mathcal{T}x_0, \mathcal{S}x_0)\},$$

so that

$$d(\mathcal{T}z, \mathcal{S}z) \leq \lambda d(\mathcal{T}x_0, \mathcal{S}x_0). \tag{3.17}$$

Since x_0 is an arbitrary point in $\mathcal{K}$, it follows from (3.17) that there exists a sequence $\{z_n\}$ in $\mathcal{K}$ such that

$$\begin{aligned} d(\mathcal{S}z_0, \mathcal{T}z_0) &\leq \lambda d(\mathcal{S}x_0, \mathcal{T}x_0) \\ d(\mathcal{S}z_1, \mathcal{T}z_1) &\leq \lambda d(\mathcal{S}z_0, \mathcal{T}z_0) \\ &\vdots \\ d(\mathcal{S}z_n, \mathcal{T}z_n) &\leq \lambda d(\mathcal{S}z_{n-1}, \mathcal{T}z_{n-1}) \end{aligned}$$

which yield that $d(\mathcal{S}z_n, \mathcal{T}z_n) \leq \lambda^{n+1} d(\mathcal{S}z_0, \mathcal{T}z_0)$ and so we have

$$d(\mathcal{S}z_n, \mathcal{T}z_n) \to 0. \tag{3.18}$$

Setting $\mathcal{K}_n = \{x \in \mathcal{K} : d(\mathcal{T}x, \mathcal{S}x) \leq \frac{1}{n}\}$ and $\mathcal{H}_n = \{x \in \mathcal{K} : d(\mathcal{T}x, \mathcal{S}x) \leq \frac{3a+1}{(1-a)n}\}$ for $n = 1, 2, 3, \ldots$.

Then (3.18) implies that $\mathcal{K}_n \neq \emptyset$, $n = 1, 2, 3, \ldots$ and $\mathcal{K}_1 \supseteq \mathcal{K}_2 \supseteq \mathcal{K}_3 \supseteq \cdots \supseteq \mathcal{K}_n \supseteq$ ...

Consequently, $\overline{\mathcal{T}\mathcal{K}_n} \neq \emptyset$ and $\overline{\mathcal{T}\mathcal{K}_n} \supseteq \overline{\mathcal{T}\mathcal{K}_{n+1}}$, $n = 1, 2, 3, \ldots$.

For any $x, y \in \mathcal{K}_n$, it follows from (3.1) that

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) &\leq a \max\{d(\mathcal{S}x, \mathcal{S}y), c[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}y, \mathcal{T}x)]\} \\ &\quad + b \max\Big\{d(\mathcal{S}x, \mathcal{T}x), d(\mathcal{S}y, \mathcal{T}y), \frac{1}{2}[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}y, \mathcal{T}x)]\Big\} \\ &\leq a \max\Big\{\frac{2}{n} + d(\mathcal{T}x, \mathcal{T}y), c\Big[\frac{2}{n} + 2d(\mathcal{T}x, \mathcal{T}y)\Big]\Big\} + b\Big[\frac{1}{n} + d(\mathcal{T}x, \mathcal{T}y)\Big]. \quad (3.19) \end{aligned}$$

Here we consider the following two possible cases of (3.19).

Case I. If $\max\{\frac{2}{n} + d(\mathcal{T}x, \mathcal{T}y), c[\frac{2}{n} + 2d(\mathcal{T}x, \mathcal{T}y)]\} = \frac{2}{n} + d(\mathcal{T}x, \mathcal{T}y)$, then from (3.19), we have

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) &\leq \frac{2a}{n} + a d(\mathcal{T}x, \mathcal{T}y) + b\Big[\frac{1}{n} + d(\mathcal{T}x, \mathcal{T}y)\Big] \\ &= \frac{2a+b}{n} + (a+b) d(\mathcal{T}x, \mathcal{T}y) \end{aligned}$$

which yields

$$d(\mathcal{T}x, \mathcal{T}y) \quad \leq \quad \frac{3a+1}{(1-a)n}.$$

Case II. If $\max\{\frac{2}{n} + d(\mathcal{T}x, \mathcal{T}y), c[\frac{2}{n} + 2d(\mathcal{T}x, \mathcal{T}y)]\} = c[\frac{2}{n} + 2d(\mathcal{T}x, \mathcal{T}y)]$, then from (3.19), we have

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) &\leq ac\Big[\frac{2}{n} + 2d(\mathcal{T}x, \mathcal{T}y)\Big] + b\Big[\frac{1}{n} + d(\mathcal{T}x, \mathcal{T}y)\Big] \\ &< a\Big[\frac{1}{n} + d(\mathcal{T}x, \mathcal{T}y)\Big] + b\Big[\frac{1}{n} + d(\mathcal{T}x, \mathcal{T}y)\Big] \\ &= \frac{a+b}{n} + (a+b)d(\mathcal{T}x, \mathcal{T}y) \end{aligned}$$

which yields

$$d(\mathcal{T}x, \mathcal{T}y) < \frac{3a+1}{(1-a)n}.$$

Thus in both cases, we get

$$d(\mathcal{T}x, \mathcal{T}y) \leq \frac{3a+1}{(1-a)n}, \qquad \text{so that } x, y \in \mathcal{H}_n.$$

Therefore, we have

$$\lim_{n\to\infty} \operatorname{diam}(\mathcal{T}\mathcal{K}_n) = \lim_{n\to\infty} \operatorname{diam}(\overline{\mathcal{T}\mathcal{K}_n}) = 0,$$

where $diam(\overline{\mathcal{T}\mathcal{K}_n})$ denotes the diameter of $\overline{\mathcal{T}\mathcal{K}_n}$. By the Cantor's intersection theorem, there exists a point $u \in \mathcal{K}$ such that $\{u\} = \cap_{n=1}^{\infty} \mathcal{T}\mathcal{K}_n$ contains exactly one point u. Since $u \in \mathcal{K}$ for each $n = 1, 2, 3, \ldots$, there exists a point $x_n \in \mathcal{K}_n$ such that $d(u, \mathcal{T}x_n) < n^{-1}$, and so $\mathcal{T}x_n \to u$. Further, as $x_n \in \mathcal{K}_n$, we have $d(\mathcal{T}x_n, \mathcal{S}x_n) < n^{-1}$ and $\mathcal{S}x_n \to u$.

Since the pair of mappings $(\mathcal{T}, \mathcal{S})$ is compatible and $\mathcal{S}$ is continuous, therefore $\mathcal{S}\mathcal{T}x_n, \mathcal{S}\mathcal{S}x_n, \mathcal{T}\mathcal{S}x_n \to \mathcal{S}u$. On setting $x = u$ and $y = \mathcal{S}x_n$ in (3.1), one gets

$$\begin{aligned} d(\mathcal{T}u, \mathcal{T}\mathcal{S}x_n) \leq\ & a\max\{d(\mathcal{S}u, \mathcal{S}\mathcal{S}x_n), c[d(\mathcal{S}u, \mathcal{T}\mathcal{S}x_n) + d(\mathcal{S}\mathcal{S}x_n, \mathcal{T}u)]\} \\ & +b\max\{d(\mathcal{S}u, \mathcal{T}u), d(\mathcal{S}\mathcal{S}x_n, \mathcal{T}\mathcal{S}x_n), \\ & \frac{1}{2}[d(\mathcal{S}u, \mathcal{T}\mathcal{S}x_n) + d(\mathcal{S}\mathcal{S}x_n, \mathcal{T}u)]\} \end{aligned}$$

which on letting $n \to \infty$, gives rise

$$d(\mathcal{T}u, \mathcal{S}u) \leq (ac+b)d(\mathcal{S}u, \mathcal{T}u) \qquad (\text{as } (ac+b) < 1)$$

a contradiction. Thus $\mathcal{S}u = \mathcal{T}u$.

Now, we proceed to show that $\mathcal{S}u = u$. Suppose that $\mathcal{S}u \neq u$. On setting $x = x_n$ and $y = \mathcal{S}x_n$ in (3.1), we have

$$\begin{aligned} d(\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n) \leq\ & a\max\{d(\mathcal{S}x_n, \mathcal{S}\mathcal{S}x_n), c[d(\mathcal{S}x_n, \mathcal{T}\mathcal{S}x_n) + d(\mathcal{S}\mathcal{S}x_n, \mathcal{T}x_n)]\} \\ & +b\max\Big\{d(\mathcal{S}x_n, \mathcal{T}x_n), d(\mathcal{S}\mathcal{S}x_n, \mathcal{T}\mathcal{S}x_n), \\ & \frac{1}{2}[d(\mathcal{S}x_n, \mathcal{T}\mathcal{S}x_n) + d(\mathcal{S}\mathcal{S}x_n, \mathcal{T}x_n)]\Big\}, \end{aligned}$$

which on making $n \to \infty$, gives rise

$$d(u, \mathcal{S}u) \leq (a+b)d(u, \mathcal{S}u) < d(u, \mathcal{S}u) \qquad \Big(\text{since } c < \frac{1}{2}\Big)$$

a contradiction. Thus $\mathcal{S}u = u$. Finally, we show that $\mathcal{T}u = u$. Suppose that $\mathcal{T}u \neq u$. On setting $x = u$ and $y = x_n$ in (3.1), we have

$$\begin{aligned} d(\mathcal{T}u, \mathcal{T}x_n) \quad &\leq \quad a\max\{d(\mathcal{S}u, \mathcal{S}x_n), c[d(\mathcal{S}u, \mathcal{T}x_n) + d(\mathcal{S}x_n, \mathcal{T}u)]\} \\ &\qquad +b\max\Big\{d(\mathcal{S}u, \mathcal{T}u), d(\mathcal{S}x_n, \mathcal{T}x_n), \\ &\qquad \frac{1}{2}[d(\mathcal{S}u, \mathcal{T}x_n) + d(\mathcal{S}x_n, \mathcal{T}u)]\Big\}, \end{aligned}$$

which on making $n \to \infty$, gives rise

$$d(\mathcal{T}u, u) \quad \leq \quad (ac + b)d(u, \mathcal{T}u) \qquad \text{(since } (ac + b) < 1),$$

a contradiction. Hence $\mathcal{T}u = u$ so that $\mathcal{T}u = \mathcal{S}u = u$ which shows that u is a common fixed point of $\mathcal{T}$ and $\mathcal{S}$ in $\mathcal{K}$. In order to prove the uniqueness of common fixed point, let v be another common fixed point of $\mathcal{T}$ and $\mathcal{S}$ so that $d(u, v) > 0$. It follows from (3.1) that

$$d(u, v) \quad = \quad d(\mathcal{T}u, \mathcal{T}v) \leq (a + b)d(u, v) < d(u, v)$$

which is a contradiction. Hence, u is the unique common fixed point of $\mathcal{T}$ and $\mathcal{S}$ in $\mathcal{K}$.

Finally, we show that $\mathcal{T}$ is continuous at the unique common fixed point u. To accomplish this, let sequence $y_n \to u$, Then (due to continuity of $\mathcal{S}$) $\mathcal{S}y_n \to \mathcal{S}u$. On setting $x = u$ and $y = y_n$ in (3.1), we have

$$\begin{aligned} d(u, \mathcal{T}y_n) = d(\mathcal{T}u, \mathcal{T}y_n) \quad &\leq \quad a\max\{d(\mathcal{S}u, \mathcal{S}y_n), c[d(\mathcal{S}u, \mathcal{T}y_n) + d(\mathcal{S}y_n, \mathcal{T}u)]\} \\ &\qquad +b\max\Big\{d(\mathcal{S}u, \mathcal{T}u), d(\mathcal{S}y_n, \mathcal{T}y_n), \\ &\qquad \frac{1}{2}[d(\mathcal{S}u, \mathcal{T}y_n) + d(\mathcal{S}y_n, \mathcal{T}u)]\Big\}, \end{aligned}$$

which on making $n \to \infty$, gives rise

$$d(u, \lim_n \mathcal{T}y_n) \leq (ac + b)d(u, \lim_n \mathcal{T}y_n)$$

so that, $\mathcal{T}y_n \to u$ (as $(ac+b) < 1$) and hence $\mathcal{T}$ is a continuous at u.This completes the proof of the theorem. □

The following two examples demonstrate Theorem 3.1.

Example 3.2. Consider $\mathcal{X} = \mathbb{R}$ equipped with usual metric wherein $\mathcal{K} = [1, \infty)$ and $\mathcal{W}(x, y, \lambda) = \lambda x + (1 - \lambda)y$. Define self maps $\mathcal{T}$ and $\mathcal{S}$ on convex metric space $(\mathcal{X}, d)$ as $\mathcal{T}x = \frac{2+x}{3}$ and $\mathcal{S}x = \frac{3x-1}{2}$ for all $x \in \mathcal{X}$. Clearly, $\mathcal{S}$ is continuous and $\mathcal{W}$- affine, Also $\mathcal{T}$ and $\mathcal{S}$ form a compatible pair of mappings on $\mathcal{X}$. Clearly $\mathcal{T}(\mathcal{K}) \subseteq \mathcal{S}(\mathcal{K})$ and for any $x, y \in \mathcal{K}$, the mappings $\mathcal{T}$ and $\mathcal{S}$ satisfy the inequality 3.1 with $a = \frac{2}{3}$, $b = \frac{1}{6}$ and $c = 0$. Notice that $Fix(\mathcal{T}) \cap Fix(\mathcal{S}) = \{1\}$ and $\mathcal{T}$ is continuous at $x = 1$.

Example 3.3. Let $\mathcal{X} = \mathbb{R}$ and $\mathcal{K} = [1, \infty)$. wherein $\mathcal{W}(x, y, \lambda) = \lambda x + (1 - \lambda)y$. Define $\mathcal{T}$ and $\mathcal{S} : \mathbb{R} \to \mathbb{R}$ as follows:

$$\mathcal{T}x \quad = \quad \begin{cases} x & \text{if } \ x < 1 \\ 2x - 1 & \text{if } \ 1 \leq x \end{cases}$$

$$\mathcal{S}x \quad = \quad \begin{cases} \frac{x}{2} & \text{if } \ x < 1 \\ 3x - 2 & \text{if } \ 1 \leq x. \end{cases}$$

Clearly, $\mathcal{T}$ and $\mathcal{S}$ are compatible mappings of $\mathcal{K}$. Also. for any $x, y \in \mathcal{K}$, the mappings $\mathcal{T}$ and $\mathcal{S}$ satisfy the inequality (3.1) with $a = \frac{2}{3}$ and $b = \frac{1}{6}$ and $c = 0$ besides $\mathcal{T}(\mathcal{K}) \subseteq \mathcal{S}(\mathcal{K})$ and $\mathcal{S}$ is $\mathcal{W}$- affine and continuous on $\mathcal{K}$. Notice that $Fix(\mathcal{T}) \cap Fix(\mathcal{S}) = \{1\}$ and $\mathcal{T}$ is continuous at $x = 1$.

Our next example exhibits that condition (3.1) is necessary in Theorem 3.1.

Example 3.4. Let $\mathcal{X} = \mathbb{R}^2$ be endowed with metric defined by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

wherein $\mathcal{W}(x, y, \lambda) = \lambda x + (1 - \lambda)y$. Define self-maps $\mathcal{T}$ and $\mathcal{S}$ on the convex metric space $(\mathcal{X}, d)$ as follows (for arbitrary (x, y) in $\mathcal{X} = \mathbb{R}^2$):

$$\mathcal{T}(x, y) = (x + 1, y) \text{ and } \mathcal{S}(x, y) = (x, y).$$

If we take $\mathcal{K} = \{(x, y) : \quad 2 \le x < \infty, \quad 0 \le y \le 1\}$, then for all $(x, y) \in \mathcal{K}$, all the conditions of Theorem 3.1 are satisfied except condition (3.1). Notice that $Fix(\mathcal{T}) \cap Fix(\mathcal{S}) = \emptyset$.

Corollary 3.5. *Let $(\mathcal{X}, d)$ be a complete convex metric space with a convex structure $\mathcal{W}$ whereas $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{X}$. Let the pair $(\mathcal{T}, \mathcal{S})$ be compatible on $\mathcal{K}$ such that for all $x, y \in \mathcal{K}$*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) \quad &\le \quad a \max\{d(\mathcal{S}x, \mathcal{S}y), c[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}y, \mathcal{T}x)]\} \\ &\quad + b \max\left\{d(\mathcal{T}x, \mathcal{S}x), d(\mathcal{T}y, \mathcal{S}y), \frac{1}{2}[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}y, \mathcal{T}x)]\right\} \end{aligned}$$

where $0 < a < 1$, $b \ge 0$, $c \ge 0$, $a + c > 0$ and $a + 2b + 4c = 1$. If $\mathcal{T}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X})$ and $\mathcal{S}$ is continuous and $\mathcal{W}$-affine in $\mathcal{K}$, then $\mathcal{T}$ and $\mathcal{S}$ have a unique common fixed point in $\mathcal{X}$.

Proof. Set $a + 4c = a_1$. Then $a_1 + 2b = 1$ and henceforth we have

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) \quad &\le \quad ad(\mathcal{S}x, \mathcal{S}y) + c.\frac{4}{1}.\frac{1}{4}[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}y, \mathcal{T}x)] \\ &\quad + b \max\left\{d(\mathcal{T}x, \mathcal{S}x), d(\mathcal{T}y, \mathcal{S}y), \frac{1}{2}[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}y, \mathcal{T}x)]\right\} \\ &\le \quad (a + 4c) \max\left\{d(\mathcal{S}x, \mathcal{S}y), \frac{1}{4}[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}y, \mathcal{T}x)]\right\} \\ &\quad + b \max\left\{d(\mathcal{T}x, \mathcal{S}x), d(\mathcal{T}y, \mathcal{S}y), \frac{1}{2}[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}y, \mathcal{T}x)]\right\}. \end{aligned}$$

Since $\frac{1}{4} \le \min\{\frac{2+a_1}{5+a_1}, \frac{2-a_1}{4}, \frac{4}{9+a_1}\}$ and $a_1 + 2b = 1$, where $a_1 = a + 4c$, the conclusion of this corollary follows from Theorem 3.1. □

If we set $c = 0$ in (3.1), then we have the following corollary which is contained in Huang and Li [14, Corollary 3.2].

Corollary 3.6. *Let $(\mathcal{X}, d)$ be a complete convex metric space with a convex structure $\mathcal{W}$ whereas $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{X}$. Let the pair $(\mathcal{T}, \mathcal{S})$ be compatible on $\mathcal{K}$ which satisfy the inequality*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) \;&\leq\; ad(\mathcal{S}x, \mathcal{S}y) \\ &+\frac{(1-a)}{2}\max\left\{d(\mathcal{T}x, \mathcal{S}x), d(\mathcal{T}y, \mathcal{S}y), \frac{1}{2}[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}y, \mathcal{T}x)]\right\} \end{aligned} \tag{3.20}$$

for all $a \in (0,1)$, where $0 < a < 1$. Suppose that $\mathcal{S}$ is continuous, $\mathcal{W}$-affine and $\mathcal{T}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X})$. Then $\mathcal{T}$ and $\mathcal{S}$ have a unique common fixed point in $\mathcal{K}$.

4. A related fixed point theorem

In this section, we use the main result of previous section to prove a new common fixed point theorem for yet another class of noncommuting mappings which are often referred as subcompatible maps.

Theorem 4.1. *Let $\mathcal{K}$ be a nonempty closed convex subset of a convex metric space $(\mathcal{X}, d)$ satisfying the property (I). Let $\mathcal{T}$ and $\mathcal{S}$ be a pair of subcompatible self-maps defined on $\mathcal{K}$. Assume that $\mathcal{S}(\mathcal{K}) = \mathcal{K}$, $q \in Fix(\mathcal{S})$, $\mathcal{S}$ is $\mathcal{W}$-affine and continuous. If $\mathcal{T}$ and $\mathcal{S}$ satisfy*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) \;&\leq\; \max\{d(\mathcal{S}x, \mathcal{S}y), c[d(seq[\mathcal{T}y, q], \mathcal{S}x) + d(seq[\mathcal{T}x, q], \mathcal{S}y)]\} \\ &+\frac{(1-k)}{2k}\max\left\{d(seq[\mathcal{T}x, q], \mathcal{S}x), d(seq[\mathcal{T}y, q], \mathcal{S}y), \right. \\ &\left. \frac{1}{2}[d(seq[\mathcal{T}y, q], \mathcal{S}x) + d(seq[\mathcal{T}x, q], \mathcal{S}y)]\right\} \end{aligned} \tag{4.1}$$

for all $x, y \in \mathcal{K}$, $0 < k < 1$ and $0 \leq c < 0.25$, then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point in $\mathcal{K}$, provided one of the following conditions holds:

(i) *$cl\mathcal{T}(\mathcal{K})$ is compact and $\mathcal{T}$ is continuous;*
(ii) *$Fix(\mathcal{S})$ is bounded and $\mathcal{T}$ is a compact map.*

Proof. Choose a sequence $\{k_n\} \subset (0,1)$ with $k_n \to 1$ as $n \to \infty$. For each $n \in \mathbb{N}$, define $\mathcal{T}_n : \mathcal{K} \to \mathcal{K}$ as follows:

$$\mathcal{T}_n x = \mathcal{W}(\mathcal{T}x, q, k_n) \tag{4.2}$$

for some $q \in \mathcal{K}$. Obviously, for each n, $\mathcal{T}_n$ maps $\mathcal{K}$ into itself as $\mathcal{K}$ is convex. The subcompatibility of the pair $(\mathcal{S}, \mathcal{T})$ and $\mathcal{W}$-affinity of $\mathcal{S}$ in the presence of $q = \mathcal{S}q$ and the property (I) (in respect of any $\{x_m\} \subset \mathcal{K}$ with $\lim_m \mathcal{T}_n x_m = \lim_m \mathcal{S}x_m = t \in \mathcal{K}$) together imply that $\lim_m \mathcal{T}_n x_m = \lim_m \mathcal{W}(\mathcal{T}x_m, q, k_n) = \lim_m \mathcal{T}_{k_n} x_m = t \in \mathcal{K}$. Now, due to subcompatibility of $\mathcal{T}$ and $\mathcal{S}$.

$$\begin{aligned} 0 \;&\leq\; \lim_m d(\mathcal{T}_n \mathcal{S}x_m, \mathcal{S}\mathcal{T}_n x_m) \\ &= \lim_m d(\mathcal{W}(\mathcal{T}\mathcal{S}x_m, q, k_n), \mathcal{W}(\mathcal{S}\mathcal{T}x_m, q, k_n)) \\ &\leq k_n \lim_m d(\mathcal{T}\mathcal{S}x_m, \mathcal{S}\mathcal{T}x_m) \\ &= 0, \end{aligned}$$

which shows that $\{\mathcal{T}_n\}$ and $\mathcal{S}$ are compatible for each n and $x_m \in \mathcal{K}$ whereas $\mathcal{T}_n(\mathcal{K}) \subseteq \mathcal{K} = \mathcal{S}(\mathcal{K})$, $\mathcal{S}$ is $\mathcal{W}$-affine and $q \in Fix(\mathcal{S})$. Also, for all $x, y \in \mathcal{K}$, one can write (in view of (4.1), (4.2) and the property (I)) that

$$\begin{aligned} d(\mathcal{T}_n x, \mathcal{T}_n y) &= d(\mathcal{W}(\mathcal{T}x, q, k_n), \mathcal{W}(\mathcal{T}y, q, k_n)) \\ &\leq k_n d(\mathcal{T}x, \mathcal{T}y) \\ &\leq k_n \Big[\max\{d(\mathcal{S}x, \mathcal{S}y), c[d(seq[\mathcal{T}y, q], \mathcal{S}x) + d(seq[\mathcal{T}x, q], \mathcal{S}y)]\} \\ &\qquad + \frac{(1-k_n)}{2k_n} \max\Big\{ d(seq[\mathcal{T}x, q], \mathcal{S}x), d(seq[\mathcal{T}y, q], \mathcal{S}y), \\ &\qquad \frac{1}{2}[d(seq[\mathcal{T}y, q], \mathcal{S}x) + d(seq[\mathcal{T}x, q], \mathcal{S}y)] \Big\} \Big] \end{aligned}$$

i.e.,

$$\begin{aligned} d(\mathcal{T}_n x, \mathcal{T}_n y) \leq\ & k_n \max\{d(\mathcal{S}x, \mathcal{S}y), c[d(\mathcal{T}_n y, \mathcal{S}x), d(\mathcal{T}_n x, \mathcal{S}y)]\} \\ & + \frac{(1-k_n)}{2} \max\Big\{ d(\mathcal{T}_n x, \mathcal{S}x), d(\mathcal{T}_n y, \mathcal{S}y), \\ & \frac{1}{2}[d(\mathcal{T}_n y, \mathcal{S}x), d(\mathcal{T}_n x, \mathcal{S}y)] \Big\} \end{aligned}$$

for all $x, y \in \mathcal{K}$ and $0 < k_n < 1$.

(i) Since $\mathcal{K}$ is closed, therefore using Theorem 3.1 (for every $n \in \mathbb{N}$), $\mathcal{T}_n$ and $\mathcal{S}$ have common fixed point x_n in $\mathcal{K}$, i.e.,

$$x_n = \mathcal{S}x_n = \mathcal{T}_n x_n = \mathcal{W}(\mathcal{T}x_n, q, \lambda_n).$$

Since $cl\mathcal{T}(\mathcal{K})$ is compact, therefore $cl\mathcal{T}_n(\mathcal{K})$ is also compact. The compactness of $\mathcal{T}(\mathcal{K})$ implies that there exists a subsequence $\{\mathcal{T}y_m\}$ of $\{\mathcal{T}y_n\}$ such that $\mathcal{T}y_m \to y$ as $m \to \infty$. Then the definition of $\mathcal{T}_m y_m$ implies $y_m \to y$ so that by the continuity of $\mathcal{T}$ and $\mathcal{S}$ we have $y \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$. Thus $Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$.

(ii) As in (i), there is a unique $y_n \in \mathcal{K}$ such that $y_n = \mathcal{T}_n y_n = \mathcal{S}y_n$. As $\mathcal{T}$ is compact and $\{y_n\}$ being in $Fix(\mathcal{S})$ is bounded, therefore $\{\mathcal{T}y_n\}$ has a subsequence $\{\mathcal{T}y_m\}$ such that $\{\mathcal{T}y_m\} \to y$ as $m \to \infty$. Then the definition of $\mathcal{T}_m y_m$ implies $y_m \to y$, so by the continuity of $\mathcal{T}$ and $\mathcal{S}$ we have $y \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$. Thus $Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$.

□

Example 4.2. Consider the set of reals $\mathcal{X} = \mathbb{R}$ as real vector space equipped with natural norm wherein $\mathcal{W}(x, y, \lambda) = \lambda x + (1-\lambda)y$. Then, clearly $\mathcal{X}$ satisfies the property (I). Define self maps $\mathcal{T}$ and $\mathcal{S}$ on convex metric space $\mathcal{X}$ as follows:

$$\mathcal{T}(x) = \frac{x}{3} \text{ and } \mathcal{S}(x) = 2x.$$

If $\mathcal{K} = [0, \infty)$, then $\mathcal{S}(\mathcal{K}) = \mathcal{K}$. Also, $q = 0 \in Fix(\mathcal{S})$, $\mathcal{S}$ is continuous and $\mathcal{W}-$ affine besides the pair $(\mathcal{T}, \mathcal{S})$ is commuting and hence subcompatible. For verification of condition (4.1), note that $d(\mathcal{T}x, \mathcal{T}y) = \frac{1}{3}|x-y|$ and $d(\mathcal{S}x, \mathcal{S}y) = 2|x-y|$ (for arbitrary $x, y \in \mathcal{K}$) which can be utilized to exhibit that the mappings $\mathcal{T}$ and $\mathcal{S}$ satisfy the inequality (4.1) with $c = 0$ and $k = \frac{1}{2}$ (also for arbitrary k such that

$0 < k < 1$). Notice that the fixed point set of the mapping $\mathcal{S}$ is bounded and $\mathcal{T}$ is compact map (as $\mathcal{T}$ is linear and $\dim\mathcal{X} < \infty$). Thus all the conditions of Theorem 4.1 are satisfied and $x = 0$ is common fixed point of $\mathcal{T}$ and $\mathcal{S}$.

Remark 4.3. Theorem 4.1 improves the corresponding result of Nashine and Khan [21, Theorem 3.3] in the three respects wherein a relatively generalized contractive condition (4.1) is used besides weakening the compactness of subset $\mathcal{K}$ and replacing the linearity of the map $\mathcal{S}$ by the affinity of the map $\mathcal{S}$ in the setting of convex metric space.

Remark 4.4. Theorem 4.1 improves the corresponding result of Nashine and Imdad [20, Theorem 2] wherein a relatively generalized contractive condition (4.1) is used besides weakening the compactness of subset $\mathcal{K}$.

If we set $c = 0$ in the Theorem 4.1, then we deduce the following corollary which substantially improves the corresponding theorems of Al-Mezel and Hussain [2], Nashine and Khan [21] and Nashine and Imdad [20].

Corollary 4.5. *Let $\mathcal{K}$ be a nonempty closed convex subset of a convex metric space $(\mathcal{X}, d)$ satisfying the property (I). Let $\mathcal{T}$ and $\mathcal{S}$ be a pair of self-maps defined on $\mathcal{K}$ which is subcompatible. Assume that $\mathcal{S}(\mathcal{K}) = \mathcal{K}$, $q \in Fix(\mathcal{S})$, $\mathcal{S}$ is $\mathcal{W}$-affine and also continuous. If $\mathcal{T}$ and $\mathcal{S}$ satisfy*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) \quad \leq \quad & d(\mathcal{S}x, \mathcal{S}y) + \frac{(1-k)}{2k} \max\{d(seq[\mathcal{T}x, q], \mathcal{S}x), d(seq[\mathcal{T}y, q], \mathcal{S}y), \\ & \frac{1}{2}[d(seq[\mathcal{T}y, q], \mathcal{S}x), d(seq[\mathcal{T}x, q], \mathcal{S}y)]\}, \end{aligned}$$

for all $x, y \in \mathcal{K}$, $0 < k < 1$, then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point in $\mathcal{K}$ under the condition (i) (or (ii)) of Theorem 4.1.

5. Applications to invariant approximation

As an application of Theorem 4.1, we derive a more general result in invariant approximation theory for subcompatible pairs (a generalized class of noncommuting pairs) in the frame work of convex metric space.

Theorem 5.1. *Let $\mathcal{T}$ and $\mathcal{S}$ be self-maps of a convex metric space $(\mathcal{X}, d)$ and $\mathcal{K}$ be a subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{K}) \subseteq \mathcal{K}$, where $\partial\mathcal{K}$ stands for the boundary of $\mathcal{K}$ and $x_0 \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$. Suppose that $\mathcal{D} = \mathcal{P}_{\mathcal{K}}(x_0)$ is nonempty closed convex subset of $\mathcal{X}$ with $\mathcal{S}(\mathcal{D}) = \mathcal{D}$, $q \in Fix(\mathcal{S})$, $\mathcal{S}$ is continuous as well as $\mathcal{W}$-affine and the pair $(\mathcal{T}, \mathcal{S})$ is subcompatible on $\mathcal{D}$. If $\mathcal{T}$ and $\mathcal{S}$ satisfy (for all $x, y \in \mathcal{D}' = \mathcal{D} \cup \{x_0\}$)*

$$(5.1) \quad d(\mathcal{T}x, \mathcal{T}y) \leq \begin{cases} d(\mathcal{S}x, \mathcal{S}x_0), & \text{if } y = x_0; \\ \max\{d(\mathcal{S}x, \mathcal{S}y), c[d(seq[\mathcal{T}x, q], \mathcal{S}y) + d(seq[\mathcal{T}y, q], \mathcal{S}x)]\} \\ +\Big(\dfrac{1-k}{2k}\Big) \max\Big\{d(seq[\mathcal{T}x, q], \mathcal{S}x), d(seq[\mathcal{T}y, q], \mathcal{S}y), \\ \dfrac{1}{2}[d(seq[\mathcal{T}x, q], \mathcal{S}y) + d(seq[\mathcal{T}y, q], \mathcal{S}x)]\Big\} & \text{if } y \in \mathcal{D} \end{cases}$$

where $0 < k < 1$ and $0 \leq c < 0.25$, then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point in $\mathcal{D}$, provided one of the following conditions holds:

(i) *$cl\mathcal{T}(\mathcal{K})$ is compact and $\mathcal{T}$ is continuous;*
(ii) *$Fix(\mathcal{S})$ is bounded and $\mathcal{T}$ is a compact map.*

Proof. Firstly, we show that $\mathcal{T}$ is a self-map on $\mathcal{D}$ i.e. $\mathcal{T} : \mathcal{D} \to \mathcal{D}$. To do this, let $y \in \mathcal{D}$, then $\mathcal{S}y \in \mathcal{D}$ as $\mathcal{S}(\mathcal{D}) = \mathcal{D}$. In case $y \in \partial\mathcal{K}$, then $\mathcal{T}y \in \mathcal{K}$ as $\mathcal{T}(\partial\mathcal{K}) \subseteq \mathcal{K}$. Owing to the fact that $\mathcal{T}x_0 = x_0 = \mathcal{S}x_0$, one may have (from (5.1))

$$d(\mathcal{T}y, x_0) = d(\mathcal{T}y, \mathcal{T}x_0) \leq d(\mathcal{S}y, \mathcal{S}x_0) = d(\mathcal{S}y, x_0) = d(x_0, \mathcal{K}),$$

which shows that $\mathcal{T}y \in \mathcal{D}$, so that $\mathcal{T}$ and $\mathcal{S}$ are self-maps on $\mathcal{D}$. Thus all the conditions of Theorem 4.1 are satisfied and hence there exists a $z \in \mathcal{D}$ such that $\mathcal{T}z = z = \mathcal{S}z$. □

We furnish the following example to demonstrate the validity of the hypotheses of Theorem 5.1.

Example 5.2. Consider the real vector space $\mathcal{X} = \mathbb{R}^2$ equipped with metric

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

for all $(x_1, x_2), (y_1, y_2) \in \mathcal{X}$ wherein $\mathcal{W}(x, y, \lambda) = \lambda x + (1 - \lambda)y$. We define $\mathcal{T}$ and $\mathcal{S}$ on the convex metric space $(\mathcal{X}, d)$ as following:

$$\mathcal{S}(x, y) = (x, y) \quad and \quad \mathcal{T}(x, y) = \begin{cases} (x, y) & \text{if } \; y \leq x \\ (x, x) & \text{if } \; x \leq y \end{cases}$$

Take $\mathcal{K} = \{(x, x) : \quad -1 \leq x \leq 1\}$ and $x_0 = (1, -1)$. Then $(-1, -1) \in Fix(\mathcal{S}) \cap Fix(\mathcal{T})$ and $\mathcal{D} = \mathcal{P}_{\mathcal{K}}(x_0)$ is the line segment joining the point $(-1, -1)$ and $(1, 1)$ which is indeed nonempty and convex. Also $\mathcal{T}$ and $\mathcal{S}$ are continuous and $\mathcal{S}$ is $\mathcal{W}-$affine, $\mathcal{S}(\mathcal{D}) = \mathcal{D}$ and $(0, 0) \in Fix(\mathcal{S})$. Also the pair $(\mathcal{T}, \mathcal{S})$ is commuting and hence subcompatible beside $\mathcal{T}(\partial\mathcal{K}) = \mathcal{K} \subseteq \mathcal{K}$. For the verification of condition (5.1), we distinguish the following two cases:

Case(I): if $y = x_0 = (1, -1)$ then

$$\begin{aligned} d(\mathcal{T}(x, x), \mathcal{T}(1, -1)) &= d((x, x), (1, -1)) = |x - 1| + |x + 1| \\ &= d(\mathcal{S}(x, x), \mathcal{S}(1, -1)) \end{aligned}$$

Case(II):

$$d(\mathcal{T}(x, x), \mathcal{T}(y, y)) = d((x, x), (y, y)) = |x - y| + |x - y| = d(\mathcal{S}(x, x), \mathcal{S}(y, y))$$

Thus all the conditions of Theorem 5.1 are satisfied. Notice that the segment joining $(-1, -1)$ and $(1, 1)$ remains fixed point under $\mathcal{T}$ and $\mathcal{S}$ both which in all substantiates Theorem 5.1. Notice that $cl\mathcal{T}(\mathcal{K})$ is compact and $\mathcal{T}$ is continuous.

Before stating our next theorem, we need to introduce the following:

$$\mathcal{D}^* = \mathcal{P}_{\mathcal{K}}(x_0) \cap \mathcal{D}^{\mathcal{S}}_{\mathcal{K}}(x_0), \;\; where \;\; \mathcal{D}^{\mathcal{S}}_{\mathcal{K}}(x_0) = \{x \in \mathcal{K} : \mathcal{S}x \in \mathcal{P}_{\mathcal{K}}(x_0)\}$$

Theorem 5.3. *Let $\mathcal{T}$ and $\mathcal{S}$ be self-maps of a convex metric space $(\mathcal{X}, d)$ and $\mathcal{K}$ be a subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{K} \cap \mathcal{K}) \subseteq \mathcal{K}$, where $\partial\mathcal{K}$ stands for the boundary of $\mathcal{K}$ and $x_0 \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$. Suppose that $\mathcal{D}^*$ is nonempty closed convex subset of $\mathcal{X}$ such* *that $\mathcal{S}(\mathcal{D}^*) = \mathcal{D}^*$, $q \in Fix(\mathcal{S})$, $\mathcal{S}$ is nonexpansive and $\mathcal{W}$-affine on $\mathcal{P}_{\mathcal{K}} \cup \{x_0\}$ besides the pair $(\mathcal{T}, \mathcal{S})$ is subcompatible on $\mathcal{D}^*$. If $\mathcal{T}$ and $\mathcal{S}$ satisfy (for all $x, y \in \mathcal{D}^* \cup \{x_0\}$)*

$$(5.2)\quad d(\mathcal{T}x, \mathcal{T}y) \le \begin{cases} d(\mathcal{S}x, \mathcal{S}x_0), & \text{if } y = x_0; \\ \max\{d(\mathcal{S}x, \mathcal{S}y), c[d(seq[\mathcal{T}x, q], \mathcal{S}y) + d(seq[\mathcal{T}y, q], \mathcal{S}x)]\} & \\ +\Big(\dfrac{1-k}{2k}\Big) \max\Big\{d(seq[\mathcal{T}x, q], \mathcal{S}x), d(seq[\mathcal{T}y, q], \mathcal{S}y), & \\ \dfrac{1}{2}[d(seq[\mathcal{T}x, q], \mathcal{S}y) + d(seq[\mathcal{T}y, q], \mathcal{S}x)]\Big\} & \text{if } y \in \mathcal{D}^*. \end{cases}$$

where $0 < k < 1$ and $0 \le c < 0.25$, then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point in $\mathcal{P}_{\mathcal{K}}(x_0)$, under condition (i)(or(ii)) of Theorem 5.1.

Proof. Let $x \in \mathcal{D}^*$. Then, $x \in \mathcal{P}_{\mathcal{K}}(x_0)$ and hence $d(x, x_0) = d(x_0, \mathcal{K})$. Notice that for any $t \in (0, 1)$,

$$d(\mathcal{W}(x, x_0, t), x_0) = d(\mathcal{W}(x, x_0, t), \mathcal{W}(x_0, x_0, t)) \le td(x, x_0) < d(x_0, \mathcal{K}).$$

Now, it follows that the segment $\{\mathcal{W}(x, x_0, t) : 0 < t < 1\}$ and the set $\mathcal{K}$ are disjoint. Thus x is not in the interior of $\mathcal{K}$ and so $x \in \partial\mathcal{K} \cap \mathcal{K}$. Since $\mathcal{T}(\partial\mathcal{K} \cap \mathcal{K}) \subset \mathcal{K}$, $\mathcal{T}x$ must be in $\mathcal{K}$. Now, proceeding on the lines of the proof of Theorem 5.1, we have $\mathcal{T}x \in \mathcal{P}_{\mathcal{K}}(x_0)$. As $\mathcal{S}$ is nonexpansive on $\mathcal{P}_{\mathcal{K}}(x_0) \cup \{x_0\}$, we have

$$d(\mathcal{S}\mathcal{T}x, x_0) \le d(\mathcal{T}x, \mathcal{T}x_0) \le d(\mathcal{S}x, \mathcal{S}x_0) = d(\mathcal{S}x, x_0) = d(x_0, \mathcal{K}).$$

Thus $\mathcal{S}\mathcal{T}x \in \mathcal{P}_{\mathcal{K}}(x_0)$ and so $\mathcal{T}x \in \mathcal{D}^{\mathcal{S}}_{\mathcal{K}}(x_0)$. Hence $\mathcal{T}x \in \mathcal{D}^*$. Consequently, $\mathcal{T}(\mathcal{D}^*) \subset \mathcal{D}^* = \mathcal{S}(\mathcal{D}^*)$. Now, in view of Theorem 5.1, $\mathcal{P}_{\mathcal{K}}(x_0) \cap Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \ne \emptyset$. □

In what follows, we observe that Example 5.2 can be utilized to demonstrate Theorem 5.3.

Example 5.4. One can easily notice that the hypotheses of Theorem 5.3 can be demonstrated by Example 5.2 because $(\partial\mathcal{K} \cap \mathcal{K}) = \mathcal{K}$, $\mathcal{T}(\mathcal{K}) = \mathcal{K}$ so that $\mathcal{T}(\partial\mathcal{K} \cap \mathcal{K}) = \mathcal{K} \subseteq \mathcal{K}$. Also $\mathcal{D}^{\mathcal{S}}_{\mathcal{K}}(x_0) = \mathcal{K} = \mathcal{P}_{\mathcal{K}}(x_0)$ and $\mathcal{D}^* = \mathcal{P}_{\mathcal{K}}(x_0) \cap \mathcal{D}^{\mathcal{S}}_{\mathcal{K}}(x_0) = \mathcal{D}$. The detailed verifications are already available in Example 5.2.

Remark 5.5. It is straight forward to notice that Theorem 5.3 is trivial if $x_0 \in \mathcal{K}$. Otherwise the disjointness of $\mathcal{K}$ with the segment $\mathcal{W}(x, x_0, t)$ is no longer necessarily true if $x_0 \in \mathcal{K}$.

Corollary 5.6. *Let $\mathcal{T}$ and $\mathcal{S}$ be self-maps of a convex metric space $(\mathcal{X}, d)$ and $\mathcal{K}$ be a subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{K} \cap \mathcal{K}) \subseteq \mathcal{K}$, where $\partial\mathcal{K}$ stands for the boundary of $\mathcal{K}$ and $x_0 \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$. Suppose that $\mathcal{D}^*$ is nonempty closed convex subset of X such that $\mathcal{S}(\mathcal{D}^*) = \mathcal{D}^*$, $q \in Fix(\mathcal{S})$, $\mathcal{S}$ is continuous and $\mathcal{W}$-affine, and the pair $(\mathcal{T}, \mathcal{S})$ is commuting on $\mathcal{D}^*$. If $\mathcal{T}$ and $\mathcal{S}$ satisfy (5.2) for all $x, y \in \mathcal{D}^* \cup \{x_0\}$, then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point in $\mathcal{P}_{\mathcal{K}}(x_0)$ under condition (i)(or(ii)) of Theorem 5.1.*

Proof. Let $x \in \mathcal{D}^*$. Then proceeding on the lines of the proof of Theorem 5.3, we obtain $\mathcal{T}x \in \mathcal{P}_{\mathcal{K}}(x_0)$. Moreover, since $\mathcal{T}$ commutes with $\mathcal{S}$ on $\mathcal{D}^*$, $\mathcal{T}$ and $\mathcal{S}$ satisfy (5.2), henceforth

$$d(\mathcal{S}\mathcal{T}x, x_0) = d(\mathcal{T}\mathcal{S}x, \mathcal{T}x_0) \le d(\mathcal{S}^2x, \mathcal{S}x_0) = d(\mathcal{S}x, x_0) = d(x_0, \mathcal{K}).$$

Thus $\mathcal{S}\mathcal{T}x \in \mathcal{P}_{\mathcal{K}}(x_0)$ and so $\mathcal{T}x \in \mathcal{D}^{\mathcal{S}}_{\mathcal{K}}(x_0)$. Thus $\mathcal{T}x \in \mathcal{D}^*$. Consequently, $\mathcal{T}(\mathcal{D}^*) \subset \mathcal{D}^* = \mathcal{S}(\mathcal{D}^*)$. Now, in view of Theorem 5.1, $\mathcal{P}_{\mathcal{K}}(x_0) \cap Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$. □

Corollary 5.7. *Let $\mathcal{T}$ and $\mathcal{S}$ be self-maps of a convex metric space $(\mathcal{X}, d)$and $\mathcal{K}$ be a subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{K} \cap \mathcal{K}) \subseteq \mathcal{K}$, where $\partial\mathcal{K}$ stands for the boundary of $\mathcal{K}$ and $x_0 \in Fix(\mathcal{T}) \cap Fix(\mathcal{S})$. Suppose that $\mathcal{D}^*$ is a nonempty closed q-starshaped subset of X such that $\mathcal{S}(\mathcal{D}^{\mathcal{S}}_{\mathcal{K}}(x_0)) \cap \mathcal{D}^* \subset \mathcal{S}(\mathcal{D}^*) \subset \mathcal{D}^*$. Further, $q \in Fix(\mathcal{S})$, $\mathcal{S}$ is continuous and $\mathcal{W}$-affine, and the pair $(\mathcal{T}, \mathcal{S})$ is commuting on $\mathcal{D}^*$. If $\mathcal{T}$ and $\mathcal{S}$ satisfy (5.2) for all $x, y \in \mathcal{D}^* \cup \{x_0\}$, then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point in $\mathcal{P}_{\mathcal{K}}(x_0)$ under condition $(i)(or(ii))$ of Theorem 5.1.*

Proof. Let $x \in \mathcal{D}^*$. Proceeding on the lines of the proof of Theorem 5.3, we obtain $\mathcal{T}x \in \mathcal{D}^*$ i.e. $\mathcal{T}(\mathcal{D}^*) \subset \mathcal{D}^*$. Also as in Theorem 6, $x \in \mathcal{D}^*$ implies that $x \in \partial\mathcal{K} \cap \mathcal{K}$ and so $\mathcal{T}(\mathcal{D}^*) \subset \mathcal{T}(\partial\mathcal{K} \cap \mathcal{K}) \subset \mathcal{S}(\mathcal{K})$. Therefore, we can choose $y \in \mathcal{K}$ such that $\mathcal{T}x = \mathcal{S}y$. As $\mathcal{S}y = \mathcal{T}x \in \mathcal{P}_{\mathcal{K}}(x_0)$, it follows that $y \in \mathcal{D}^{\mathcal{S}}_{\mathcal{K}}(x_0)$. Consequently, $\mathcal{T}(\mathcal{D}^*) \subset \mathcal{S}(\mathcal{D}^{\mathcal{S}}_{\mathcal{K}}(x_0)) \subset \mathcal{P}_{\mathcal{K}}(x_0)$. Therefore, $\mathcal{T}(\mathcal{D}^*) \subset \mathcal{S}(\mathcal{D}^{\mathcal{S}}_{\mathcal{K}}) \cap \mathcal{D}^* \subset \mathcal{S}(\mathcal{D}^*) \subset \mathcal{D}^*$. Now, in view of Theorem 5.1, $\mathcal{P}_{\mathcal{K}}(x_0) \cap Fix(\mathcal{T}) \cap Fix(\mathcal{S}) \neq \emptyset$. □

Remark 5.8. It is straight forward to observe that $\mathcal{S}(\mathcal{P}_{\mathcal{K}}(x_0)) \subset \mathcal{P}_{\mathcal{K}}(x_0)$ implies $\mathcal{P}_{\mathcal{K}}(x_0) \subset \mathcal{D}^{\mathcal{S}}_{\mathcal{K}}(x_0)$ and henceforth $\mathcal{D}^* = \mathcal{P}_{\mathcal{K}}(x_0)$. Consequently, Theorem 5.3, and Corollaries 5.6 and 5.7 remain valid when $\mathcal{D}^* = \mathcal{P}_{\mathcal{K}}(x_0)$.

Remark 5.9. Similar remarks can be outlined in respect of best approximation results of Nashine and Khan [21] and Nashine and Imdad [20] as mentioned in the Remarks 4.3 and 4.4.

Remark 5.10. Theorem 5.1 as well as Corollary 5.7 improves Theorem 6 of Beg et al. [4] owing to the fact that we have employed relatively generalized nonexpansive subcompatible pair of mappings as opposed to relatively contractive commuting pair.

Remark 5.11. Theorem 5.1 together with Corollary 5.7 improves Theorem 3.2 of Al-Thagafi [3], Theorem 3 of Sahab et al. [22] and corresponding relevant results contained in Singh [24, 25] as we have utilized relatively generalized nonexpansive subcompatible pair of mappings in the setting of convex metric space.

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