Journal of Nonlinear and Convex Analysis
Volume 15, Number 6, 2014, 1105–1111

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ASYMPTOTIC CONTRACTIONS OF BROWDER'S AND EDELSTEIN'S TYPE

SHAHRAM SAEIDI

Abstract. In this paper, we investigate asymptotic versions of some known contractive conditions on mappings in metric spaces. We present more general fixed point results for asymptotic contractions of Browder's type in a complete metric space and Edelstein's type in a general metric space.

1. Introduction

Let (M, d) be a metric space. For a mapping $T : M \to M$ and $x \in M$ the orbit starting at x is denoted by $\mathcal{O}(x) = \{x, Tx, T^2x, \dots, T^nx, \dots\}$. The set $\mathcal{O}(x, y)$ is the union of two orbits starting at x and y, $\mathcal{O}(x, y) = \mathcal{O}(x) \cup \mathcal{O}(y)$.

Inspired by Browder [4], Walter [14] obtained a result that may be stated as follows.

Theorem 1.1 ([14]). *Let M be a complete metric space and suppose $T : M \to M$ has bounded orbits. If there exists a continuous, increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ for which $\phi(t) < t$, for all $t > 0$, and for each $x \in M$ there exists $n(x) \in \mathbb{N}$ such that for all $n \geq n(x)$ and $y \in M$,*

$$d(T^nx, T^ny) \leq \phi(diam(\mathcal{O}(x, y))). \tag{1.1}$$

Then, there exists $v \in M$ such that $\lim_{n\to\infty} T^nx = v$, for each $x \in M$. If, in addition, T is continuous at v, then v is a unique fixed point of T.

We recall that a mapping $T : M \to M$ is said to be contractive if $d(Tx, Ty) < d(x, y)$, for all $x, y \in M$, $x \neq y$. It is worth pointing out that in the literature numerous examples could be found showing that completeness and boundedness of a metric space do not ensure the existence of fixed points of contractive mappings. However, it is well-known that contractive mappings always have fixed points in compact metric spaces.

The following classical result is due to Edelstein [6]:

Theorem 1.2 ([6]). *Let (M, d) be a metric space, $x \in M$ and suppose $T : M \to M$ is a contractive mapping for which $\{T^nx\}$ has some convergent subsequence, say $\{T^{n_i}x\}$. Then, $v = \lim_{i\to\infty} T^{n_i}x$ is a unique fixed point of T.*

Many authors have extended Edelstein's result in various ways (see e.g., [2, 8, 10]). Bailey [2] investigated a mapping $T : M \to M$ satisfying the following condition in the context of a compact metric space (M, d):

$$0 < d(x, y) \Longrightarrow \liminf_{n\to\infty} d(T^nx, T^ny) < d(x, y). \tag{1.2}$$

2010 *Mathematics Subject Classification.* 47H09, 47H10, 54E45, 54E50.
Key words and phrases. Asymptotic fixed point theory, contractions, metric space.

On the other hand, asymptotic versions of some classical fixed point theorems have been considered recently in, e.g., [1, 3, 7, 9, 11, 12].

In this paper, by providing a new approach, we prove more general asymptotic versions of Theorem 1.1 with weaker conditions. As a corollary, we replace the continuity of ϕ by upper semicontinuity from the right and remove the monotonicity condition in Theorem 1.1. Moreover, in the continuation of Edelstein' theorem, we present several modifications in the assumptions, in Bailey's point of view.

2. Asymptotic contractions of Browder's type

In this section, we give generalizations of Theorem 1.1. In preparation for our results, we first establish the following lemma.

Lemma 2.1. *Let M be a complete metric space, $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an upper semicontinuous function from the right satisfying $\phi(t) < t$ for all $t > 0$ and $T : M \to M$ with a bounded orbit, say $\mathcal{O}(x)$, such that, for each $y \in \mathcal{O}(x)$,*

$$\limsup_{n\to\infty} \sup_{k\in\mathbb{N}} \{d(T^n y, T^{n+k} y) - \phi(diam(\mathcal{O}(y)))\} \le 0. \tag{2.1}$$

Then, there exists $v \in M$ such that $\lim_{n\to\infty} T^n(x) = v$. If, in addition, T is continuous at v, then v is a fixed point of T.

Remark 2.2. Because $\mathcal{O}(y, T^k y)) = \mathcal{O}(y)$, it is easy to see that condition (2.1) is weaker than (1.1).

Proof. Note that

$$0 \le diam(\mathcal{O}(T^{m+1}x)) \le diam(\mathcal{O}(T^m x)), \quad \forall m \in \mathbb{N}.$$

So, $\mathcal{D} := \lim_{m\to\infty} diam(\mathcal{O}(T^m x))$ exists. If we show that $\mathcal{D} = 0$, then we have proved that $\{T^n(x)\}$ is Cauchy sequence.

Note that there exists a strictly increasing sequence $\{n_i\}$ and a sequence $\{k_i\}$ of natural numbers, for which

$$\lim_{i\to\infty} d(T^{n_i}x, T^{n_i+k_i}x) = \mathcal{D}. \tag{2.2}$$

Because ϕ a is an upper semicontinuous function from the right, given $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$\phi(diam(\mathcal{O}(T^{m_0}x))) < \phi(\mathcal{D}) + \frac{\varepsilon}{2}. \tag{2.3}$$

For $y_0 := T^{m_0}x$, we have

$$d(T^n y_0, T^{n+k} y_0) \le \phi(diam(\mathcal{O}(y_0))) + r_n(y_0), \quad \forall n, k \in \mathbb{N}, \tag{2.4}$$

where,

$$r_n(y_0) = \sup\{d(T^n y_0, T^{n+k} y_0) - \phi(diam(\mathcal{O}(y_0))) : \quad k \in \mathbb{N}\},$$

and by (2.1), $\limsup_{n\to\infty} r_n(y_0) \le 0$. So, we can determine an integer n_o such that

$$r_n(y_0) \le \frac{\varepsilon}{2}, \quad \forall n \ge n_0. \tag{2.5}$$

Hence, from (2.3), (2.4) and (2.5), we obtain

$$d(T^{n+m_0}x, T^{n+k+m_0}x) = d(T^n y_0, T^{n+k} y_0)$$

$$\begin{aligned} &\leq \phi(diam(\mathcal{O}(y_0))) + r_n(y_0) \\ &< \phi(\mathcal{D}) + \frac{\varepsilon}{2} + r_n(y_0) \\ &\leq \phi(\mathcal{D}) + \varepsilon \end{aligned}$$

for all $n \geq n_0$ and $k \in \mathbb{N}$. That is,

$$d(T^n x, T^{n+k} x) \leq \phi(\mathcal{D}) + \varepsilon, \qquad \forall n \geq n_0 + m_0 \text{ and } \forall k \in \mathbb{N}.$$

Then, by (2.2), we get $\mathcal{D} \leq \phi(\mathcal{D}) + \varepsilon$. Because ε is an arbitrary positive real number, we have $\mathcal{D} \leq \phi(\mathcal{D})$, which implies that $\mathcal{D} = 0$ (since $\phi(t) < t$ for all $t > 0$). Hence, $\{T^n(x)\}$ is Cauchy sequence. Since M is complete, $\lim_{n\to\infty} T^n(x) = v \in M$. □

Now, we are ready to get our main results of this section.

Theorem 2.3. *Let M be a complete metric space, $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an upper semicontinuous function from the right satisfying $\phi(t) < t$ for all $t > 0$ and $\phi_n : \mathbb{R}_+ \to \mathbb{R}_+$ a sequence of functions converging pointwise to ϕ. Let $T : M \to M$ be a mapping with a bounded orbit, say $\mathcal{O}(x)$, such that for each $y \in \mathcal{O}(x)$ there exists $n(y) \in \mathbb{N}$ such that for all $n \geq n(y)$ and $z \in \mathcal{O}(y)$,*

$$d(T^n y, T^n z) \leq \phi_n(diam(\mathcal{O}(y))). \tag{2.6}$$

Then, there exists $v \in M$ such that $\lim_{n\to\infty} T^n(x) = v$. If, in addition, T is continuous at v, then v is a fixed point of T.

Proof. It suffices to show that condition (2.1) of Lemma 2.1 holds. For $y \in \mathcal{O}(x)$ and $n \geq n(y)$, we have, using (2.6),

$$\begin{aligned} &\sup\{d(T^n y, T^{n+k} y) - \phi(diam(\mathcal{O}(y))) : \quad k \in \mathbb{N}\} \\ &\quad = \sup_{z \in \mathcal{O}(y)} \{d(T^n y, T^n z) - \phi(diam(\mathcal{O}(y)))\} \\ &\quad \leq \phi_n(diam(\mathcal{O}(y))) - \phi(diam(\mathcal{O}(y))). \end{aligned}$$

Now, because ϕ_n converges pointwise to ϕ, the above inequality implies (2.1). □

Theorem 2.4. *Let M be a complete metric space, $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an upper semicontinuous function from the right satisfying $\phi(t) < t$ for all $t > 0$ and $\phi_n : \mathbb{R}_+ \to \mathbb{R}_+$ a sequence of functions converging pointwise to ϕ. Suppose $T : M \to M$ has bounded orbits and for each $x \in M$ there exists $n(x) \in \mathbb{N}$ such that for all $n \geq n(x)$ and $y \in M$,*

$$d(T^n x, T^n y) \leq \phi_n(diam(\mathcal{O}(x, y))). \tag{2.7}$$

Then, there exists $v \in M$ such that $\lim_{n\to\infty} T^n(x) = v$, for each $x \in M$. If, in addition, T is continuous at v, then v is a unique fixed point of T.

Proof. Because (2.7) implies (2.6), it follows from Theorem 2.3 that $\lim_{n\to\infty} T^n(x)$ exists , for each $x \in M$. Assume that, for $x, y \in M$, $T^n x \to v$ and $T^n y \to u$. Then, by (2.7),

$$\begin{aligned} d(u, v) &= \lim_{n\to\infty} d(T^n x, T^n y) = \lim_{n\to\infty} d(T^n(T^m x), T^n(T^m y)) \\ &\leq \lim_{n\to\infty} \phi_n(diam(\mathcal{O}(T^m x, T^m y))) = \phi(diam(\mathcal{O}(T^m x, T^m y))), \quad \forall m \in \mathbb{N}. \end{aligned}$$

But, it is easy to verify that $diam(\mathcal{O}(T^m x, T^m y)) \downarrow d(u,v)$, as $m \to \infty$. Hence, by the upper semicontinuity of ϕ from the right, we obtain

$$d(u,v) \leq \limsup_{m\to\infty} \phi(diam(\mathcal{O}(T^m x, T^m y))) \leq \phi(d(u,v)).$$

Therefore, $d(u,v) = 0$. That is, for each $x \in M$, $\{T^n x\}$ converges to a fixed element $v \in M$. □

Compared to Walter's result, Theorem 2.4 has the merit of holding under a more general asymptotic contractive condition. Also, the continuity condition of $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is replaced by upper semicontinuity from the right.

In the following, we give an example illustrating that Theorem 2.4 is an essential generalization of Walter's result.

Example 2.5. Let $M = \mathbb{R}_+$ with the usual metric and let $\{\ell_n\}$ be an arbitrary sequence in the segment $(0,1)$. Now, define $T, \phi : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$T(x) = \phi(x) = [x] - \ell_{[x]}, \quad \text{for } x \geq 1,$$

and $T(x) = \phi(x) = 0$, for $0 \leq x < 1$. Then, it is easy to verify that T and ϕ satisfy the hypotheses in Theorem 2.4. That is,

(i) $\phi(t) < t$ for all $t > 0$;
(ii) ϕ is an upper semicontinuous function;
(iii) T has bounded orbits and $\lim_{n\to\infty} T^n(x) = 0$, for each $x \in \mathbb{R}_+$;
(iv) $|T^n x - T^n y| \leq \phi(diam(\mathcal{O}(x,y)))$, for all $x, y \in \mathbb{R}_+$.

But, ϕ is not a continuous function and Walter's theorem (Theorem 1.1) can not be applied here.

3. Asymptotic contractions of Edelstein's type

We recall that a mapping T from a metric space M into itself is called *weakly asymptotically nonexpansive* (see [3]) if it satisfies the condition

$$\limsup_{n\to\infty} d(T^n x, T^n y) \leq d(x,y), \quad \text{for each } x, y \in M.$$

It is worth mentioning that Tingley [13] has constructed an example of a bounded closed convex C in Hilbert space and a continuous but fixed point free $T : C \to C$ which satisfies $lim_n \|T^n x - T^n y\| = 0$.

At first, we present two lemmas which are essential to prove the main results of this section.

Lemma 3.1. *Let (M,d) be a metric space and $T : M \to M$ a weakly asymptotically nonexpansive mapping. Then, for all $x, y \in M$, the limit $\lim_{n\to\infty} d(T^n x, T^n y)$ exists.*

Proof. Let $x, y \in M$. Because T is weakly asymptotically nonexpansive, we have

$$\begin{aligned}\limsup_{n\to\infty} d(T^n x, T^n y) &= \limsup_{n\to\infty} d(T^{n+m} x, T^{n+m} y) \\ &\leq d(T^m x, T^m y), \quad \text{for all } m \geq 0.\end{aligned}$$

Consequently,

$$\limsup_{n\to\infty} d(T^n x, T^n y) \leq \liminf_{m\to\infty} d(T^m x, T^m y).$$

This completes the proof. □

Lemma 3.2. *Let (M,d) be a metric space, $x \in M$ and $T : M \to M$ a mapping for which $\omega(\mathcal{O}(x)) \neq \varnothing$, where $\omega(\mathcal{O}(x)) = \{y \in M : \lim_k d(T^{n_k}x, y) = 0,\ \text{for some } n_k \to \infty\}$. Suppose T is weakly asymptotically nonexpansive on $\mathcal{O}(x)$ and T^N is continuous on $\omega(\mathcal{O}(x))$, the cluster points of $\mathcal{O}(x)$, for some integer $N \geq 1$. Then,*

$$d(T^{kN}p, T^{(k+1)N}p) = d(q, T^N q), \quad \forall p, q \in \omega(\mathcal{O}(x)) \text{ and } \forall k \geq 0.$$

Proof. Because $T : \mathcal{O}(x) \to \mathcal{O}(x)$ is weakly asymptotically nonexpansive, it follows from Lemma 3.1 that the limit

$$\alpha := \lim_{n\to\infty} d(T^n x, T^{n+N}x)$$

exists. Hence, by continuity of T^N on $\omega(\mathcal{O}(x))$, we have

$$\alpha = d(q, T^N q), \quad \forall q \in \omega(\mathcal{O}(x)).$$

On the other hand, it follows by continuity that $T^{kN}p \in \omega(\mathcal{O}(x))$, for all $p \in \omega(\mathcal{O}(x))$ and $k \geq 1$. Therefore, the desired result follows. □

Lemma 3.3. *Let (M,d) be a metric space, $x \in M$ and $T : M \to M$ a mapping for which $\omega(\mathcal{O}(x)) \neq \varnothing$. Suppose, for some integer $N \geq 1$, T^N is continuous on $\omega(\mathcal{O}(x))$ and $T^N : \omega(\mathcal{O}(x)) \to \omega(\mathcal{O}(x))$ satisfies condition (1.2). If T is weakly asymptotically nonexpansive on $\mathcal{O}(x)$, then $\omega(\mathcal{O}(x)) = \{v\}$ for which v is a fixed point of T^N.*

Proof. Taking $S = T^N$, we have, from Lemma 3.2, that

$$d(S^k p, S^{(k+1)}p) = d(q, Sq), \quad \forall p, q \in \omega(\mathcal{O}(x)) \text{ and } \forall k \geq 0.$$

In particular,

$$d(S^k p, S^{(k+1)}p) = d(p, Sp), \quad \forall p \in \omega(\mathcal{O}(x)) \text{ and } \forall k \geq 0.$$

Hence, because S satisfies condition (1.2), it follows that

$$T^N p = Sp = p, \quad \forall p \in \omega(\mathcal{O}(x)).$$

We show that $\omega(\mathcal{O}(x))$ is singleton. Suppose, for contradiction, that $p, q \in \omega(\mathcal{O}(x))$ and $p \neq q$. Then, because $Sp = p$ and $Sq = q$, we get

$$d(p,q) = \liminf_{n\to\infty} d(S^n p, S^n q) < d(p,q),$$

a contradiction. Therefore $\omega(\mathcal{O}(x))$ is a singleton, say $\omega(\mathcal{O}(x)) = \{v\}$, and $T^N v = v$. □

Lemma 3.4. *Let (M,d) be a metric space, $x \in M$ and $T : M \to M$ a mapping for which $\omega(\mathcal{O}(x)) \neq \varnothing$. Suppose, for some integer $N \geq 1$, T^N is continuous on $\omega(\mathcal{O}(x))$ and the restriction of T^N to $X = \cup_{k=0}^{N} T^k(\omega(\mathcal{O}(x)))$ satisfies condition (1.2). If T is weakly asymptotically nonexpansive on $\mathcal{O}(x)$, then $\omega(\mathcal{O}(x)) = \{v\}$ for which $Tv = v$.*

Proof. By Lemma 3.3, it follows that $\omega(\mathcal{O}(x)) = \{v\}$ and $T^N v = v$. Hence, $X = \{v, Tv, T^2v, \dots, T^{N-1}v\}$ and $T(X) \subseteq X$. Suppose, for contradiction, that $Tv \neq v$. Then, by letting

$$0 < \delta = \min\{d(T^i v, T^j v) : \ 0 \leq i, j \leq N-1 \text{ and } T^i v \neq T^j v\},$$

we can find $0 \leq k < l \leq N-1$ for which $\delta = d(T^k v, T^l v)$. Then, it is easy to see that

$$d(T^k v, T^l v) = \delta \leq d(T^n(T^k v), T^n(T^l v)), \quad \forall n \in \mathbb{N}.$$

Thus,

$$d(T^k v, T^l v) \leq \liminf_{n\to\infty} d(T^{nN}(T^k v), T^{nN}(T^l v)) < d(T^k v, T^l v),$$

a contradiction. Therefore $Tv = v$. □

As a direct consequence of Lemma 3.4, we get the following theorem which is the main result of this section.

Theorem 3.5. *Let (M, d) be a metric space, $x \in M$ and $T : M \to M$ a mapping for which $\omega(\mathcal{O}(x)) \neq \varnothing$. Suppose some iterate of T is continuous and satisfies condition (1.2). If T is weakly asymptotically nonexpansive on $\mathcal{O}(x)$, then $\omega(\mathcal{O}(x)) = \{v\}$ for which v is a fixed point of an iterate of T.*

Corollary 3.6. *Let (M, d) be a metric space and $T : M \to M$ a mapping satisfying*

$$0 < d(x, y) \Longrightarrow \limsup_{n\to\infty} d(T^n x, T^n y) < d(x, y). \qquad (3.1)$$

Let, for some $x \in M$, $\omega(\mathcal{O}(x)) \neq \varnothing$ and suppose T^N is continuous on $\omega(\mathcal{O}(x))$, for some integer $N \geq 1$. Then, T has a unique fixed point v and $\lim_{n\to\infty} T^n x = v$.

Proof. By Lemma 3.4, T has a fixed point $v \in \omega(\mathcal{O}(x))$. Now, for any real number $\varepsilon > 0$, there exist $m_0 \in \mathbb{N}$ such that $d(T^{m_0}x, v) < \varepsilon$. Hence, by (3.1), we obtain

$$\limsup_{n\to\infty} d(T^n x, v) = \limsup_{n\to\infty} d(T^n(T^{m_0}x), v) \leq d(T^{m_0}x, v) < \varepsilon.$$

Because $\varepsilon > 0$ is arbitrary, it follows that $\lim_{n\to\infty} T^n x = v$. It suffices to prove the uniqueness. Suppose that $Tu = u$ and $u \neq v$. Then,

$$d(u, v) = \limsup_{n\to\infty} d(T^n u, T^n v) < d(p, q),$$

a contradiction. Therefore, T has a unique fixed point. □

Example 3.7. Let $T : c_0 \to c_0$ be the left shift operator defined by

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots).$$

One can easily see that T satisfies (3.1), but, it is not contractive.

Corollary 3.8. *Let (M, d) be a compact metric space and $T : M \to M$ a mapping satisfying (3.1). If T^N is continuous, for some integer $N \geq 1$, then T has a unique fixed point v in M. Moreover, for each $x \in M$, the sequence of Picard iterates, $\{T^n x\}$, converges in norm to v.*

Remark 3.9. Even in a complete M, one can find a map without fixed points, for which $\lim_{n\to\infty} d(T^n x, T^n y) = 0$, for any x, y in M: Let $M = [0, \infty)$ with the usual metric and $Tx = \ln(1 + e^x)$.

Acknowledgments

I would like to thank the referee for his/her thorough review and very useful comments and suggestions that helped improve the clarity of this paper.

References

[1] J. Anuradha and P. Veeramani, *Proximal pointwise contraction*, Topology Appl. **156** (2009) 2942–2948.
[2] D. F. Bailey, *Some theorems on contractive mappings*, J. London Math. Soc. **41** (1966), 101–106.
[3] E. M. Briseid, *Fixed points of generalized contractive mappings*, J. Nonlinear Convex Anal. **9** (2008), 181–204.
[4] F. Browder, *Remarks on fixed point theorems of contractive type*, Nonlinear Anal. **3** (1979), 657–661.
[5] T. D. Benavides and P. L. Ramirez, *Structure of the fixed point set and common fixed points of asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc **129** (2001) 3549–3557.
[6] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37** (1962), 74–79.
[7] J. Jachymski and I. Jozwik, *On Kirk's asymptotic contractions*, J. Math. Anal. Appl. **300** (2004), 147–159.
[8] M. A. Khamsi and W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, Pure and Applied Mathematics, Wiley-Interscience, New York, NY, USA, 2001.
[9] W. A. Kirk, *Fixed points of asymptotic contractions*, J. Math. Anal. Appl.**277** (2003), 645–650.
[10] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **224** (1977), 257–290.
[11] T. Suzuki, *A definitive result on asymptotic contractions*, J. Math. Anal. Appl. **335** (2007), 707–715.
[12] T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness*, Proc. Amer. Math. Soc. **136** (2008), 1861–1869.
[13] D. Tingley, *An asymptotically nonexpansive commutative semigroup with no fixed points*, Proc. Amer. Math. Soc. **97** (1986), 107–113.
[14] W. Walter, *Remarks on a paper by F. Browder about contraction*, Nonlinear Anal. **5** (1981), 21–25.

Manuscript received July 10, 2012
revised March 22, 2013

Shahram Saeidi
Department of Mathematics, University of Kurdistan, Sanandaj 416, Kurdistan, Iran
E-mail address: shahram_saeidi@yahoo.com, sh.saeidi@uok.ac.ir