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ON ROBUST CONVEX MULTIOBJECTIVE OPTIMIZATION

DAISHI KUROIWA[†] AND GUE MYUNG LEE[*]

Abstract. The robust approach (the worst-case approach) for convex multiobjective optimization problem (UCMP) with uncertainty data is considered. Using the robust approach, we define three kinds of robust efficient solutions for an uncertain convex multiobjective optimization problem (UCMP) which consists of more than two objective functions with uncertainty data and constraint functions with uncertainty data. The purpose of this paper is to give a necessary and sufficient constraint qualification for the three kinds of robust efficient solutions for (UCMP). We give a formula for calculating the subdifferential of certain maximum function, and then we obtain results concerned with optimality conditions for the three kinds of robust efficient solutions of (UCMP). Moreover, we give examples illustrating that our main result is very useful for properly and weakly robust efficient solutions for (UCMP). Finally, we give the closedness constraint qualification for (UCMP) and show that under the constraint qualification, the optimality conditions hold.

1. Introduction and preliminaries

Consider an uncertain convex multiobjective optimization problem:

$$\text{(UCMP)} \qquad \text{minimize} \qquad (f_1(x,u_1),\ldots,f_l(x,u_l))$$
$$\text{subject to} \qquad g_j(x,v_j) \leq 0,\ j=1,\ldots,m,$$

where $f_i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $f_i(\cdot,u_i)$ is convex for each $i=1,\ldots,l$ and uncertain parameter $u_i \in \mathcal{U}_i$, $g_j : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ and $g_j(\cdot,v_j)$ is convex for each $j=1,\ldots,m$ and uncertain parameter $v_j \in \mathcal{V}_j$, and every $\mathcal{U}_i$ in $\mathbb{R}^p$ and $\mathcal{V}_j$ in $\mathbb{R}^q$ are some convex compact sets.

Recently, robust optimization has emerged as a powerful deterministic approach for studying (single-objective) optimization problem under uncertainty, and uncertain scalar optimization problems have been studied by many authors ([1, 2, 6, 7]). In particular, Kuroiwa and Lee ([10]) investigated scalarizations and optimality conditions for robust multiobjective optimization problems.

In this paper, we treat the robust approach for (UCMP), which is the worst-case approach for (UCMP), in the same way as [10]. We associate with the uncertain multiobjective optimization problem (UCMP) its robust counterpart:

$$\text{(RCMP)} \qquad \text{minimize} \qquad (\max_{u_1\in\mathcal{U}_1} f_1(x,u_1),\ldots,\max_{u_l\in\mathcal{U}_l} f_l(x,u_l))$$

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$$\text{subject to} \qquad \max_{v_j \in \mathcal{V}_j} g_j(x, v_j) \leq 0, \ j = 1, \ldots, m.$$

A vector $x \in \mathbb{R}^n$ is a robust feasible solution of (UCMP) if $\max_{v_j \in \mathcal{V}_j} g_j(x, v_j) \leq 0$ for all $j = 1, \ldots, m$. Let F be the set of all the robust feasible solutions of (UCMP), that is

$$F = \{x \in \mathbb{R}^n \mid \max_{v_j \in \mathcal{V}_j} g_j(x, v_j) \leq 0, \quad \text{for all} \quad j = 1, \ldots, m\}.$$

Following solution concepts for multiobjective optimization problem in [14], we define three solution concepts for (UCMP). A robust feasible solution $\bar{x}$ of (UCMP) is said to be

(i) a robust efficient solution of (UCMP) if there does not exist a robust feasible solution x of (UCMP) such that

$$\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \leq \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i), \quad \text{for all } i = 1, \ldots, l, \text{ and}$$
$$\max_{u_j \in \mathcal{U}_j} f_j(x, u_j) < \max_{u_j \in \mathcal{U}_j} f_j(\bar{x}, u_j), \quad \text{for some } j = 1, \ldots, l.$$

(ii) a weakly robust efficient solution of (UCMP) if there does not exist a robust feasible solution x of (UCMP) such that

$$\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) < \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i), \text{ for all } i = 1, \ldots, l.$$

(iii) a properly robust efficient solution of (UCMP) if it is a robust efficient solution of (UCMP) and if there is a number $M > 0$ such that for all $i = 1, \ldots, l$ and $x \in F$ satisfying $\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) < \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)$, there exists an index $j = 1, \ldots, l$ such that

$$\max_{u_j \in \mathcal{U}_j} f_j(\bar{x}, u_j) < \max_{u_j \in \mathcal{U}_j} f_j(x, u_j), \quad \text{and}$$
$$\frac{\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(x, u_i)}{\max_{u_j \in \mathcal{U}_j} f_j(x, u_j) - \max_{u_j \in \mathcal{U}_j} f_j(\bar{x}, u_j)} \leq M.$$

(RCMP) can be interpreted as the worst-case of (UCMP) in the following sence: we denote the set of all efficient solutions of (RCMP) by Eff(RCMP) and the set of all optimal values of (RCMP) by

$$\text{Val(RCMP)} = \{(\max_{u_1 \in \mathcal{U}_1} f_1(x, u_1), \ldots, \max_{u_l \in \mathcal{U}_l} f_l(x, u_l)) \mid x \in \text{Eff(RCMP)}\}.$$

Also, for given uncertain parameters $u = (u_1, \ldots, u_l) \in \Pi_{i=1}^l \mathcal{U}_i$ and $v = (v_1, \ldots, v_m) \in \Pi_{j=1}^m \mathcal{V}_J$, we denote the problem by $(\text{UCMP})_{u,v}$, the set of all feasible solutions of $(\text{UCMP})_{u,v}$ by F_v, all efficient solutions of $(\text{UCMP})_{u,v}$ by $\text{Eff(UCMP)}_{u,v}$, and the set of all optimal values of $(\text{UCMP})_{u,v}$ by

$$\text{Val(UCMP)}_{u,v} = \{(f_1(x, u_1), \ldots, f_l(x, u_l)) \mid x \in \text{Eff(UCMP)}_{u,v}\}.$$

Under the well-known domination property for $(\text{UCMP})_{u,v}$, see [14]: for each $x \in F_v$, there exists $\hat{x} \in \text{Eff(UCMP)}_{u,v}$ such that

$$f_i(\hat{x}, u_i) \leq f_i(x, u_i), \text{ for all } i = 1, \ldots, l,$$

we have the following observation: if $\bar{x} \in$Eff(RCMP) then there exists $\hat{x} \in \text{Eff(UCMP)}_{u,v}$ such that

$$f_i(\hat{x}, u_i) \leq \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i), \text{ for all } i = 1, \dots, l.$$

In this mean, (RCMP) can be interpreted as the worst-case of (UCMP). In addition, it is also written as $\mathrm{Val(UCMP)}_{u,v} \leq^{(\mathrm{iii})} \mathrm{Val(RCMP)}$, by using *set-relation* $\leq^{(\mathrm{iii})}$, which is a relation between two sets, see [11]. The study of *set optimization*, which is set-valued optimization based on the *set-relations*, has been developed rapidly, see [3, 4, 8, 9].

In this paper, we give a necessary and sufficient constraint qualification for the three robust efficient solutions for (UCMP). To the purpose, we give a formula for calculating the subdifferential of certain maximum function, and then we get results concerned with optimality conditions for the three robust efficient solutions for (UCMP). Moreover, we give examples showing that our main result is very useful for properly and weakly efficient robust efficient solutions for (UCMP). Finally, we give the closedness constraint qualification for (UCMP) and show the optimality conditions hold under the constraint qualification.

Let us first recall some notation and preliminary results which will be used throughout the paper. For a function $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the effective domain and the epigraph of h are given by

$$\mathrm{dom} h = \{x \in \mathbb{R}^n \mid h(x) < +\infty\}$$

and

$$\mathrm{epi} h = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid h(x) \leq r\},$$

respectively. h is said to be proper if $\mathrm{dom}h$ is nonempty, and h is said to be convex if $\mathrm{epi}h$ is a convex set, or equivalently,

$$h((1-\alpha)x + \alpha y) \leq (1-\alpha)h(x) + \alpha h(y)$$

for all $x, y \in \mathbb{R}^n$, and $\alpha \in (0, 1)$. Moreover, h is concave if $-h$ is convex.

For any proper convex function $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the subdifferential of h at $\bar{x} \in \mathrm{dom}h$ is given by

$$\partial h(\bar{x}) = \{v \in \mathbb{R}^n \mid h(x) \geq h(\bar{x}) + \langle v, x - \bar{x}\rangle, \text{ for all } x \in \mathbb{R}^n\},$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^n$, and the ϵ-subdifferential of h at $\bar{x}$ is defined by

$$\partial_\epsilon h(\bar{x}) = \{v \in \mathbb{R}^n \mid h(x) \geq h(\bar{x}) + \langle v, x - \bar{x}\rangle - \epsilon, \text{ for all } x \in \mathbb{R}^n\}.$$

The conjugate function $h^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of h is defined by

$$h^*(x^*) = \sup\{\langle x^*, x\rangle - h(x) \mid x \in \mathbb{R}^n\},$$

for any $x^* \in \mathbb{R}^n$.

Proposition 1.1 ([5]). *If $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a continuous convex function and if $a \in \mathrm{dom}h$, then*

$$\mathrm{epi} h^* = \bigcup_{\epsilon \geq 0} \{(v, \langle v, a\rangle + \epsilon - h(a)) \mid v \in \partial_\epsilon h(a)\}.$$

Now, let us recall the normal cone of convex sets, which is important to consider necessary and sufficient robust optimality conditions.

Definition 1.2. Let C be a closed convex set in $\mathbb{R}^n$ and $x \in C$. Then $N_C(x) = \{v \in \mathbb{R}^n \mid \langle v, y - x\rangle \leq 0, \text{ for all } y \in C\}$ is called the normal cone to C at x.

2. Subgradients of maximum functions

In this section, we give the subgradient formula for certain maximum function, which is important to characterize properly and weakly robust efficient solutions of (UCMP). To the purpose, we give the following propositions and lemma.

Proposition 2.1. *Let $\mathcal{U}$ be a nonempty compact convex subset of $\mathbb{R}^p$, and $\phi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ a convex-concave function, that is, $\phi(\cdot, y)$ is a convex function for each $y \in \mathbb{R}^p$ and $\phi(x, \cdot)$ is a concave function for each $x \in \mathbb{R}^n$. Then for each $\bar{x} \in \mathbb{R}^n$,*

$$\bigcup_{u \in \mathcal{U}(\bar{x})} \partial\phi(\cdot, u)(\bar{x})$$

is a convex set, where

$$\mathcal{U}(\bar{x}) = \left\{ \bar{u} \in \mathcal{U} \;\middle|\; \phi(\bar{x}, \bar{u}) = \max_{u \in \mathcal{U}} \phi(\bar{x}, u) \right\}.$$

Proof. At first, we have that $\mathcal{U}(\bar{x})$ is convex because $\mathcal{U}(\bar{x})$ is the intersection of a level set of concave function and $\mathcal{U}$. Also it is nonempty because $\phi(\bar{x}, \cdot)$ is continuous and $\mathcal{U}$ is compact. Now, we show that $\bigcup_{u \in \mathcal{U}(\bar{x})} \partial\phi(\cdot, u)(\bar{x})$ is convex. For any $y_1, y_2 \in \bigcup_{u \in \mathcal{U}(\bar{x})} \partial\phi(\cdot, u)(\bar{x})$ and $\alpha \in (0, 1)$, there exist $\bar{u}_1, \bar{u}_2 \in \mathcal{U}(\bar{x})$ such that $y_1 \in \partial\phi(\cdot, \bar{u}_1)(\bar{x})$ and $y_2 \in \partial\phi(\cdot, \bar{u}_2)(\bar{x})$. Since $\mathcal{U}(\bar{x})$ is convex, $(1-\alpha)\bar{u}_1 + \alpha\bar{u}_2 \in \mathcal{U}(\bar{x})$. Then for each $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\phi(x, (1-\alpha)\bar{u}_1 + \alpha\bar{u}_2) \\
&\geq (1-\alpha)\phi(x, \bar{u}_1) + \alpha\phi(x, \bar{u}_2) \\
&\geq (1-\alpha)(\phi(\bar{x}, \bar{u}_1) + \langle y_1, x - \bar{x} \rangle) + \alpha(\phi(\bar{x}, \bar{u}_2) + \langle y_2, x - \bar{x} \rangle) \\
&= (1-\alpha)\phi(\bar{x}, \bar{u}_1) + \alpha\phi(\bar{x}, \bar{u}_2) + \langle (1-\alpha)y_1 + \alpha y_2, x - \bar{x} \rangle \\
&= (1-\alpha)\max_{u \in \mathcal{U}} \phi(\bar{x}, u) + \alpha \max_{u \in \mathcal{U}} \phi(\bar{x}, u) + \langle (1-\alpha)y_1 + \alpha y_2, x - \bar{x} \rangle \\
&= \max_{u \in \mathcal{U}} \phi(\bar{x}, u) + \langle (1-\alpha)y_1 + \alpha y_2, x - \bar{x} \rangle \\
&= \phi(\bar{x}, (1-\alpha)\bar{u}_1 + \alpha\bar{u}_2) + \langle (1-\alpha)y_1 + \alpha y_2, x - \bar{x} \rangle .
\end{aligned}$$

The inequalities are due to concavity of $\phi(\bar{x}, \cdot)$, $y_1 \in \partial\phi(\cdot, \bar{u}_1)(\bar{x})$, $y_2 \in \partial\phi(\cdot, \bar{u}_2)(\bar{x})$, and $\bar{u}_1, \bar{u}_2, (1-\alpha)\bar{u}_1 + \alpha\bar{u}_2 \in \mathcal{U}(\bar{x})$. Thus

$$(1-\alpha)y_1 + \alpha y_2 \in \partial\phi(\cdot, (1-\alpha)\bar{u}_1 + \alpha\bar{u}_2)(\bar{x}) \subset \bigcup_{u \in \mathcal{U}(\bar{x})} \partial\phi(\cdot, u)(\bar{x}).$$

This completes the proof. □

Lemma 2.2 ([13])**.** *Assume that $\phi : \mathbb{R}^n \times Y \to \mathbb{R} \cup \{+\infty\}$ satisfies the following conditions:*

- (i) *$\phi(\cdot, y)$ is convex for all $y \in Y$;*
- (ii) *$\phi(x, \cdot)$ is upper semicontinuous for all x in a certain neighborhood of a point x_0;*
- (iii) *The set $Y \subset \mathbb{R}^m$ is compact.*

Then

$$\partial\left(\max_{y\in Y}\phi(\cdot,y)\right)(x_0)\supset \operatorname{conv}\bigcup_{y\in Y(x_0)}\partial\phi(\cdot,y)(x_0),$$

where

$$Y(x_0)=\left\{y\in Y\ \middle|\ \phi(x_0,y)=\max_{y\in Y}\phi(x_0,y)\right\}.$$

If, in addition, the function $f(\cdot,y)$ is continuous at x_0 for all $y\in Y$, then

$$\partial\left(\max_{y\in Y}\phi(\cdot,y)\right)(x_0)= \operatorname{conv}\bigcup_{y\in Y(x_0)}\partial\phi(\cdot,y)(x_0).$$

Proposition 2.3. *Let $\mathcal{U}_1,\mathcal{U}_2,\ldots,\mathcal{U}_l$ be nonempty convex and compact subsets of $\mathbb{R}^p$, $f_1,f_2,\ldots,f_l:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}$ be convex-concave functions. For each $\bar{x}\in F$ and $\lambda_i\geq 0$, $i=1,2,\ldots,l$,*

$$\partial\left(\max_{u\in\mathcal{U}}\sum_{i=1}^{l}\lambda_i f_i(\cdot,u_i)\right)(\bar{x})=\bigcup_{u\in\mathcal{U}(\bar{x})}\sum_{i=1}^{l}\lambda_i\partial f_i(\cdot,u_i)(\bar{x}),$$

where $\mathcal{U}=\prod_{i=1}^{l}\mathcal{U}_i$ and

$$\mathcal{U}(\bar{x})=\left\{\bar{u}\in\mathcal{U}\ \middle|\ \sum_{i=1}^{l}\lambda_i f_i(\bar{x},\bar{u}_i)=\max_{u\in\mathcal{U}}\sum_{i=1}^{l}\lambda_i f_i(\bar{x},u_i)\right\}.$$

Proof. For given $\lambda_i\geq 0$, $i=1,2,\ldots,l$, define $\phi:\mathbb{R}^n\times\mathbb{R}^{lp}\to\mathbb{R}$ as follows:

$$\phi(x,u)=\sum_{i=1}^{l}\lambda_i f_i(x,u_i),\quad x\in\mathbb{R}^n, u=(u_1,\ldots,u_q)\in\mathbb{R}^{lp}.$$

It is clear that ϕ is a convex-concave continuous function. From Lemma 2.2 and Proposition 2.1, we have

$$\begin{aligned}\partial\left(\max_{u\in\mathcal{U}}\sum_{i=1}^{l}\lambda_i f_i(\cdot,u_i)\right)(\bar{x}) &= \partial\left(\max_{u\in\mathcal{U}}\phi(\cdot,u)\right)(\bar{x})\\ &= \operatorname{conv}\bigcup_{u\in\mathcal{U}(\bar{x})}\partial\phi(\cdot,u)(\bar{x})\\ &= \bigcup_{u\in\mathcal{U}(\bar{x})}\partial\phi(\cdot,u)(\bar{x})\\ &= \bigcup_{u\in\mathcal{U}(\bar{x})}\sum_{i=1}^{l}\lambda_i\partial f_i(\cdot,u_i)(\bar{x}).\end{aligned}$$

This completes the proof. □

3. A necessary and sufficient constraint qualification for robust optimality conditions

Now, we give a necessary and sufficient constraint qualification for optimality conditions for the three kinds of robust efficient solutions of (UCMP).

For each $j = 1, 2, \ldots, m$, let $\mathcal{V}_j$ be a nonempty convex compact set in $\mathbb{R}^q$ and $g_j : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ a convex-concave function. Recall that F is the constraint set of (UCMP), that is,

$$F = \{x \in \mathbb{R}^n \mid \max_{v_j \in \mathcal{V}_j} g_j(x, v_j) \leq 0, \text{ for all } j = 1, \ldots, m\}.$$

At first we observe that the following equality condition:

$$N_F(\bar{x}) = \bigcup_{\substack{(\mu,v)\in\mathbb{R}^m_+\times\Pi^m_{j=1}\mathcal{V}_j \\ \sum_{j=1}^m \mu_j g_j(\bar{x},v_j)=0}} \sum_{j=1}^m \mu_j \partial g_j(\cdot, v_j)(\bar{x}).$$

The following inclusion is always true: for any $\bar{x} \in F$,

$$N_F(\bar{x}) \supset \bigcup_{\substack{(\mu,v)\in\mathbb{R}^m_+\times\Pi^m_{j=1}\mathcal{V}_j \\ \sum_{j=1}^m \mu_j g_j(\bar{x},v_j)=0}} \sum_{j=1}^m \mu_j \partial g_j(\cdot, v_j)(\bar{x}).$$

Indeed, let $(\mu, v) \in \mathbb{R}^m_+ \times \Pi^m_{j=1}\mathcal{V}_j$ satisfying $\sum_{j=1}^m \mu_j g_j(\bar{x}, v_j) = 0$. For any $y \in \sum_{j=1}^m \mu_j \partial g_j(\cdot, v_j)(\bar{x})$, there exist $y_j \in \partial g_j(\cdot, v_j)(\bar{x})$ $(j = 1, 2, \ldots, m)$ such that $y = \sum_{j=1}^m \mu_j y_j$. For fixed $x \in F$,

$$0 \geq g_j(x, v_j) \geq g_j(\bar{x}, v_j) + \langle y_j, x - \bar{x}\rangle$$

for each $j = 1, 2, \ldots, m$, and then

$$0 \geq \sum_{j=1}^m \mu_j g_j(\bar{x}, v_j) + \left\langle \sum_{j=1}^m \mu_j y_j, x - \bar{x} \right\rangle = \langle y, x - \bar{x}\rangle.$$

Consequently we have $y \in N_F(\bar{x})$. Therefore the above equality condition is equivalent to

$$N_F(\bar{x}) \subset \bigcup_{\substack{(\mu,v)\in\mathbb{R}^m_+\times\Pi^m_{j=1}\mathcal{V}_j \\ \sum_{j=1}^m \mu_j g_j(\bar{x},v_j)=0}} \sum_{j=1}^m \mu_j \partial g_j(\cdot, v_j)(\bar{x}).$$

We can see that the condition is a necessary and sufficient constraint qualification for the three kinds of robust efficient solutions of (UCMP) in the following theorem.

Theorem 3.1. *Let $\mathcal{V}_j \subset \mathbb{R}^m$ be a convex compact set in $\mathbb{R}^q$ and $g_j : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ is a convex-concave function for each $j = 1, 2, \ldots, m$, and $\bar{x} \in F$. Then, the following statements are equivalent:*

(i) $N_F(\bar{x}) = \bigcup_{\substack{(\mu,v)\in\mathbb{R}^m_+\times\Pi^m_{j=1}\mathcal{V}_j \\ \sum_{j=1}^m \mu_j g_j(\bar{x},v_j)=0}} \sum_{j=1}^m \mu_j \partial g_j(\cdot, v_j)(\bar{x})$ *holds.*

(ii) *For all convex-concave functions* $f_1, f_2, \ldots, f_l : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, *and for all nonempty compact convex subsets* $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_l$ *of* $\mathbb{R}^p$, $\bar{x}$ *is a weakly efficient solution of (RCMP):*

$$\begin{cases} \text{Minimize} & \left(\max\limits_{u_1 \in \mathcal{U}_1} f_1(x, u_1), \ldots, \max\limits_{u_l \in \mathcal{U}_l} f_l(x, u_l)\right) \\ \text{subject to} & \max\limits_{v_j \in \mathcal{V}_j} g_j(x, v_j) \leq 0, \ j = 1, \ldots, m, \end{cases}$$

if and only if there exist $\bar{\lambda}_1, \ldots, \bar{\lambda}_l \geq 0$, *all non zero,* $(\bar{u}_1, \ldots, \bar{u}_l) \in \Pi_{i=l}^{l} \mathcal{U}_i$, $\bar{\mu}_1, \ldots, \bar{\mu}_m \geq 0$, *and* $(\bar{v}_1, \ldots, \bar{v}_m) \in \Pi_{j=l}^{m} \mathcal{V}_j$ *such that*

- $0 \in \sum_{i=1}^{l} \bar{\lambda}_i \partial f_i(\cdot, \bar{u}_i)(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \partial g_j(\cdot, \bar{v}_j)(\bar{x})$,
- $\bar{\lambda}_i f_i(\bar{x}, \bar{u}_i) = \max_{u_i \in \mathcal{U}_i} \bar{\lambda}_i f_i(\bar{x}, u_i)$, *for all* $i = 1, \ldots, l$, *and*
- $\bar{\mu}_j g_j(\bar{x}, \bar{v}_j) = 0$, *for all* $j = 1, \ldots, m$.

(iii) *For all convex-concave functions* $f_1, f_2, \ldots, f_l : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, *and for all nonempty compact convex subsets* $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_l$ *of* $\mathbb{R}^p$, *if* $\bar{x}$ *is an efficient solution of (RCMP):*

$$\begin{cases} \text{Minimize} & \left(\max\limits_{u_1 \in \mathcal{U}_1} f_1(x, u_1), \ldots, \max\limits_{u_l \in \mathcal{U}_l} f_l(x, u_l)\right) \\ \text{subject to} & \max\limits_{v_j \in \mathcal{V}_j} g_j(x, v_j) \leq 0, \ j = 1, \ldots, m, \end{cases}$$

then there exist $\bar{\lambda}_1, \ldots, \bar{\lambda}_l \geq 0$, *all non zero,* $(\bar{u}_1, \ldots, \bar{u}_l) \in \Pi_{i=l}^{l} \mathcal{U}_i$, $\bar{\mu}_1, \ldots, \bar{\mu}_m \geq 0$, *and* $(\bar{v}_1, \ldots, \bar{v}_m) \in \Pi_{j=l}^{m} \mathcal{V}_j$ *such that*

- $0 \in \sum_{i=1}^{l} \bar{\lambda}_i \partial f_i(\cdot, \bar{u}_i)(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \partial g_j(\cdot, \bar{v}_j)(\bar{x})$,
- $\bar{\lambda}_i f_i(\bar{x}, \bar{u}_i) = \max_{u_i \in \mathcal{U}_i} \bar{\lambda}_i f_i(\bar{x}, u_i)$, *for all* $i = 1, \ldots, l$, *and*
- $\bar{\mu}_j g_j(\bar{x}, \bar{v}_j) = 0$, *for all* $j = 1, \ldots, m$.

(iv) *For all convex-concave functions* $f_1, f_2, \ldots, f_l : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, *and for all nonempty compact convex subsets* $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_l$ *of* $\mathbb{R}^p$, $\bar{x}$ *is a properly efficient solution of (RCMP):*

$$\begin{cases} \text{Minimize} & \left(\max\limits_{u_1 \in \mathcal{U}_1} f_1(x, u_1), \ldots, \max\limits_{u_l \in \mathcal{U}_l} f_l(x, u_l)\right) \\ \text{subject to} & \max\limits_{v_j \in \mathcal{V}_j} g_j(x, v_j) \leq 0, \ j = 1, \ldots, m, \end{cases}$$

if and only if there exist $\bar{\lambda}_1, \ldots, \bar{\lambda}_l > 0$, $(\bar{u}_1, \ldots, \bar{u}_l) \in \Pi_{i=l}^{l} \mathcal{U}_i$, $\bar{\mu}_1, \ldots, \bar{\mu}_m \geq 0$, *and* $(\bar{v}_1, \ldots, \bar{v}_m) \in \Pi_{j=l}^{m} \mathcal{V}_j$ *such that*

- $0 \in \sum_{i=1}^{l} \bar{\lambda}_i \partial f_i(\cdot, \bar{u}_i)(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \partial g_j(\cdot, \bar{v}_j)(\bar{x})$,
- $f_i(\bar{x}, \bar{u}_i) = \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)$, *for all* $i = 1, \ldots, l$, *and*
- $\bar{\mu}_j g_j(\bar{x}, \bar{v}_j) = 0$, *for all* $j = 1, \ldots, m$.

(v) *For all convex functions* $f_1, f_2, \ldots, f_l : \mathbb{R}^n \to \mathbb{R}$, $\bar{x}$ *is a weakly efficient solution of (RCMP):*

$$\begin{cases} \text{Minimize} & (f_1(x), \ldots, f_l(x)) \\ \text{subject to} & \max\limits_{v_j \in \mathcal{V}_j} g_j(x, v_j) \leq 0, \ j = 1, \ldots, m, \end{cases}$$

if and only if there exist $\bar{\lambda}_1, \ldots, \bar{\lambda}_l \geq 0$, *all non zero,* $\bar{\mu}_1, \ldots, \bar{\mu}_m \geq 0$, *and* $(\bar{v}_1, \ldots, \bar{v}_m) \in \Pi_{j=l}^{m} \mathcal{V}_j$ *such that*

- $0 \in \sum_{i=1}^{l} \bar{\lambda}_i \partial f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \partial g_j(\cdot, \bar{v}_j)(\bar{x})$, *and*
- $\bar{\mu}_j g_j(\bar{x}, \bar{v}_j) = 0$, *for all* $j = 1, \ldots, m$.

(vi) *For all convex functions* $f_1, f_2, \ldots, f_l : \mathbb{R}^n \to \mathbb{R}$, *if* $\bar{x}$ *is an efficient solution of (RCMP):*

$$\begin{cases} \text{Minimize} & (f_1(x), \ldots, f_l(x)) \\ \text{subject to} & \max\limits_{v_j \in \mathcal{V}_j} g_j(x, v_j) \le 0, \ j = 1, \ldots, m, \end{cases}$$

then there exist $\bar{\lambda}_1, \ldots, \bar{\lambda}_l \ge 0$, *all non zero,* $\bar{\mu}_1, \ldots, \bar{\mu}_m \ge 0$, *and* $(\bar{v}_1, \ldots, \bar{v}_m) \in \Pi_{j=l}^{m} \mathcal{V}_j$ *such that*

- $0 \in \sum_{i=1}^{l} \bar{\lambda}_i \partial f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \partial g_j(\cdot, \bar{v}_j)(\bar{x})$, *and*
- $\bar{\mu}_j g_j(\bar{x}, \bar{v}_j) = 0$, *for all* $j = 1, \ldots, m$.

(vii) *For all convex functions* $f_1, f_2, \ldots, f_l : \mathbb{R}^n \to \mathbb{R}$, $\bar{x}$ *is a properly efficient solution of (RCMP):*

$$\begin{cases} \text{Minimize} & (f_1(x), \ldots, f_l(x)) \\ \text{subject to} & \max\limits_{v_j \in \mathcal{V}_j} g_j(x, v_j) \le 0, \ j = 1, \ldots, m, \end{cases}$$

if and only if there exist $\bar{\lambda}_1, \ldots, \bar{\lambda}_l > 0$, $\bar{\mu}_1, \ldots, \bar{\mu}_m \ge 0$, *and* $(\bar{v}_1, \ldots, \bar{v}_m) \in \Pi_{j=l}^{m} \mathcal{V}_j$ *such that*

- $0 \in \sum_{i=1}^{l} \bar{\lambda}_i \partial f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \partial g_j(\cdot, \bar{v}_j)(\bar{x})$, *and*
- $\bar{\mu}_j g_j(\bar{x}, \bar{v}_j) = 0$, *for all* $j = 1, \ldots, m$.

(viii) *For all convex function* $f : \mathbb{R}^n \to \mathbb{R}$, $\bar{x}$ *is a minimum solution of (P):*

$$\begin{cases} \text{Minimize} & f(x) \\ \text{subject to} & \max\limits_{v_j \in \mathcal{V}_j} g_j(x, v_j) \le 0, \ j = 1, \ldots, m, \end{cases}$$

if and only if there exist $\bar{\mu}_1, \ldots, \bar{\mu}_m \ge 0$, *and* $(\bar{v}_1, \ldots, \bar{v}_m) \in \Pi_{j=l}^{m} \mathcal{V}_j$ *such that*

- $0 \in \partial f(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \partial g_j(\cdot, \bar{v}_j)(\bar{x})$, *and*
- $\bar{\mu}_j g_j(\bar{x}, \bar{v}_j) = 0$, *for all* $j = 1, \ldots, m$.

Proof. At first, we show [(i)⇒(ii)]. Assume (i), and let $f_1, f_2, \ldots, f_l : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ be convex-concave functions, and let $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_l$ be nonempty compact convex subsets in $\mathbb{R}^p$. By using Gordan's alternative theorem in [12], $\bar{x}$ is a weakly efficient solution of (RCMP):

$$\begin{cases} \text{Minimize} & \left(\max\limits_{u_1 \in \mathcal{U}_1} f_1(x, u_1), \ldots, \max\limits_{u_l \in \mathcal{U}_l} f_l(x, u_l) \right) \\ \text{subject to} & \max\limits_{v_j \in \mathcal{V}_j} g_j(x, v_j) \le 0, \ j = 1, \ldots, m \end{cases}$$

if and only if there exists $\bar{\lambda}_1, \ldots, \bar{\lambda}_l \ge 0$, all non zero, such that for all $x \in F$,

$$\sum_{i=1}^{l} \bar{\lambda}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) \le \sum_{i=1}^{l} \bar{\lambda}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i).$$

Equivalently,

$$0 \in \partial \left(\sum_{i=1}^{l} \bar{\lambda}_i \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i) + \delta_F \right)(\bar{x}) = \partial \left(\max_{u \in \mathcal{U}} \sum_{i=1}^{l} \bar{\lambda}_i f_i(\cdot, u_i) \right)(\bar{x}) + N_F(\bar{x}),$$

where $\mathcal{U} = \Pi_{i=l}^{l} \mathcal{U}_i$. By using Proposition 2.3,

$$\partial \left(\max_{u \in \mathcal{U}} \sum_{i=1}^{l} \bar{\lambda}_i f_i(\cdot, u_i) \right)(\bar{x}) = \bigcup_{u \in \mathcal{U}(\bar{x})} \sum_{i=1}^{l} \bar{\lambda}_i \partial f_i(\cdot, u_i)(\bar{x}),$$

where

$$\mathcal{U}(\bar{x}) = \left\{ \bar{u} \in \mathcal{U} \;\middle|\; \sum_{i=1}^{l} \bar{\lambda}_i f_i(\bar{x}, \bar{u}_i) = \max_{u \in \mathcal{U}} \sum_{i=1}^{l} \bar{\lambda}_i f_i(\bar{x}, u_i) \right\}.$$

From this and condition (i), we have

$$0 \in \bigcup_{u \in \mathcal{U}(\bar{x})} \sum_{i=1}^{l} \bar{\lambda}_i \partial f_i(\cdot, u_i)(\bar{x}) + \bigcup_{\substack{(\mu, v) \in \mathbb{R}_+^m \times \Pi_{j=1}^m \mathcal{V}_j \\ \sum_{j=1}^m \mu_j g_j(\bar{x}, v_j) = 0}} \sum_{j=1}^{m} \mu_j \partial g_j(\cdot, v_j)(\bar{x}),$$

that is, there exist $\bar{u} \in \mathcal{U}(\bar{x})$ and $(\bar{\mu}, \bar{v}) \in \mathbb{R}_+^m \times \Pi_{j=1}^m \mathcal{V}_j$ such that

- $0 \in \sum_{i=1}^{l} \bar{\lambda}_i \partial f_i(\cdot, \bar{u}_i)(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \partial g_j(\cdot, \bar{v}_j)(\bar{x})$, and
- $\sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j) = 0$.

From the above, we can check easily that $\bar{\lambda}_i f_i(\bar{x}, \bar{u}_i) = \max_{u_i \in \mathcal{U}_i} \bar{\lambda}_i f_i(\bar{x}, u_i)$ for all $i = 1, \ldots, l$, and $\bar{\mu}_j g_j(\bar{x}, \bar{v}_j) = 0$ for all $j = 1, \ldots, m$.

Proof of [(i)⇒(iv)] is similar to [(i)⇒(ii)], and omitted. Also, proofs of [(ii)⇒(v)⇒(viii)], [(ii)⇒(iii)⇒(vi)⇒(viii)], and [(iv)⇒(vii)⇒(viii)], are clear and omitted.

Finally, we show [(viii)⇒(i)]. If $y \in N_F(\bar{x})$, then $\langle -y, \bar{x} \rangle \leq \langle -y, x \rangle$ for each $x \in F$. Let $f(x) = \langle -y, x \rangle$. From the assumption (viii), there exist $\bar{\mu}_1, \ldots, \bar{\mu}_m \geq 0$, and $(\bar{v}_1, \ldots, \bar{v}_m) \in \Pi_{j=l}^m \mathcal{V}_j$ such that

- $0 \in \partial f(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \partial g_j(\cdot, \bar{v}_j)(\bar{x})$,
- $\bar{\mu}_j g_j(\bar{x}, \bar{v}_j) = 0$, for all $j = 1, 2, \ldots, m$.

Since $\partial f(\bar{x}) = \{-y\}$, we have

$$y \in \sum_{j=1}^{m} \bar{\mu}_j \partial g_j(\cdot, \bar{v}_j)(\bar{x}).$$

Also $\sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j) = 0$ is clear, then we have

$$y \in \bigcup_{\substack{(\mu, v) \in \mathbb{R}_+^m \times \Pi_{j=1}^m \mathcal{V}_j \\ \sum_{j=1}^m \mu_j g_j(\bar{x}, v_j) = 0}} \sum_{j=1}^{m} \mu_j \partial g_j(\cdot, v_j)(\bar{x}).$$

This completes the proof. □

Remark 3.2. The interesting feature of the Karush-Kuhn-Tucker system for robust optimization problem, which was appeared in (ii)-(viii) of Theorem 3.1, is that the number of Lagrangian multipliers coincides with the number of constraint functions.

Now we give examples illustrating that Theorem 3.1 is very useful for properly and weakly robust efficient solutions for (RCMP).

Example 3.3. Let $g_1 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by $g_1(x, v_1) = v_1(|x|-1)$, and $\mathcal{V}_1 = [0, 1]$. Then we can check that $F = [-1, 1]$ and

$$N_F(\bar{x}) = \bigcup_{\substack{(\mu,v)\in\mathbb{R}_+\times\mathcal{V}_1 \\ \mu g_1(\bar{x},v)=0}} \mu\partial g_1(\cdot, v)(\bar{x}) = \begin{cases} (-\infty, 0] & \text{if } \bar{x} = -1, \\ \{0\} & \text{if } \bar{x} \in (-1, 1), \\ [0, +\infty) & \text{if } \bar{x} = 1, \end{cases}$$

and then condition (i) of Theorem 3.1 holds. Therefore, for all multivalued convex-concave functions, properly and weakly robust efficient solutions of (RCMP) can be characterized. For example, let $f_1, f_2 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by $f_1(x, u_1) = u_1(2+x)$, $f_2(x, u_2) = u_2(2 - x)$, and $\mathcal{U}_1 = \mathcal{U}_2 = [0, 1]$. A feasible solution $\bar{x}$ of (RCMP) is a weakly efficient solution of (RCMP) if and only if there exist $\bar{\lambda}_1, \bar{\lambda}_2 \geq 0$, all non zero, $(\bar{u}_1, \bar{u}_2) \in \mathcal{U}_1 \times \mathcal{U}_2$, $\bar{\mu}_1 \geq 0$, and $\bar{v}_1 \in \mathcal{V}_1$ such that

- $0 \in \bar{\lambda}_1\partial f_1(\cdot, \bar{u}_1)(\bar{x}) + \bar{\lambda}_2\partial f_2(\cdot, \bar{u}_2)(\bar{x}) + \bar{\mu}_1\partial g_1(\cdot, \bar{v}_1)(\bar{x})$,
- $\bar{\lambda}_i f_i(\bar{x}, \bar{u}_i) = \max_{u_i\in\mathcal{U}_i} \bar{\lambda}_i f_i(\bar{x}, u_i)$, for all $i = 1, 2$, and
- $\bar{\mu}_1 g_1(\bar{x}, \bar{v}_1) = 0$.

We can check these conditions always hold when $(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\mu}_1) = (1, 1, 0)$ and $(\bar{u}_1, \bar{u}_2, \bar{v}_1) = (1, 1, 1)$, for instance. This shows that every feasible solution of (RCMP) is a weakly efficient solution of (RCMP).

Example 3.4. Let $g_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $g_1(x, y) = \begin{cases} 0, & x \leq 0, \\ yx^2, & x > 0, \end{cases}$ and $\mathcal{V}_1 = [0, 1]$. Then, we can check that $F = (-\infty, 0]$, $N_F(0) = [0, \infty)$, but

$$\bigcup_{\substack{\mu\geq 0,\ v_1\in\mathcal{V}_1 \\ \mu g_1(0,v_1)=0}} \partial(\mu g_1)(\cdot, v_1)(0) = \{0\}.$$

In this case, condition (i) of Theorem 3.1 does not hold when $\bar{x} = 0$. So, it follows from Theorem 3.1 that we can give objective functions such that properly and weakly robust efficient solutions of (RCMP) can not be characterized by the Karush-Kuhn-Tucker systems. For example, let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = u_1x$, $f_2(x) = u_2x$, and $\mathcal{U}_1 = \mathcal{U}_2 = [1, 2]$. Then $\bar{x} = 0$ is a weakly robust efficient solution of (RCMP) but the following formula has no solutions: $\bar{\lambda}_1 \geq 0$, $\bar{\lambda}_2 \geq 0$, $(\bar{\lambda}_1, \bar{\lambda}_2) \neq (0, 0)$, $\bar{\mu}_1 \geq 0$, and $0 \in \bar{\lambda}_1\partial f_1(\cdot, \bar{u}_1)(\bar{x}) + \bar{\lambda}_2\partial f_2(\cdot, \bar{u}_2)(\bar{x}) + \bar{\mu}_1\partial g_1(\cdot, \bar{v}_1)(\bar{x})$.

4. Closedness constraint qualification for robust optimality conditions

In this last section, we give sufficient conditions for (i) of Theorem 3.1. Consider the following closedness constraint qualification:

$$\bigcup_{(\mu,v)\in\mathbb{R}^m_+\times\prod_{j=1}^m V_j} \operatorname{epi}\Bigg(\sum_{j=1}^m \mu_j g_j(\cdot, v_j)\Bigg)^* \text{ is closed.}$$

The closedness constraint qualification is found in [7], which is the robust version of the one in [5].

To the purpose, we introduce the following lemma, which is the robust version of an alternative theorem and can be obtained from Theorem 2.4 and Proportion 2.3 in [7].

Lemma 4.1 (Robust Theorem of the Alternative)**.** *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a convex function, let* $g_j : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ *be a convex-concave function, and let* $\mathcal{V}_j$ *be a nonempty convex and closed subset of* $\mathbb{R}^q$ *for each* $j = 1, 2, \ldots, m$*. Suppose that* $F = \{x \in \mathbb{R}^n \mid g_j(x, v_j) \leq 0, \text{ for all } j = 1, \ldots, m \text{ and } v_j \in \mathcal{V}_j\}$ *is not empty. Then exact one of the following two statements holds:*

(i) $\exists x \in \mathbb{R}^n$ *s.t.* $f(x) < 0,\ g_j(x, v_j) \leq 0,\ \forall v_j \in \mathcal{V}_j,\ \forall j = 1, \ldots, m,$

(ii) $(0, 0) \in \text{epi} f^* + \text{cl} \bigcup_{(\mu,v) \in \mathbb{R}^m_+ \times \prod_{j=1}^m V_j} \text{epi}\Big(\sum_{j=1}^m \mu_j g_j(\cdot, v_j)\Big)^*.$

Proposition 4.2. *Under the same assumptions as Theorem 3.1, the above closedness constraint qualification implies (i) of Theorem 3.1.*

Proof. For any $y \in N_F(\bar{x})$, we define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \langle -y, x - \bar{x}\rangle$. Since $0 \leq \langle -y, x - \bar{x}\rangle = f(x)$ for any $x \in F$, by using Lemma 4.1 and the closedness condition, we have

$$(0, 0) \in \text{epi} f^* + \bigcup_{(\mu,v) \in \mathbb{R}^m_+ \times \prod_{j=1}^m V_j} \text{epi}\Big(\sum_{j=1}^m \mu_j g_j(\cdot, v_j)\Big)^*.$$

Notice that $\text{epi} f^* = \{(-y,\ \langle -y, \bar{x}\rangle + \beta) \mid \beta \geq 0\}$, and $\text{epi}(\sum_{j=1}^m \mu_j g_j(\cdot, v_j))^* = \cup_{\epsilon \geq 0}\{(w, \langle w, \bar{x}\rangle + \epsilon - \sum_{j=1}^m \mu_j g_j(\bar{x}, v_j)) \mid w \in \partial_\epsilon(\sum_{j=1}^m \mu_j g_j(\cdot, v_j))(\bar{x})\}$ by using Proposition 1.1. Thus, there exist $v_j \in \mathcal{V}_j$, $\mu_j \geq 0$ $(j = 1, \ldots, m)$, $\epsilon \geq 0$, $w \in \partial_\epsilon(\sum_{j=1}^m \mu_j g_j(\cdot, v_j))(\bar{x})$, and $\beta \geq 0$ such that

$$0 = -y + w \quad \text{and} \quad 0 = \langle -y, \bar{x}\rangle + \beta + \langle w, \bar{x}\rangle + \epsilon - \sum_{j=1}^m \mu_j g_j(\bar{x}, v_j).$$

This shows $y \in \partial_\epsilon(\sum_{j=1}^m \mu_j g_j(\cdot, v_j))(\bar{x})$ and $0 = \beta + \epsilon - \sum_{j=1}^m \mu_j g_j(\bar{x}, v_j)$. Since $\beta \geq 0$, $\epsilon \geq 0$ and $-\sum_{j=1}^m \mu_j g_j(\bar{x}, v_j) \geq 0$, we have

$$\beta = \epsilon = \sum_{j=1}^m \mu_j g_j(\bar{x}, v_j) = 0.$$

This completes the proof. □

Remark 4.3. Under the same assumptions as Theorem 3.1, the Slater type strict feasibility condition for (UCMP), that is, there exists $x_0 \in \mathbb{R}^n$ such that $g_j(x_0, v_j) < 0$ for any $j = 1, 2, \ldots, m$ and $v_j \in \mathcal{V}_j$, implies the closedness condition, see [7], but it is clear that the reverse implication is not true.

Also, the closedness condition implies (i) of Theorem 3.1, but the reverse implication is also not true. Indeed, let g_1 be the same in Example 3.4 and $\bar{x} = -1$, then (i) of Theorem 3.1 holds, but the closedness condition fails.

When the Slater type condition or the closedness are not satisfied, solutions of (RCMP) can not be characterized at some feasible points and some objective functions, as seen in Example 3.4.

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References

[1] A. Beck and A. Ben-Tal, *Duality in robust optimization: primal worst equals dual best*, Oper. Res. Lett. **37** (2009), 1–6.

[2] D. Bertsimas, D. Pachamanova and M. Sim, *Robust linear optimization under general norms*, Oper. Res. Lett. **32** (2004), 510–516.

[3] C. Gutiérrez, E. Miglierina, E. Molho and V. Novo, *Pointwise well-posedness in set optimization with cone proper sets*, Nonlinear Anal. **75** (2012), 1822–1833.

[4] E. Hernández and L. Rodríguez-Marín, *Duality in set optimization with set-valued maps*, Pac. J. Optim. **3** (2007), 245–255.

[5] V. Jeyakumar, *Asymptotic dual conditions characterizing optimality for convex programs*, J. Optim. Theory Appl. **93** (1997), 153–165.

[6] V. Jeyakumar and G. Y. Li, *Robust Farkas' lemma for uncertain linear systems with applications*, Positivity **15** (2011), 331–342.

[7] V. Jeyakumar and G. Y. Li, *Strong duality in robust convex programming: complete characterizations*, SIAM J. Optim. **20** (2010), 3384–3407.

[8] D. Kuroiwa, *Existence of efficient points of set optimization with weighted criteria*, J. Nonlinear Convex Anal. **4** (2003), 117–123.

[9] D. Kuroiwa, *On derivatives of set-valued maps and optimality conditions for set optimization*, J. Nonlinear Convex Anal. **10** (2009), 41–50.

[10] D. Kuroiwa and G. M. Lee, *On robust multiobjective optimization*, Vietnam J. Math. **40** (2012), 305–317.

[11] D. Kuroiwa, T. Tanaka and T. X. D. Ha, *On cone convexity of set-valued maps*, J. Nonlinear Anal. **30** (1997), 1487–1496.

[12] O. L. Mangasarian, *Nonlinear Programming*, McGraw-Hill, Inc, New York, 1969.

[13] A. Ruszczyński, *Nonlinear Optimization*, Princeton University Press, Princeton, 2006

[14] Y. Sawaragi, H. Nakayama and T. Tanino, *Theory of Multiobjective Optimization*, Academic Press, Inc., New York, 1985.

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Daishi Kuroiwa
Department of Mathematics, Shimane University, Matsue 690-8504, Japan
E-mail address: kuroiwa@math.shimane-u.ac.jp

Gue Myung Lee
Department of Applied Mathematics, Pukyong National University, Busan 608-737, Korea
E-mail address: gmlee@pknu.ac.kr