Journal of Nonlinear and Convex Analysis
Volume 15, Number 6, 2014, 1113–1123

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

COMMON FIXED POINTS AND INVARIANT APPROXIMATIONS FOR NONCOMMUTING CONTRACTION MAPPINGS IN STRONGLY CONVEX METRIC SPACES

SUMIT CHANDOK, JIN LIANG*, AND DONAL O'REGAN

Abstract. In this paper, we obtain some new common fixed point theorems for noncommuting contraction mappings, which generalize some known results.

1. Introduction and preliminaries

Common fixed point theorems for various nonlinear operators were investigated extensively in recent years (cf., e.g., [1, 2, 5, 8, 11, 14–16, 18]). Stimulated by these studies, in this paper we establish some new common fixed point theorems for noncommuting contraction mappings. First, we recall some basic concepts and notations.

For a metric space (X, d), a continuous mapping $W : X \times X \times [0, 1] \to X$ is said to be a **convex structure** on X if for all $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

holds for all $u \in X$. The metric space (X, d) together with a convex structure is called a **convex metric space** ([17]).

A metric space (X, d) is said to be **strongly convex** (see [13]) if for each pair of points $x, y \in X$ and every $\lambda \in [0, 1]$, there exists exactly one point $z \in X$ such that $d(x, z) = (1 - \lambda)d(x, y)$ and $d(z, y) = \lambda d(x, y)$.

A convex metric space (X, d) is said to be **strongly convex** (see [12, 17]) if for each pair $x, y \in X$ and every $\lambda \in [0, 1]$, there exists exactly one point $z = W(x, y, \lambda) \in X$ such that $(d(x, z) =)d(x, W(x, y, \lambda)) = (1 - \lambda)d(x, y)$ and $(d(z, y) =)d(W(x, y, \lambda), y) = \lambda d(x, y)$.

Let (X, d) be a metric space and $x, y \in X$. We say that the point z is between x and y if $d(x, z) + d(z, y) = d(x, y)$. For any two points x, y of X, the set $\{z \in X : d(x, z) + d(z, y) = d(x, y)\}$ (i.e. the set of all those points which lie between x and y) is called the segment $s[x, y]$.

For a convex metric space (X, d), $x, y \in X$ and $0 \leq t \leq 1$, $d(x, y) = d(x, W(x, y, t)) + d(W(x, y, t), y)$ (see [17]-Proposition 3).

If (X, d) is a complete convex metric space, then each two distinct points x, y of X are joined by a segment $s[x, y]$ (see [3], p. 41). For a complete strongly convex metric spaces (X, d), each two distinct points x, y of X are joined by exactly one segment $s[x, y]$ (see [3], p. 49-50).

2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 47N20.
Key words and phrases. Common fixed point, noncommuting contraction mappings, convex metric space, starshaped set.
This work was supported by the NSF of China (11171210).
*Corresponding author.

A subset K of a metric space (X, d) is said to be convex (see [13]) if for every $x, y \in K$, any point between x and y is also in K i.e. for each x and y in K, the segment $s[x, y]$ lies in K.

A subset K of a convex metric space (X, d) is said to be a **convex set** ([17]) if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$. A set K is said to be **p-starshaped** ([9]) where $p \in K$, provided $W(x, p, \lambda) \in K$ for all $x \in K$ and $\lambda \in [0, 1]$ i.e. the segment

$$[p, x] = \{W(x, p, \lambda) : 0 \leq \lambda \leq 1\}$$

joining p to x is contained in K for all $x \in K$. K is said to be **starshaped** if it is p-starshaped for some $p \in K$.

Clearly, each convex set is starshaped but not conversely.

A convex metric space (X, d) is said to satisfy **Property (I)** ([9]) if for all $x, y, q \in X$ and $\lambda \in [0, 1]$,

$$d(W(x, q, \lambda), W(y, q, \lambda)) \leq \lambda d(x, y).$$

A normed linear space and each of its convex subsets are simple examples of convex metric spaces with W given by

$$W(x, y, \lambda) = \lambda x + (1 - \lambda) y$$

for $x, y \in X$ and $0 \leq \lambda \leq 1$. There are many convex metric spaces which are not normed linear spaces (see [9,17]). Property (I) is always satisfied in a normed linear space.

Example 1.1 ([7]). Consider a closed subset X of the unit ball $S = \{\|x\| = 1\}$ in a Hilbert space H, such that X has diameter $\delta(X) \leq \sqrt{2}$ and is geodesically connected, i.e., the point

$$W(x, y, \lambda) = \frac{\lambda x + (1 - \lambda) y}{\|\lambda x + (1 - \lambda) y\|}$$

lies in X whenever $x, y \in X$ and $\lambda \in [0, 1]$. The metric space we obtain by measuring distances in X through central angles, i.e., with the metric $d[x, y] = \cos^{-1}(x, y)$ for every $x, y \in X$, turns out be a convex metric space (whose convex sets are exactly the goedesically connected subsets of X).

Example 1.2 ([12]). Let (X, ρ) be a closed ball of $S_{2,r}$ ($S_{2,r}$ is the 2-dimensional spherical space of radius r) of radius ρ with

$$\frac{\pi r}{4} < \rho < \frac{\pi r}{2}.$$

Then X is a strongly convex metric space. The elements of this spherical space are all the ordered 3-tuples $x = (x_1, x_2, x_3)$ of real numbers with

$$x_1^2 + x_2^2 + x_3^2 = r^2.$$

Distance is defined for each pair of elements x, y to be the smallest non-negative number xy such that

$$\cos\left(\frac{xy}{r}\right) = \frac{x_1 y_1 + x_2 y_2 + x_3 y_3}{r^2}.$$

For a non-empty subset M of a metric space (X, d) and $x \in X$, an element $y \in M$ is said to be a **best approximant** to x or a **best M-approximant** to x if

$$d(x, y) = dist(x, M) = \inf\{d(x, y) : y \in M\}.$$

The set of all such $y \in M$ is denoted by $P_M(x)$.

For a convex subset M of a convex metric space (X, d), a mapping $g : M \to X$ is said to be **affine** if for all $x, y \in M$,

$$g(W(x, y, \lambda)) = W(gx, gy, \lambda), \quad \forall\ \lambda \in [0, 1].$$

The mapping g is said to be **affine with respect to** $p \in M$ if

$$g(W(x, p, \lambda)) = W(gx, gp, \lambda)$$

for all $x \in M$ and $\lambda \in [0, 1]$.

Suppose (X, d) is a metric space, M a nonempty subset of X, and S, T be self mappings of M. The mapping T is said to be

(i) S**-contraction** if there exists a $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(Sx, Sy)$;

(ii) S**-nonexpansive** if $d(Tx, Ty) \leq d(Sx, Sy)$ for all $x, y \in M$.

If S is the identity mapping, then T is called **contraction**, **nonexpansive** respectively in (i) and (ii).

A point $x \in M$ is a **common fixed (coincidence) point** of S and T if

$$x = Sx = Tx \quad (Sx = Tx).$$

The set of fixed points (respectively, coincidence points) of S and T is denoted by $F(S, T)$ (respectively, $C(S, T)$). In particular we write $F(T) = F(S, T)$ when S is the identity mapping, that is, $F(T)$ stands for the set of fixed points of the mapping T.

The ordered pair (S, T) of two self mappings of (X, d) is called a **Banach operator pair**, if the set $F(T)$ of fixed points of a mapping T is S-invariant, i.e., $S(F(T)) \subset F(T)$.

The pair (S, T) is said to be **commuting** on M if

$$STx = TSx, \quad \forall\ x \in M.$$

It is obvious that if a pair (S, T) is commuting, then it must be a Banach operator pair but not conversely.

In [10], Jr Gregus proved the following fixed point theorem in Banach spaces.

Theorem 1.3. *Let C be a closed convex subset of a Banach space X and T a self mapping of C satisfying*

$$\|Tx - Ty\| \leq a\|x - y\| + p\|Tx - x\| + p\|Ty - y\|$$

for all $x, y \in C$, where $0 < a < 1$, $p \geq 1$ and $a + 2p = 1$. Then T has a unique fixed point.

Later, many results which are closely related to the theorem above have appeared in the literature. In 2000, Ćirić [6] proved the following result which generalizes the theorem above with another contractive condition.

Theorem 1.4. *Let C be a closed convex subset of a complete convex metric space (X, d) and $T : C \to C$ a mapping satisfying, for all $x, y \in C$*

$$d(Tx, Ty) \leq a \max\{d(x, y), c[d(x, Ty) + d(y, Tx)]\} + b \max\{d(x, Tx), d(y, Ty)\}$$

where $0 < a < 1$, $a + b = 1$, $0 \leq c \leq \frac{4-a}{8-a}$. Then T has a unique fixed point.

The purpose of this paper is to prove common fixed point results for maps S and T without the assumptions of linearity or affinity of either T or S and when the underlying spaces are convex metric spaces. We prove some results on the existence of common fixed points for noncommuting mappings with some contractive conditions and as applications some results on best approximation are obtained. The results proved in this paper generalize some previous results in the literature.

2. Main results

2.1. Common fixed point theorems with Ćirić type contraction mappings.

In this section we prove some common fixed point theorems for noncommuting Ćirić type contraction mappings in convex metric spaces.

Theorem 2.1. *Let M be a convex subset of a complete strongly convex metric space (X, d), T, I self mappings of M and $T(F(I)) \subseteq F(I)$. Assume that T and I satisfy*

$$\begin{aligned} d(Tx, Ty) \leq a \max\{&d(Ix, Iy), c[d(Ix, Ty) + d(Iy, Tx)]\} \\ &+ b \max\{d(Tx, Ix), d(Ty, Iy)\} \end{aligned}$$

for all $x, y \in M$, where $0 < a < 1$, $a + b = 1$, and $0 \leq c \leq \frac{4-a}{8-a}$. If I is nonexpansive and $F(I)$ is nonempty then there is a unique common fixed point of T and I.

Proof. First we show that $F(I)$ is closed. Let x be a limit point of $F(I)$. Then there exists a sequence $< x_n >$ in $F(I)$ such that $< x_n > \to x$. Note that I is continuous so

$$Ix = I(\lim x_n) = \lim Ix_n = \lim x_n = x,$$

we have $x \in F(I)$ and hence $F(I)$ is closed. Now we show that $F(I)$ is convex. Let $x, y \in F(I)$, and $0 \leq t \leq 1$. Then $x, y \in M$ and so $W(x, y, t) \in M$ since M is convex. Since I is nonexpansive and X is a convex metric space we have

$$\begin{aligned} d(x, y) &= d(Ix, Iy) \\ &\leq d(Ix, IW(x, y, t)) + d(IW(x, y, t), Iy) \\ &\leq d(x, W(x, y, t)) + d(W(x, y, t), y) \\ &\leq (1 - t)d(x, y) + td(x, y) \\ &= d(x, y), \end{aligned}$$

and since $x, y \in F(I)$ we get

$$d(x, y) = d(Ix, IW(x, y, t)) + d(IW(x, y, t), Iy) = d(x, IW(x, y, t)) + d(IW(x, y, t), y),$$

and so $IW(x, y, t)$ lies on the segment joining x and y. For complete strongly convex metric spaces (X, d), each two distinct points x, y of X are joined by exactly one segment $s[x, y]$. For strongly convex metric spaces (X, d), each two distinct points $x, y \in X$ and every $\lambda \in [0, 1]$, there exists exactly one point $z \in X$ such that

$z = W(x, y, \lambda)$ with $d(x, z) = (1 - \lambda)d(x, y)$ and $d(y, z) = \lambda d(x, y)$ (so $d(x, y) = d(x, z) + d(z, y)$ i.e. $z \in s[x, y]$). Thus there exists a $\lambda \in [0, 1]$ with

$$IW(x, y, t) = W(x, y, \lambda).$$

We claim that $\lambda = t$. Since

$$d(IW(x, y, t), x) = d(W(x, y, \lambda), x) = (1 - \lambda)d(x, y)$$

and

$$d(IW(x, y, t), y) = d(W(x, y, \lambda), y) = \lambda d(x, y),$$

$$\lambda d(x, y) = d(IW(x, y, t), y) \geq d(x, y) - d(x, IW(x, y, t))$$

which implies

$$\begin{aligned}(1 - \lambda)d(x, y) &= d(x, IW(x, y, t)) = d(Ix, IW(x, y, t)) \\ &\leq d(x, W(x, y, t)) = (1 - t)d(x, y).\end{aligned}$$

This implies that $\lambda \geq t$. Similarly, we can prove that $\lambda d(x, y) \leq td(x, y)$, $\lambda \leq t$. Therefore $\lambda = t$ i.e. $IW(x, y, t) = W(x, y, t)$ and so $W(x, y, t) \in F(I)$. Hence $F(I)$ is convex. Further for $x, y \in F(I)$, we have

$$\begin{aligned}d(Tx, Ty) &\leq a \max\{d(Ix, Iy), c[d(Ix, Ty) + d(Iy, Tx)]\} \\ &\quad + b \max\{d(Tx, Ix), d(Ty, Iy)\} \\ &= a \max\{d(x, y), c[d(x, Ty) + d(y, Tx)]\} \\ &\quad + b \max\{d(Tx, x), d(Ty, y)\},\end{aligned}$$

so it follows from Theorem 1.4 (with $C = F(I)$) that T has a unique fixed point y in $F(I)$ and consequently $M \cap F(T) \cap F(I)$ is a singleton. □

Theorem 2.2. *Let M be a subset of a complete convex metric space (X, d), and S, T are self mappings of M. Suppose that $F(S)$ is nonempty, closed and convex and $T(F(S)) \subseteq F(S)$. If T satisfies*

$$\begin{aligned}d(Tx, Ty) &\leq a \max\{d(Sx, Sy), c[d(Sx, Ty) + d(Sy, Tx)]\} \\ &\quad + b \max\{d(Tx, Sx), d(Ty, Sy)\}\end{aligned}$$

for all $x, y \in M$, where $0 < a < 1$, $a + b = 1$, and $0 \leq c \leq \frac{4-a}{8-a}$, then there is a common fixed point of T and S.

Proof. For $x, y \in F(S)$, we have

$$\begin{aligned}d(Tx, Ty) &\leq a \max\{d(Sx, Sy), c[d(Sx, Ty) + d(Sy, Tx)]\} \\ &\quad + b \max\{d(Tx, Sx), d(Ty, Sy)\} \\ &= a \max\{d(x, y), c[d(x, Ty) + d(y, Tx)]\} \\ &\quad + b \max\{d(Tx, x), d(Ty, y)\}.\end{aligned}$$

From Theorem 1.4, (with $C = F(S)$) we deduce that T has a unique fixed point z in $F(S)$ and consequently $M \cap F(T) \cap F(S)$ is a singleton. □

2.2. **Common fixed point theorems and invariant approximation.** In this section we prove the existence of some common fixed points of best approximations for noncommuting Ćirić type contraction mappings.

We begin the section with the following result.

Proposition 2.3. *If C is a convex subset of a convex metric space (X, d) and $x \in X$ then $P_C(x)$ is closed and convex.*

Proof. Let $y, z \in P_C(x)$ and $\lambda \in [0, 1]$. Note

$$\begin{aligned} d(x, W(y, z, \lambda)) &\leq \lambda d(x, y) + (1-\lambda) d(x, z) \\ &= \lambda dist(x, C) + (1-\lambda) dist(x, C) \\ &= dist(x, C) \\ &\leq d(x, W(y, z, \lambda)) \text{ as } W(y, z, \lambda) \in C. \end{aligned}$$

Therefore, $d(x, W(y, z, \lambda)) = dist(x, C)$ and so $W(y, z, \lambda) \in P_C(x)$. Thus $P_C(x)$ is convex and it is easy to see its closed. $\square$

The following result extends the corresponding theorems of [4], [5] and [14].

Theorem 2.4. *Let C be a convex subset of a complete strongly convex metric space (X, d), and $S, T : X \to X$ be mappings such that $u \in F(S) \cap F(T)$ for some $u \in X$ and $T(\partial C \cap C) \subseteq C$. Suppose that $D = P_C(u)$ and $F(S)$ are nonempty, S is nonexpansive on D, $T(F(S) \cap D) \subseteq F(S) \cap D$ and T satisfies*

$$d(Tx, Ty) \leq \begin{cases} d(Sx, Su) & , \ if \ y = u \\ M(x, y) & , \ if \ y \in D \end{cases} \qquad (*)$$

where

$$\begin{aligned} M(x, y) \quad = \quad & a \max\{d(Sx, Sy), c[dist(Sx, Ty) + dist(Sy, Tx)]\} + \\ & b \max\{dist(Sx, Tx), dist(Sy, Ty)\}, \end{aligned}$$

for all $x, y \in C$, $0 < a < 1$, $a + b = 1$, and $0 \leq c \leq \frac{4-a}{8-a}$. Then there is a common fixed point of T and S in $P_C(u)$.

Proof. Let $d(u, C) = 0$. Then $u \in D \cap F(S) \cap F(T)$ (note $u \in D$ since $d(u, u) = 0 = d(u, C)$).

Let $d(u, C) \neq 0$. Let $x \in D = P_C(u)$. For any $\lambda \in (0, 1)$, we have

$$d(W(u, x, \lambda), u) \leq \lambda d(u, u) + (1-\lambda) d(x, u) = (1-\lambda) d(x, u) < dist(u, C).$$

This implies that $W(y, u, \lambda) \notin M$ for any λ, $0 < \lambda < 1$. Therefore the open line segment $\{W(u, x, \lambda) : 0 < \lambda < 1\}$ and the set C are disjoint. Thus x is not in the interior of C and so $x \in \partial C \cap C$. Since $T(\partial C \cap C) \subset C$, Tx must be in C. Also since S is nonexpansive on D, $u \in F(T) \cap F(S)$, and from inequality (*), we have

$$d(Tx, u) = d(Tx, Tu) \leq d(Sx, Su) \leq d(x, u) = dist(u, C).$$

This implies that $Tx \in P_C(u)$. Consequently, $P_C(u)$ is T-invariant, S-invariant, closed and convex. Hence the result follows from Theorem 2.1 (with $M = D$). $\square$

Let $G_\circ$ denote the class of closed convex subsets containing a point $x_\circ$ of a convex metric space (X,d). For $M \in G_\circ$ and $p \in X$, let

$$M_p = \{x \in M : d(x, x_\circ) \leq 2d(p, x_\circ)\},$$

let

$$P_M(p) = \{x \in M : d(p, x) = dist(p, M)\}$$

be the set of best approximants to p in M, and

$$C_M^S(p) = \{x \in M : Sx \in P_M(p)\}.$$

Note $P_M(p) \subseteq M_p$ since if $x \in P_M(p)$ then

$$d(x, x_0) \leq d(x, p) + d(p, x_0) = dist(p, M) + d(p, x_0) \leq 2d(p, x_0).$$

The next theorem extend and generalize the corresponding results of [5] and [14].

Theorem 2.5. *Let S and T be self mappings of a complete strongly convex metric space (X,d), $u \in F(S) \cap F(T)$ and $M \in G_\circ$ such that $T(M_u) \subseteq S(M) \subseteq M$. Suppose that $cl(S(M_u))$ is compact, and T, S satisfy $d(Tx, u) \leq d(Sx, u)$, $d(Sx, u) \leq d(x, u)$ for all $x \in M_u$. Then*

i) *$P_M(u)$ is nonempty, closed and convex,*
ii) *$T(P_M(u)) \subseteq S(P_M(u)) \subseteq P_M(u)$, if $d(Sx, Su) = d(x, u)$ for all $x \in C_M^S(u)$, and*
iii) *there is a common fixed point of T and S in $P_M(u)$, if S is nonexpansive on $P_M(u)$, $F(S)$ is nonempty, $T(F(S) \cap P_M(u)) \subseteq F(S) \cap P_M(u)$ and T satisfies*

$$\begin{aligned} d(Tx, Ty) \leq \; & a \max\{d(Sx, Sy), c[dist(Sx, Ty) + dist(Sy, Tx)]\} + \\ & b \max\{dist(Sx, Tx), dist(Sy, Ty)\} \end{aligned}$$

for all $x, y \in P_M(u)$, $0 < a < 1$, $a + b = 1$, and $0 \leq c \leq \frac{4-a}{8-a}$.

Proof. If $u \in M$ then the result is clear. So assume that $u \notin M$. If $x \in M \backslash M_u$, then

$$d(x, x_\circ) > 2d(u, x_\circ)$$

and so

$$d(u, x) \geq d(x, x_\circ) - d(u, x_\circ) > d(u, x_\circ) \geq dist(u, M).$$

Thus

$$\alpha = dist(u, M_u) = dist(u, M) \leq d(u, x_\circ).$$

Since $cl(S(M_u))$ is compact, and the distance function is continuous, there exists $z \in cl(S(M_u))$ such that

$$\beta = dist(u, cl(S(M_u))) = d(u, z).$$

Hence

$$\begin{aligned} \alpha &= dist(u, M_u) \leq dist(u, cl(S(M_u))) \text{ as } T(M_u) \subseteq S(M) \subseteq M \Rightarrow clS(M_u) \subseteq M \\ &= \beta \\ &\leq dist(u, S(M_u)) \\ &\leq d(u, Sx) \\ &\leq d(u, x) \end{aligned}$$

for all $x \in M_u$. Therefore $\alpha \leq \beta \leq dist(u, M_u) = dist(u, M) = \alpha$ so $\alpha = \beta = dist(u, M)$, i.e.

$$dist(u, M) = dist(u, cl(S(M_u))) = d(u, z),$$

i.e. $z \in P_M(u)$ and so $P_M(u)$ is nonempty. The closedness and convexity of $P_M(u)$ follows from that of M. This proves (i).

To prove (ii) let $z \in P_M(u)$. Then since $u \in F(S)$, $z \in P_M(u) \subseteq M_u$ and $d(Sx, u) \leq d(x, u)$ for $x \in M_u$ we have

$$d(Sz, u) \leq d(z, u) = dist(u, M).$$

This implies that $Sz \in P_M(u)$ and so $S(P_M(u)) \subseteq P_M(u)$. Let $y \in T(P_M(u))$. Since $T(M_u) \subseteq S(M)$ and $P_M(u) \subseteq M_u$ (easy to check), there exists $z \in P_M(u)$ and $x_1 \in M$ such that $y = Tz = Sx_1$. Further we have (note $d(Tw, u) \leq d(Sw, u)$ and $d(Sw, u) \leq d(w, u)$ for $w \in M_u$)

$$d(Sx_1, u) = d(Tz, u) \leq d(Sz, u) \leq d(z, u) = dist(u, M).$$

Thus $Sx_1 \in P_M(u)$ and $x_1 \in C_M^S(u)$. Also, as $Sx_1 \in M$ and $dist(u, M) \leq d(Sx_1, u)$, it follows that $dist(u, M) = d(Sx_1, u)$. Since $d(Sx_1, Su) = d(x_1, u)$ and $u \in F(S)$ we have

$$d(x_1, u) = d(Sx_1, u) = dist(u, M),$$

$x_1 \in P_M(u)$ and $y = Sx_1 \in S(P_M(u))$. Hence $T(P_M(u)) \subseteq S(P_M(u))$ and so (ii) holds.

Proceeding as in Theorem 2.1 we can prove the convexity and closedness of $F(S)$. Hence the conclusion (iii) follows from Theorem 2.1 (with $M = P_M(u)$). □

2.3. **Common fixed point theorem with generalized contraction mappings.** In this section we prove a common fixed point theorem for noncommuting generalized contraction mappings.

We begin the section with the following result of Al-Thagafi and Shahzad [1] which will be used in the sequel.

Lemma 2.6 ([1, Lemma 3.1]). *Let C be a nonempty subset of a metric space (X, d) and $T : C \to C$. If $clT(C) \subseteq C$, $clT(C)$ is complete and*

$$d(Tx, Ty) \leq k \max\{d(x, y), d(Tx, x), d(Ty, y), d(Tx, y), d(Ty, x)\},$$

for all $x, y \in C$ and some $k \in [0, 1)$, then $F(T)$ is a singleton.

Lemma 2.7. *Let C be a nonempty subset of a metric space (X, d), T, f, g self mappings of C, $cl\ T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$. Suppose that $cl\ (T(C))$ is complete and T, f, g satisfy for all $x, y \in C$ and some $0 \leq k < 1$*

$$d(Tx, Ty) \leq k \max\{d(fx, gy), d(Tx, fx), d(Ty, gy), d(Tx, gy), d(Ty, fx)\}.$$

If $F(f) \cap F(g)$ is nonempty and $cl\ T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$, then there is a common fixed point of T, f and g.

Proof. Now $cl\ T(F(f) \cap F(g))$, being a closed subset of the complete set $cl\ T(C)$, is complete. Further for all $x, y \in F(f) \cap F(g)$, we have

$$\begin{aligned} d(Tx, Ty) &\leq k \max\{d(fx, gy), d(Tx, fx), d(Ty, gy), d(Tx, gy), d(Ty, fx)\} \\ &= k \max\{d(x, y), d(Tx, x), d(Ty, y), d(Tx, y), d(Ty, x)\} \end{aligned}$$

Hence T is a generalized contraction on $F(f) \cap F(g)$ and

$$cl\ T(F(f) \cap F(g)) \subseteq F(f) \cap F(g).$$

So by Lemma 2.6, T has a unique fixed point z in $F(f) \cap F(g)$ and consequently $F(T) \cap F(f) \cap F(g)$ is a singleton. □

Remark 2.8. i) If $f = g$, then Theorem 3.2 of Al-Thagafi and Shahzad [1] is a particular case of Lemma 2.7.

ii) Lemma 2.7 also generalizes Lemma 2.10 of Hussain [11].

The following result extends the corresponding results of [1, 4, 5, 11] and [14].

Theorem 2.9. *Let C be a nonempty subset of a convex metric space (X, d) with Property (I), and T, g and h be self mappings of C. Suppose that $F(g) \cap F(h)$ is nonempty, and cl $T(F(g) \cap F(h)) \subseteq F(g) \cap F(h)$, cl $(T(C))$ is compact, T is continuous and satisfies*

$$\begin{aligned} d(Tx, Ty) \quad \leq \quad & \max\{d(hx, gy), dist(hx, [q, Tx]), dist(gy, [q, Ty]), \\ & dist(hx, [q, Ty]), dist(gy, [q, Tx])\} \end{aligned} \tag{2.1}$$

for all $x, y \in C$, some $q \in F(g) \cap F(h)$. Further, if C and $F(g) \cap F(h)$ are q-starshaped, then T, g and h have a common fixed point.

Proof. For each $n \in \mathbb{N}$, define $T_n : C \to C$ by

$$T_n(x) = W(Tx, q, k_n), \quad \text{for each } x \in C,$$

where $\langle k_n \rangle$ is a sequence in $(0, 1)$ such that $k_n \to 1$; note each T_n is a self mapping since C is q-starshaped. Since

$$cl\ T(F(g) \cap F(h)) \subseteq F(g) \cap F(h)$$

and $F(g) \cap F(h)$ is q-starshaped, and $q \in F(g) \cap F(h)$ we have

$$T_n(F(g) \cap F(h)) \subseteq F(g) \cap F(h)$$

for each n. We now claim that

$$cl(T_n(F(g) \cap F(h))) \subseteq F(g) \cap F(h)$$

for each n. To see this let $y \in cl(T_n(F(g) \cap F(h)))$. Then there exists a sequence $\{y_m\}$ in $T_n(F(g) \cap F(h))$ with $y_m \to y$. We must show $y \in F(g) \cap F(h)$. Note $y_m = T_n(x_m) = W(Tx_m, q, k_n)$ for $x_m \in F(g) \cap F(h)$. The compactness of $clT(C)$ implies there is a subsequence $\{Tx_{m_i}\}$ with $Tx_{m_i} \to z$. Note $T(x_{m_i}) \in T(F(g) \cap F(h))$ so $z \in cl(T(F(g) \cap F(h))) \subseteq F(g) \cap F(h)$. Also $y_{m_i} = W(Tx_{m_i}, q, k_n)$ together with the continuity of W yields $y = W(z, q, k_n)$. Finally $z, q \in F(g) \cap F(h)$ and $F(g) \cap F(h)$ is q-starshaped guarantees that $y \in F(g) \cap F(h)$. Thus our claim is proved. Note

$$\begin{aligned} d(T_n x, T_n y) &= d(W(Tx, q, k_n), W(Ty, q, k_n)) \\ &\leq k_n d(Tx, Ty) \\ &\leq k_n \max\{d(hx, gy), dist(hx, [q, Tx]), dist(gy, [q, Ty]), \\ &\qquad dist(hx, [q, Ty]), dist(gy, [q, Tx])\} \\ &\leq k_n \max\{d(hx, gy), d(hx, T_n x), d(gy, T_n y), d(hx, T_n y), d(gy, T_n x)\} \end{aligned}$$

for all $x, y \in C$. As $cl\ (T(C))$ is compact, $cl\ (T_n(C))$ is compact for each n and hence complete. Now by Lemma 2.7, there exists $x_n \in C$ such that x_n is common fixed point of g, h and T_n for each n. The compactness of $cl(T(C))$ implies there exists a subsequence $\{Tx_{n_i}\}$ of $\{Tx_n\}$ such that $Tx_{n_i} \to z \in cl\ T(C)$. Since $\{Tx_n\}$ is a sequence in $T(F(g) \cap F(h))$, then $z \in cl\ T(F(g) \cap F(h)) \subseteq F(g) \cap F(h)$. Now, as $k_{n_i} \to 1$, we have

$$x_{n_i} = T_{n_i}x_{n_i} = W(Tx_{n_i}, q, k_{n_i}) = W(z, q, 1) \to z,$$

(note $d(z, W(z, q, 1)) \leq d(z, z)$) and since T is continuous, we have $Tz = z$ and hence $F(T) \cap F(g) \cap F(h) \neq \emptyset$. □

Remark 2.10. Notice (2.1) could be replaced by the less restrictive condition

$$d(Tx, Ty) \leq max\{d(dx, gy), d(hx, T_nx), d(gy, T_ny), d(hx, T_ny), d(gy, T_nx)\}$$

where q is as in the statement of Theorem 2.9 and T_n is as in the proof of Theorem 2.9.

Remark 2.11. We note that the assumption of linearity or affinity for the map f is necessary in almost all known results in the literature about common fixed points of mappings T, f under the conditions of commuting and noncommuting, weakly commuting, R-subweakly commuting, compatibility and Banach operator pair, but the results in this paper are independent of these. Moreover, the results in this paper improve the results of Chandok and Narang [5].

References

[1] M. A. Al-Thagafi and N. Shahzad, *Banach operator pairs, common fixed points, invariant approximations, and ∗-nonexpansive multimaps*, Nonlinear Analysis **69** (2008), 2733–2739.

[2] A. Amini-Harandi, A. P. Farajzadeh, D. O'Regan and R. P. Agarwal, *Coincidence point, best approximation, and best proximity theorems for condensing set-valued maps in hyperconvex metric spaces*, Fixed Point Theory Appl. 2008, Art. ID 543154, 8 pp.

[3] L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Clarendon Press, Oxford, 1943.

[4] S. Chandok and T. D. Narang, *Some common fixed point theorems for Banach operator pairs with applications in best approximation*, Nonlinear Anal. **73** (2010), 105–109.

[5] S. Chandok and T. D. Narang, *Common fixed points and invariant approximation for Banach operator pairs with Ćirić type nonexpansive mappings*, submitted.

[6] Lj. B. Ćirić, *On a generalization of Gregus fixed point theorem*, Czechoslovak Math. J. **50** (2000), 449–458.

[7] H. V. Machado, *A characterization of convex subsets of normed spaces*, Kodai Math. Sem. Rep. **25** (1973), 307–320.

[8] C. S. Ge, J. Liang, D. O'Regan and T. J. Xiao, *Random coincidence points and random fixed points of multifunctions in metric spaces*, Taiwanese J. Math. **13** (2009), 899–912.

[9] M. D. Guay, K. L. Singh and J. H. M. Whitfield, *Fixed point theorems for nonexpansive mappings in convex metric spaces*, Proc. Conference on nonlinear analysis, S.P.Singh and J. H. Bury (eds), Vol. 80, Marcel Dekker, New York, 1982, pp. 179–189.

[10] M. Gregus, Jr, *A fixed point theorem in Banach space*, Boll. Unione Mat. Ital. Sez. A Mat. Soc. Cult. **517** (1980), 193–198.

[11] N. Hussain, *Common fixed points in best approximation for Banach operator pairs with Ciric̀ type I-contractions*, J. Math. Anal. Appl. **338** (2008), 1351–1363.

[12] R. Khalil, *Best approximation in metric spaces*, Proc. Amer. Math. Soc. **103** (1988), 579–586.

[13] T. D. Narang, *Best approximation and strict convexity of metric spaces*, Arch. Math **17** (1981), 87–90.

[14] T.D. Narang and S. Chandok, *Common fixed points and invariant approximation of R-subweakly commuting maps in convex metric spaces*, Ukrainian Math. J. **62** (2010), 1367–1376.
[15] D. O'Regan and M. Zima, *Leggett-Williams theorems for coincidences of multivalued operators*, Nonlinear Anal. **68** (2008), 2879–2888.
[16] N. Shahzad, *Invariant approximations, generalized I-contractions, and R-subweakly commuting maps*, Fixed Point Theory Appl. **1** (2005), 79–86.
[17] W. Takahashi, *A convexity in metric space and nonexpansive mappings I*, Kodai Math. Sem. Rep. **22** (1970), 142–149.
[18] H. Zegeye and N. Shahzad, *A hybrid scheme for finite families of equilibrium, variational inequality and fixed point problems*, Nonlinear Anal. **74** (2011), 263–272.

Manuscript received Proof ,
revised ,

S. Chandok
Department of Mathematics, Dr. B.R. Ambedkar National Institute of Technology, Jalandhar-144011, India
E-mail address: chansok.s@gmail.com

Jin Liang
Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, People's Republic of China
E-mail address: jinliang@sjtu.edu.cn

Donal O'Regan
Department of Mathematics, National University of Ireland, Galway, Ireland
E-mail address: donal.oregan@nuigalway.ie