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A CONSTRUCTIVE PROOF OF THE SKLAR'S THEOREM ON COPULAS

W. ANAKKAMATEE, S. DHOMPONGSA*, AND S. TASENA

Abstract. Using the fact that the family of shuffles of min is dense in the family of all copulas under supremum norm and using a well-known result of W. Takahashi [10], we are able to give a new and constructive proof of the Sklar's Theorem on copulas.

1. Introduction

Let F be a distribution function, i.e., $F : \overline{\mathbb{R}} \to [0,1]$ is nondecreasing, $F(-\infty) = 0$, $F(\infty) = 1$, and F is right continuous. For each $u \in [0,1]$, put

$$F^{-1}(u) := \inf\{x : F(x) \geq u\}.$$

Thus $FF^{-1}(u) = u$ for all $u \in R_F$ (the range of F). For a joint distribution H with marginals $F_1, \ldots, F_d$, Sklar's Theorem simply states that the function $C_H(u_1, \ldots, u_d) := H(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))$ for $u_i \in R_i(= R_{F_i})$ $(i = 1, \ldots, d)$ is a subcopula.

Definition 1.1. Let $U_1, U_2, \ldots, U_d$ be nonempty subsets of $\mathbb{R}$. For a function $H : U_1 \times U_2 \times \cdots U_d \to \mathbb{R}$, let $S = [a,b] := [a_1, b_1] \times \cdots \times [a_d, b_d]$ be a d-box all of whose vertices are in the domain of H. Then the *H-volume* of S is defined by

$$V_H(S) := \sum_v sign(v) H(v),$$

where the sum is taken over the 2^d vertices v of the box $[a,b]$; here

$$sign(v) = \begin{cases} 1, & \text{if } v_j = a_j \text{ for an even number of indices,} \\ -1, & \text{if } v_j = a_j \text{ for an odd number of indices.} \end{cases}$$

The function H is *d-increasing* if $V_H(S) \geq 0$ for any such d-box S.

Definition 1.2. A d-dimensional *subcopula* is a function C with the following properties:

(1) $C : U_1 \times U_2 \times \cdots \times U_d \to [0,1]$, where U_i, $i = 1, \ldots, d$ are subsets of $[0,1]$ which contain $\{0,1\}$,
(2) $C(x_1, \ldots, x_d) = 0$ if $x_i = 0$ for some i,
(3) C is d-increasing,

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*Corresponding author.
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(4) C has uniform margins, i.e., for each i, $C(1,\ldots,1,x_i,1,\ldots,1) = x_i$ for all $x_i \in U_i$.

A *copula* is a subcopula with domain I^d $(= [0,1]^d)$.

Theorem 1.3 (Sklar's Theorem [8])**.** *Let H be a joint distribution with univariate marginals $F_1, F_2, \ldots, F_d$. Then there exists a d-copula C such that, for every point $x \in \mathbb{R}^d$,*

$$H(x) = C(F_1(x_1), F_2(x_2), \ldots, F_d(x_d)).$$

If the marginals $F_1, F_2, \ldots, F_d$ are continuous, then the copula C is uniquely defined.

Observe that $H(x_1,\ldots,x_d) = C(F_1(x_1), F_2(x_2), \ldots, F_d(x_d))$ for $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$ if and only if $C(u_1,\ldots,u_d) = H(F_1^{-1}(u_1),\ldots,F_d^{-1}(u_d))$ for all $u_i \in R_{F_i}$, $i = 1,\ldots,d$. Thus C is a subcopula defined on $R_{F_1} \times \cdots \times R_{F_d}$.

Sklar announced his theorem in 1959 [8] and gave its proof in 1974 [7]. Another proofs can be found respectively in [1–3, 5, 6].

It is the purpose of this paper to present another proof of the Sklar's Theorem which is constructive. The method of the proof is based on the standard argument in proving the denseness of the space of shuffles of min in the class of all copulas. The result immediately implies a well-known fact that every subcopula is extendable to a copula. Some examples are given where the algorithms involved can be easily derived from the proof of the main theorem.

2. Preliminaries

We recall that a copula C induces a probability measure μ_C defined on the Borel σ-algebra $(\mathfrak{B}(I^d))$ of I^d so that

$$\mu_C(S) := V_C(S),$$

for every nonempty d-box $S \subset I^d$ $(I = [0,1])$. The measure μ_C satisfies the property:

$$\mu_C(A_1 \times \cdots \times A_d) = \lambda(A_i) \tag{2.1}$$

whenever, $A_j = I$ for all $j \neq i$, where λ is the Lebesgue measure on I. On the other hand, given any measure μ on $\mathfrak{B}(I^d)$ with property (2.1), it induces a copula C_μ via the formula

$$C_\mu(u) := \mu([0,u])$$

for all $u = (u_1,\ldots,u_d) \in I^d$, where $[0,u] = [0,u_1] \times \cdots \times [0,u_d]$.

For example, if μ is the measure that corresponds to the Min copula $M(u,v) = u \wedge v$, then the measure μ is uniformly distributed along the main diagonal D of I^2, i.e., $\mu(A) = \lambda(\pi_I(A \cap D))$, where π_I is the projection from I^2 onto I. Here the main diagonal means the one joining the points $(0,0)$ and $(1,1)$. Start with this measure μ on I^2, cut the I^2 vertically into k strips. Rearrange (or shuffle) the strips to obtain a new copy of I^2 according to a permutation σ on $\{1,\ldots,k\}$. The resulting mass distribution will correspond to a copula Min_σ called a shuffle of min.

Consider the cube $I^d \subset \mathbb{R}^d$. For each $i = 1, \ldots, d$, let $0 = u_{i0} < u_{i1} < \cdots < u_{ik_i} = 1$ be a partition of I. Set $U = \prod_{i=1}^d \{u_{i1}, \ldots, u_{ik_i}\}$ and order points in U lexicographically. Put $\overline{m}_{ij} = u_{ij} - u_{i(j-1)}$ for $j = 1, \ldots, k_i$. Now suppose a function $\overline{H} : U \to [0, 1]$ be such that

i) $\overline{H}(u_{11'}, \ldots, u_{dd'}) = u_{ii'}$ if $u_{jj'} = 1$ for all $j \neq i$, and
ii) $\overline{H}$ is d-increasing.

Let $\overline{h}(u_{11'}, \ldots, u_{dd'}) = V_{\overline{H}}(S(u))$ if $u_{ii'} > 0$ for all i, where $S(u)$ is the box $\prod_{i=1}^d [u_{i(i'-1)}, u_{ii'}]$. Thus $\overline{h}$ is a probability mass function of the distribution function $\overline{H}$ whose i^{th} marginal having $\{\overline{m}_{i1}, \ldots, \overline{m}_{ik_i}\}$ as the set of its probability mass values. That is,

$$\sum_{u \in U_{ij}} \overline{h}(u) = \overline{m}_{ij}$$

for all $i = 1, \ldots, d$ and $j = 1, \ldots, k_i$ where $U_{ij} := \{(u_1, \ldots, u_d) \in U : u_i = u_{ij}\}$.

The cube I^d now comprises of $k_1 \cdots k_d$ nonempty boxes $S(u)$, $u \in U$. Put $\pi_i = \prod_{j \neq i} k_j$ and let $u = (u_{11'}, u_{22'}, \ldots, u_{dd'}) \in U$. We divide the box $S(u)$ into $\prod_{i=1}^d \pi_i$ small boxes as follow. We start the process by dividing the interval $[u_{i(i'-1)}, u_{ii'}]$ for each i into π_i subintervals whose lengths are all numbers $\overline{h}(u)$ for $u \in U_{ii'}$. Thus the box $S(u)$ is partitioned into $\prod_{i=1}^d \pi_i$ subboxes one of which is a cube $Q(u)$ having $\overline{h}(u)$ as the length of its edges. It is straightforward to see that the projection of all the cubes $Q(u), u \in U$ onto each I cover the whole I. Moreover, these cubes can be used to formulate a copula $C_{\overline{H}}$ with the property that $C_{\overline{H}}(u) = \overline{H}(u)$ for each $u \in U$. Indeed for every $u \in U$, let μ_0 be a measure on $Q(u)$ so that it is uniformly distributed along any chosen (but fixed) diagonal D of $Q(u)$, i.e., $\mu_0(A) = \lambda(\pi_I(A \cap D))$. Then we define a measure μ on $\mathfrak{B}(I^d)$ by the relation

$$\mu(A \cap S(u)) = \mu_0(A \cap Q(u)).$$

Clearly, μ is a measure on $\mathfrak{B}(I^d)$ and satisfies the property (2.1).

In summary, it is shown that every discrete distribution function defined on a lattice U above can be extended to a copula $C_{\overline{H}}$.

Given a joint distribution H of some random variables having $F_1, \ldots, F_d$ as its marginals. It is easy to see that

$$(2.2) \qquad |H(x_1, \ldots, x_d) - H(y_1, \ldots, y_d)| \leq \sum_{i=1}^d \left(F_i(x_i \vee y_i) - F_i(x_i \wedge y_i)\right).$$

Thus every copula $C \in \mathfrak{C}_d$, the set of all d-copulas, is nonexpansive as a mapping from I^d into $I \subset I^d$ under L^1-norm since C can be considered as a distribution with uniform marginals. That is,

$$|C(u_1, \ldots, u_d) - C(v_1, \ldots, v_d)| \leq \sum_{i=1}^d (u_i \vee v_i - u_i \wedge v_i) = \sum_{i=1}^d |u_i - v_i|.$$

In the course of the proof of our main Theorem, we will apply the following result of Takahashi [10]. Let (X, d) be a metric space. Following [10], a mapping $W : X \times X \times [0,1] \to X$ is a *convex structure* on X if

$$d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1-\lambda) d(u, y)$$

for all $u, x, y \in X$ and $\lambda \in [0,1]$. Let K be a compact convex subset of a Banach space and X be the set of all nonexpansive mappings of K into itself. Then for each pair of elements A and B of X, define a metric d by $d(A,B) = \sup\{\|Ax - Bx\| : x \in K\}$ whence X is a metric space with d. Define a mapping $W : X \times X \times [0,1] \to X$ by

$$W(A, B; \lambda)(x) = \lambda Ax + (1-\lambda)Bx$$

for $x \in K$ and $\lambda \in [0,1]$.

Theorem 2.1 ([10])**.** *The set X is a compact convex metric space with respect to metric d and the convex structure W.*

Denote by $C(I^d)$ the family of all real valued continuous functions on I^d. Now consider the metric space $(C(I^d), d_\infty)$, where d_∞ is the distance given, for all f_1 and f_2 in $C(I^d)$, by

$$d_\infty(f_1, f_2) = \sup_{u \in I^d} |f_1(u) - f_2(u)|.$$

Thus the following result is immediately a consequence of Theorem 2.1 since $\mathfrak{C}_d$ is closed in $(C(I^d), d_\infty)$.

Theorem 2.2 ([3, Theorem 3.3])**.** *The set $\mathfrak{C}_d$ is a compact subset in $(C(I^d), d_\infty)$.*

3. Main results

Theorem 3.1 (Sklar's Theorem)**.** *Let H be a joint distribution with univariate marginals $F_1, F_2, \ldots, F_d$. Then there exists a d-copula C such that, for every point $x \in \mathbb{R}^d$,*

$$H(x) = C(F_1(x_1), F_2(x_2), \ldots, F_d(x_d)).$$

If the marginals $F_1, F_2, \ldots, F_d$ are continuous, then the copula C is uniquely defined.

Proof. For each n, divide I into 2^n subintervals each of whose length is $\frac{1}{2^n}$. Form a set D_{in} consisting of points one from each subinterval which also lies in R_i. We require the sequence $\{D_{in}\}$ to be increasing and write $D_{in} = \{u_{i0}, u_{i1}, \ldots, u_{ik_i(n)}\}$ in an increasing order. Observe that $D_{in} = R_i$ for all large n whenever F_i is discrete. By adding 0 and 1 into D_{in}, we may assume that $0 = u_{i0}$ and $u_{ik_i(n)} = 1$.

For each n, let $U_n = \prod_{i=1}^n D_{in}$ and define $\overline{H}_n$ on U_n by

$$\overline{H}_n(u_{11'}, \ldots, u_{dd'}) = H(F_1^{-1}(u_{11'}), \ldots, F_d^{-1}(u_{dd'})). \tag{3.1}$$

Then $\overline{H}_n$ satisfies conditions i) and ii) for $\overline{H}$. Obtain the corresponding $\overline{h}_n$ as well as a copula $C_{\overline{H}_n}$ satisfying

$$C_{\overline{H}_n} = \overline{H}_n \tag{3.2}$$

on U_n. Observe that

$$C_{\overline{H}_{n+k}}(u) = C_{\overline{H}_n}(u) \tag{3.3}$$

for all $u \in U_n$ and $k \geq 1$. As $\mathfrak{C}_d$ is compact, we assume that the sequence $\{C_{\overline{H}_n}\}$ converges to C for some $C \in \mathfrak{C}_d$ under the supremum norm. It follows from (3.2) and (3.3) that

$$C(u) = C_{\overline{H}_n}(u) = \overline{H}_n(u) \tag{3.4}$$

for all $n \in \mathbb{N}$ and $u \in U_n$.

Finally, if a $d-$tuple $(x_1, \ldots, x_d) \in \mathbb{R}^d$, then we choose a sequence $\{u_n\}$ with $u_n \in U_n$ for each n and $u_n \searrow (F_1(x_1), \ldots, F_d(x_d))$. Write $u_n = (u_{1n}, \ldots, u_{dn})$. Since the sequence $\{F_i^{-1}(u_{in})\}$ is decreasing, it converges to a point x_i^*. Thus $F_i(x_i^*) = F_i(x_i)$ for each i. Now, from (2.2), (3.1) and (3.4),

$$\begin{aligned} C(F_1(x_1), \ldots, F_d(x_d)) &= \lim_n C(u_{1n}, \ldots, u_{dn}) \\ &= \lim_n \overline{H}_n(u_{1n}, \ldots, u_{dn}) \\ &= \lim_n H(F_1^{-1}(u_{1n}), \ldots, F_d^{-1}(u_{dn})) \\ &= H(x_1^*, \ldots, x_d^*) = H(x_1, \ldots, x_d). \end{aligned}$$

Clearly, if $F_1, .., F_d$ are continuous, the sequence $\{C_{\overline{H}_n}\}$ converges to the copula C and consequently C is uniquely determined . □

The following result is an immediate consequence of Theorem 3.1 since each subcopula is a $d-$distribution.

Corollary 3.2 (The extension theorem for d-subcopulas [9]). *Every d-subcopula can be extended to a d-copula, i.e., given any d-subcopula $\overline{C}$ there is a d-copula C such that*

$$C(u_1, \ldots, u_d) = \overline{C}(u_1, \ldots, u_d)$$

for all $(u_1, \ldots, u_d)$ in the domain of $\overline{C}$.

Remark 3.3. The copula $C_{\overline{H}_n}$ obtained in the proof of Theorem 3.1 is a shuffle of min for $d = 2$. We present here a corresponding permutation of the shuffle. We start with a distribution $\overline{H}(u, v)$ defined for $u = u_1, u_2, \ldots, u_{k_1}$, $v = v_1, v_2, \ldots, v_{k_2}$, where $0 = u_0 < u_1 < u_2 < \cdots < u_{k_1} = 1$, $0 = v_0 < v_1 < v_2 < \cdots < v_{k_1} = 1$ and $\overline{H}(u_i, 1) = u_i$, $\overline{H}(1, v_j) = v_j$. As usual, define $\overline{h}(u_i, v_j)$ to be the volume $V_{\overline{H}}(S(u_i, v_j))$.

Order the k_1k_2 numbers $\overline{h}(u_i, v_i)$ for $i = 1, \ldots, k_1$ and $j = 1, \ldots, k_2$ following the order of (i, j) which is ordered lexicographically. Let $0 = t_0 < t_1 < \cdots < t_{k_1k_2} = 1$ be a partition of $[0, 1]$ so that $t_k - t_{k-1}$ is the k^{th} number in the list of $\{\overline{h}(u_i, v_j)\}$.

For each $i = 1, \ldots, k_1$, divide $[u_{i-1}, u_i]$ at points u_{ik} $(k = 1, \ldots, k_2 - 1)$ so that $u_{ik} < u_{i(k+1)}$ and $u_{i(k+1)} - u_{ik} = \overline{h}(u_i, v_k)$. Do the same for $[v_{j-1}, v_j]$ and obtain points of division called v_{js}, $s = 1, \ldots, k_1 - 1$. Obtain a copula $C_{\overline{H}}$ as indicated in the proof of Theorem 3.1. Here we use the main diagonal of the box $[(u_{i(j-1)}, v_{j(i-1)}), (u_{ij}, v_{ji})]$. We now slice I^2 into k_1k_2 pieces along the lines

$$x = t_k$$

for each $k = 1, \dots, k_1k_2$. Define a permutation σ on $\{1, 2, \dots, k_1k_2\}$ by

$$\sigma(k) = (\overline{k \mod k_1} - 1)k_2 + \lceil \frac{k}{k_1} \rceil,$$

where $\overline{k \mod k_1} \in \{1, \dots, k_1\}$ and $\lceil \frac{k}{k_1} \rceil$ is the smallest integer bigger than $\frac{k}{k_1}$. Following an observation below, we see that the copula $C_{\overline{H}}$ is Min_σ.

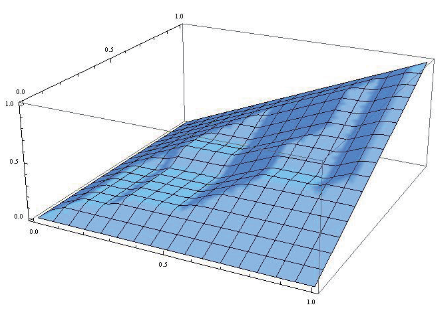
$$\begin{array}{rcl}
1 & \mapsto & 1 \\
2 & \mapsto & k_2 + 1 \\
 & \vdots & \\
k_1 & \mapsto & (k_1 - 1)k_2 + 1 \\
\hline
k_1 + 1 & \mapsto & 2 \\
k_1 + 2 & \mapsto & k_2 + 2 \\
 & \vdots & \\
2k_1 = k_1 + k_1 & \mapsto & (k_1 - 1)k_2 + 2 \\
\hline
 & \vdots & \\
\hline
(k_2 - 1)k_1 + 1 & \mapsto & k_2 \\
(k_2 - 1)k_1 + 2 & \mapsto & k_2 + k_2 \\
 & \vdots & \\
k_2k_1 = (k_2 - 1)k_1 + k_1 & \mapsto & (k_1 - 1)k_2 + k_2 = k_1k_2
\end{array}$$

We end the paper by presenting examples to demonstrate the achievement of the algorithm described above.

Example 3.4. The table displays the joint distribution $H(x, y)$ of a pair of random variables with values $x = 0, 1, 2, 3$ and $y = 0, 1, 2$.

H	$x = 0$	$x = 1$	$x = 2$	$x = 3$
$y = 0$	0.1	0.15	0.35	0.45
$y = 1$	0.2	0.25	0.55	0.65
$y = 2$	0.2	0.45	0.85	1

Here is the copula C_H corresponding to H.



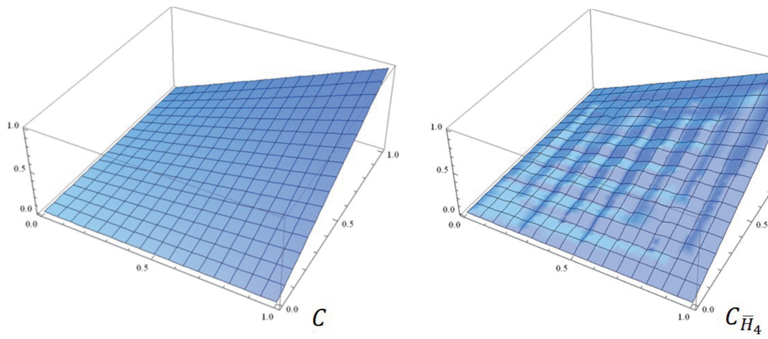
Example 3.5. Consider the density function.

$$h(x, y) = \begin{cases} x + y, & 0 < x < 1, 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

We see that the copula corresponding to this density function can be given by the formula:

$$C(u,v) = \frac{1}{2}\left(\sqrt{2u+\frac{1}{4}}-\frac{1}{2}\right)\left(\sqrt{2v+\frac{1}{4}}-\frac{1}{2}\right)\left(\sqrt{2u+\frac{1}{4}}+\sqrt{2v+\frac{1}{4}}-1\right)$$

for all $u, v \in [0,1]$. The figure compares the copula C with the copula generated by our method.



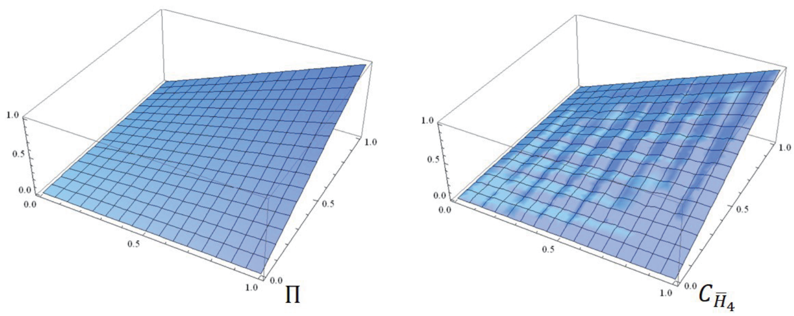
Example 3.6. Consider the density function.

$$h(x,y) = \frac{1}{4}e^{-|x|-|y|}, \quad (x, y \in \mathbb{R}).$$

We see that the copula corresponding to this density function can be given by the formula:

$$\Pi(u,v) = uv$$

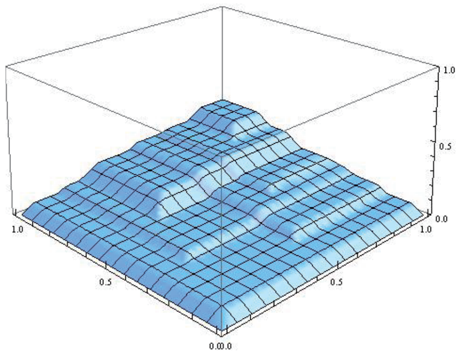
for all $u, v \in [0,1]$.



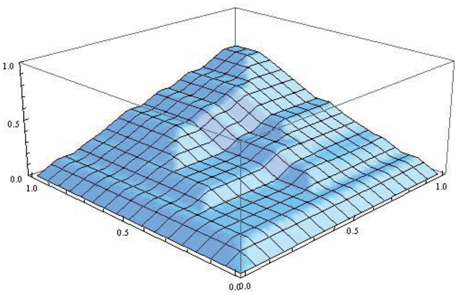
Example 3.7. The table displays the joint distribution $H(x,y,z)$ of random variables X, Y, Z with values $x = 0,1,2$, $y = 0,1$ and $z = 0,1,2$.

	H	$x = 0$	$x = 1$	$x = 2$
$z = 0$	$y = 0$	0.05	0.08	0.16
	$y = 1$	0.09	0.14	0.29
$z = 1$	$y = 0$	0.09	0.13	0.24
	$y = 1$	0.19	0.36	0.65
$z = 2$	$y = 0$	0.16	0.15	0.31
	$y = 1$	0.31	0.57	1

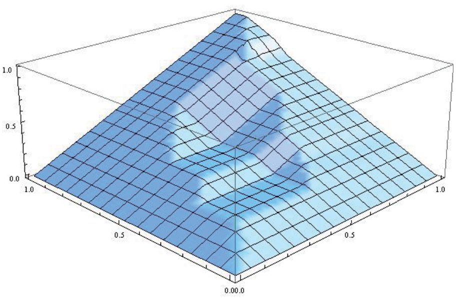
The figures demonstrate cross sections of C_H at $z = 0$, $z = 1$, and $z = 2$, respectively.



$z = 0$



$z = 1$



$z = 2$

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References

[1] A. Burchard and H. Hajaiej, *Rearrangement inequalities for functionals with monotone integrands*, J. Funct. Anal. **233** (2006), 561–582.

[2] H. Carley and M.D. Taylor, *A new proof of Sklar's theorem*, Distributions with Given Marginals and Statistical Modelling, Kluwer Academic Publishers, Dordrecht, 2002, pp. 2934.

[3] F. Durente and J. Fernández-Sánchez, C. Sempi, *Sklar's theorem obtained via regularization techniques*, Nonlinear Anal. **75** (2012), 769–774.

[4] P. Mikusiński and M. D. Taylor, *Some approximations of n-copulas*, Metrika **72** (2010), 385–414.

[5] L. Rüschendorf, *On the distributional transform, Sklar's Theorem, and the empirical copula process*, J. Stat. Plan. Infer. **139** (2009), 3921–3927.

[6] M. Scarsini, *Copulae of probability measures on product spaces*, J. Multivar. Anal. **31** (1989), 201–219.

[7] B. Schweizer and A. Sklar, *Operations on distribution functions not derivable from operations on random variables*, Studia Math. **52** (1974), 43–52.

[8] A. Sklar, *Fonctions de répartition ándimensions et leurs marges*, Publ. Inst. Stat. Univ. Paris **8** (1959), 229–231.

[9] A. Sklar, *Random Variables, Distribution Functions, and Copulas: A Personal Look Backward and Forward*, Distributions with Fixed Marginals and Related Topics, vol. 28, Institute of Mathematical Statistics, 1996, pp. 1-14.

[10] W. Takahashi, *A convexity in metric space and nonexpansive mappings*, Kodai Math. Sem. Rep. **22** (1970), 142–149.

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W. Anakkamatee
Department of Mathematics, Faculty of Science, Chiang Mai University,
Chiang Mai, 50200, Thailand
E-mail address: watcharapong_a@cmu.ac.th

S. Dhompongsa
Department of Mathematics, Faculty of Science, Chiang Mai University,
Chiang Mai, 50200, Thailand
E-mail address: sompongd@chiangmai.ac.th

S. Tasena
Department of Mathematics, Faculty of Science, Chiang Mai University,
Chiang Mai, 50200, Thailand
E-mail address: santi.tasena@cmu.ac.th