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ATTRACTIVE POINT AND MEAN CONVERGENCE THEOREMS FOR SEMIGROUPS OF MAPPINGS WITHOUT CONTINUITY IN HILBERT SPACES

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Abstract. In this paper, using the theory of invariant means, we first prove an attractive point and fixed point theorem for commutative semigroups of mappings without continuity which generalizes theorems of Takahashi and Takeuchi [19] and Atsushiba and Takahashi [1] in a Hilbert space. We also obtain a mean convergence theorem of Baillon's type [2] for the semigroups of mappings without continuity. Using this result, we also prove their mean convergence theorems in a Hilbert space.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into itself. Then we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [19] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;
(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|,\ \forall x \in C\}$.

We know from [19] that $A(T)$ is closed and convex. This property is important. A mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We know that if C is a bounded, closed and convex subset of H and $T : C \to C$ is nonexpansive, then $F(T)$ is nonempty. Furthermore, from Baillon [2] we know the first nonlinear mean convergence theorem in a Hilbert space: Let C be a bounded, closed and convex subset of H and let $T : C \to C$ be nonexpansive. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$; see also [21]. An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping F is said to be *firmly nonexpansive* if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle$$

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for all $x, y \in C$; see, for instance, Browder [3] and Goebel and Kirk [5]. Recently, Kocourek, Takahashi and Yao [8] defined a broad class of generalized hybrid mappings containing nonexpansive mappings, nonspreading mappings [9,10] and hybrid mappings [18] in a Hilbert space. A mapping $T : C \to C$ is called *generalized hybrid* [8] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$; see also [12]. We call such a mapping an (α, β)-*generalized hybrid* mapping. Kocourek, Takahashi and Yao [8] proved a fixed point theorem for such mappings in a Hilbert space. Furthermore, they proved a nonlinear mean convergence theorem of Baillon's type [2] in a Hilbert space. In 2011, Takahashi and Takeuchi [19] proved the following fixed point and mean convergence theorem without convexity for generalized hybrid mappings in a Hilbert space.

Theorem 1.1. *Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a generalized hybrid mapping from C into itself. Let $\{v_n\}$ and $\{b_n\}$ be sequences defined by*

$$v_1 \in C, \quad v_{n+1} = Tv_n, \quad b_n = \tfrac{1}{n}\textstyle\sum_{k=1}^{n} v_k$$

for all $n \in \mathbb{N}$. If $\{v_n\}$ is bounded, then the following hold:

(1) *$A(T)$ is nonempty, closed and convex;*
(2) *$\{b_n\}$ converges weakly to $u_0 \in A(T)$, where $u_0 = \lim_{n\to\infty} P_{A(T)}v_n$ and $P_{A(T)}$ is the metric projection of H onto $A(T)$.*

Very recently, Atsushiba and Takahashi [1] defined the set of all common attractive points of a family of mappings of C into itself and then they proved an attractive point and mean convergence theorem for commutative semigroups of nonexpansive mappings in a Hilbert space.

In this paper, motivated by Takahashi and Takeuchi [19] and Atsushiba and Takahashi [1], we prove an attractive point and fixed point theorem for commutative semigroups of mappings without continuity which generalizes the attractive point theorems of [19] and [1]. We also obtain a mean convergence theorem of Baillon's type [2] for the semigroups of mappings without continuity. Using this result, we also prove mean convergence theorems of [19] and [1].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let A be a nonempty subset of H. We denote by $\overline{co}A$ the closure of the convex hull of A. In a Hilbert space, it is known that

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha \|x\|^2 + (1 - \alpha) \|y\|^2 - \alpha(1 - \alpha) \|x - y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$. Furthermore, we have that

$$2 \langle x - y, z - w\rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2 \tag{2.2}$$

for all $x, y, z, w \in H$; see [17]. From (2.2), we have that

$$\langle (x-y)+(x-w), y-w\rangle = \|x-w\|^2 - \|x-y\|^2 \tag{2.3}$$

for all $x, y, w \in H$. Let C be a nonempty subset of H. A mapping $T : C \to C$ is *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\|, \quad \forall x \in C,\ y \in F(T).$$

An $(\alpha,\ \beta)$-generalized hybrid mapping in Introduction is nonexpansive for $\alpha = 1$ and $\beta = 0$. It is *nonspreading* [9, 10] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is *hybrid* [18] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous mappings. For example, we can give the following example [6] of nonspreading mappings. Let H be a real Hilbert space. Set $E = \{x \in H : \|x\| \leq 1\}$, $D = \{x \in H : \|x\| \leq 2\}$ and $C = \{x \in H : \|x\| \leq 3\}$. Define a mapping $S : C \to C$ as follows:

$$Sx = \begin{cases} 0, & x \in D, \\ P_E x, & x \notin D, \end{cases}$$

where P_E is the metric projection of H onto E. Then the mapping S is a nonspreading mapping which is not continuous. Putting $x = u$ with $u = Tu$ in (1.1), we have that for any $y \in C$,

$$\alpha\|u - Ty\|^2 + (1-\alpha)\|u - Ty\|^2 \leq \beta\|u - y\|^2 + (1-\beta)\|u - y\|^2$$

and hence

$$\|u - Ty\| \leq \|u - y\|, \quad \forall u \in F(T),\ y \in C. \tag{2.4}$$

That is, a generalized hybrid mapping with a fixed point is quasi-nonexpansive. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see Itoh and Takahashi [7]. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \to z$. Since C is weakly closed, we have $z \in C$. Furthermore, from

$$\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \to 0,$$

z is a fixed point of T and so $F(T)$ is closed. Let us show that $F(T)$ is convex. For $x, y \in F(T)$ and $\alpha \in [0, 1]$, put $z = \alpha x + (1-\alpha)y$. Then, we have from (2.1) that

$$\begin{aligned} \|z - Tz\|^2 &= \|\alpha x + (1-\alpha)y - Tz\|^2 \\ &= \alpha\|x - Tz\|^2 + (1-\alpha)\|y - Tz\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\ &\leq \alpha\|x - z\|^2 + (1-\alpha)\|y - z\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\ &= \alpha(1-\alpha)^2\|x - y\|^2 + (1-\alpha)\alpha^2\|x - y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\ &= \alpha(1-\alpha)(1-\alpha+\alpha-1)\|x - y\|^2 \\ &= 0. \end{aligned}$$

This implies $Tz = z$. So, $F(T)$ is convex. The following result proved by Takahashi and Takeuchi [19] is also important.

Lemma 2.1. *Let H be a real Hilbert space, let C be a nonempty subset of H and let T be a mapping from C into H. Then $A(T)$ is a closed and convex subset of H.*

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For the proof of existence of a Banach limit and its other elementary properties, see [15].

3. Semitopological semigroups and invariant means

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S to S are continuous. In the case when S is commutative, we denote st by $s + t$. Let $B(S)$ be the Banach space of all bounded real-valued functions on S with supremum norm and let $C(S)$ be the subspace of $B(S)$ of all bounded real-valued continuous functions on S. Let μ be an element of $C(S)^*$ (the dual space of $C(S)$). We denote by $\mu(f)$ the value of μ at $f \in C(S)$. Sometimes, we denote by $\mu_t(f(t))$ or $\mu_t f(t)$ the value $\mu(f)$. For each $s \in S$ and $f \in C(S)$, we define two functions $l_s f$ and $r_s f$ as follows:

$$(l_s f)(t) = f(st) \quad \text{and} \quad (r_s f)(t) = f(ts)$$

for all $t \in S$. An element μ of $C(S)^*$ is called a *mean* on $C(S)$ if $\mu(e) = \|\mu\| = 1$, where $e(s) = 1$ for all $s \in S$. We know that $\mu \in C(S)^*$ is a mean on $C(S)$ if and only if

$$\inf_{s\in S} f(s) \le \mu(f) \le \sup_{s\in S} f(s), \quad \forall f \in C(S).$$

A mean μ on $C(S)$ is called *left invariant* if $\mu(l_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. Similarly, a mean μ on $C(S)$ is called *right invariant* if $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. A left and right invariant invariant mean on $C(S)$ is called an *invariant* mean on $C(S)$. If $S = \mathbb{N}$, an invariant mean on $C(S) = B(S)$ is a Banach limit on l^∞. The following theorem is in [15, Theorem 1.4.5].

Theorem 3.1 ([15])**.** *Let S be a commutative semitopological semigroup. Then there exists an invariant mean on $C(S)$, i.e., there exists an element $\mu \in C(S)^*$ such that $\mu(e) = \|\mu\| = 1$ and $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$.*

Let S be a semitopological semigroup. For any $f \in C(S)$ and $c \in \mathbb{R}$, we write

$$f(s) \to c, \quad \text{as} \quad s \to \infty_R$$

if for each $\varepsilon > 0$ there exists an $\omega \in S$ such that

$$|f(tw) - c| < \varepsilon, \quad \forall t \in S.$$

We denote $f(s) \to c$, as $s \to \infty_R$ by

$$\lim_{s\to\infty_R} f(s) = c, \quad \text{or} \quad \lim_{s} f(s) = c.$$

When S is commutative, we also denote $s \to \infty_R$ by $s \to \infty$.

Theorem 3.2 ([15]). *Let $f \in C(S)$ and $c \in \mathbb{R}$. If*

$$f(s) \to c, \quad \text{as} \quad s \to \infty_R,$$

then $\mu(f) = c$ for all right invariant mean μ on $C(S)$.

Theorem 3.3 ([15]). *If $f \in C(S)$ fulfills*

$$f(ts) \leq f(s), \quad \forall t, s \in S,$$

then

$$f(t) \to \inf_{w\in S} f(w), \quad \text{as} \quad t \to \infty_R.$$

Theorem 3.4 ([15]). *Let S be a commutative semitopological semigroup and let $f \in C(S)$. Then the following are equivalent:*

(i) $f(s) \to c, \quad$ as $\quad s \to \infty$;

(ii) $\sup_w \inf_t f(t + w) = \inf_w \sup_t f(t + w) = c$.

Let H be a Hilbert space and let C be a nonempty subset of H. Let S be a semitopological semigroup and let $\mathcal{S} = \{T_s : s \in S\}$ be a family of mappings of C into itself. Then $\mathcal{S} = \{T_s : s \in S\}$ is called a *continuous representation* of S as mappings on C if $T_{st} = T_sT_t$ for all $s, t \in S$ and $s \mapsto T_sx$ is continuous for each $x \in C$. We denote by $F(\mathcal{S})$ the set of common fixed points of $T_s,\ s \in S$, i.e.,

$$F(\mathcal{S}) = \cap\{F(T_s) : s \in S\}.$$

A continuous representation $\mathcal{S} = \{T_s : s \in S\}$ of S as mappings on C is called a *nonexpansive semigroup* on C if each $T_s,\ s \in S$ is nonexpansive, i.e.,

$$\|T_sx - T_sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

The following definition [13] is crucial in the nonlinear ergodic theory of abstract semigroups. Let $u : S \to H$ be a continuous function such that $\{u(s) : s \in S\}$ is bounded and let μ be a mean on $C(S)$. Then there exists a unique point $z_0 \in \overline{co}\{u(s) : s \in S\}$ such that

$$\mu_s\langle u(s), y\rangle = \langle z_0, y\rangle, \quad \forall y \in H. \tag{3.1}$$

In fact, since $\{u(s) : s \in S\}$ is bounded and μ is a mean on $C(S)$, we can define a real-valued function g as follows:

$$g(y) = \mu_s\langle u(s), y\rangle, \quad \forall y \in H.$$

We have that for any $y, z \in H$ and $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} g(\alpha y + \beta z) &= \mu_s\langle u(s), \alpha y + \beta z\rangle \\ &= \alpha\mu_s\langle u(s), y\rangle + \beta\mu_s\langle u(s), z\rangle \\ &= \alpha g(y) + \beta g(z). \end{aligned}$$

Then g is a linear functional of H into $\mathbb{R}$. Furthermore we have that for any $y \in H$,

$$\begin{aligned} |g(y)| &= |\mu_s \langle u(s), y\rangle| \\ &\le \|\mu_s\| \sup_s |\langle u(s), y\rangle| \\ &\le \|\mu_s\| \sup_s \|u(s)\| \, \|y\| \\ &= (\sup_s |\|u(s)\|) \, \|y\|. \end{aligned}$$

Put $K = \sup_s \|u(s)\|$. We have that

$$|g(y)| \le K\|y\|, \quad \forall y \in H.$$

Then g is bounded. By the Riesz theorem, there exists $z_0 \in H$ such that

$$g(y) = \langle z_0, y\rangle, \quad \forall y \in H. \tag{3.2}$$

It is obvious that such $z_0 \in H$ is unique. Furthermore we have $z_0 \in \overline{co}\{u(s) : s \in S\}$. In fact, if $z_0 \notin \overline{co}\{u(s) : s \in S\}$, then there exists $y_0 \in H$ from the separation theorem such that

$$\langle z_0, y_0\rangle < \inf \big\{\langle z, y_0\rangle : z \in \overline{co}\{u(s) : s \in S\}\big\}.$$

Using the property of a mean, we have that

$$\begin{aligned} \langle z_0, y_0\rangle &< \inf \big\{\langle z, y_0\rangle : z \in \overline{co}\{u(s) : s \in S\}\big\} \\ &\le \inf\{\langle u(s), y_0\rangle : s \in S\} \\ &\le \mu_s\langle u(s), y_0\rangle \\ &= \langle z_0, y_0\rangle. \end{aligned}$$

This is a contradiction. Thus we have $z_0 \in \overline{co}\{u(s) : s \in S\}$. We call such z_0 the *mean vector* of u for μ. In particular, if $\mathcal{S} = \{T_s : s \in S\}$ is a continuous representation of S as mappings on C such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$ and $u(s) = T_s x$ for all $s \in S$, then there exists $z_0 \in H$ such tat

$$\mu_s\langle T_s x, y\rangle = \langle z_0, y\rangle, \quad \forall y \in H.$$

We denote such z_0 by $T_\mu x$.

Motivated by Takahashi and Takeuchi [19], Atsushiba and Takahashi [1] defined the set $A(\mathcal{S})$ of all common attractive points of a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings of C into itself, i.e.,

$$A(\mathcal{S}) = \cap\{A(T_s) : s \in S\}.$$

A net $\{\mu_\alpha\}$ of means on $C(S)$ is said to be *asymptotically invariant* if for each $f \in C(S)$ and $s \in S$,

$$\mu_\alpha(f) - \mu_\alpha(l_s f) \to 0 \quad \text{and} \quad \mu_\alpha(f) - \mu_\alpha(r_s f) \to 0.$$

See [4] and [15] for more details.

4. Mean vector and attractive point theorems

In this section, we first prove an important result for mean vectors in a Hilbert space. This result will be used in Section 5. Furthermore, using mean vectors, we prove attractive point and fixed point theorems for commutative semigroups of mappings without continuity in a Hilbert space.

Theorem 4.1. *Let S be a semitopological semigroup and let $C(S)$ be the Banach space of all bounded real-valued continuous functions on S with supremum norm. Let D be a nonempty, closed and convex subset of a Hilbert space H. Let $u : S \to H$ be a continuous function such that $\{u(s) : s \in S\} \subset D$ is bounded and let μ be a mean on $C(S)$. If $g : D \to \mathbb{R}$ is defined by*

$$g(z) = \mu_s \|u(s) - z\|^2, \quad \forall z \in D,$$

then g is continuous and there exists a unique $z_0 \in D$ such that

$$g(z_0) = \min\{g(z) : z \in D\}.$$

Furthermore, such z_0 is the mean vector of $\{u(s) : s \in S\}$ for μ.

Proof. For a bounded set $\{u(s)\} \subset D$ and a mean μ on $C(S)$, we know from [16] that a function $g : D \to \mathbb{R}$ defined by

$$g(z) = \mu_s \|u(s) - z\|^2, \quad \forall z \in D$$

is continuous. We also know from (3.1) that there exists the mean vector z_0 of $\{u(s)\}$ for μ, that is, there exists $z_0 \in \overline{co}\{u(s) : s \in S\}$ such that

$$\mu_s \langle u(s), y\rangle = \langle z_0, y\rangle, \quad \forall y \in H.$$

Since D is closed and convex and $\{u(s)\} \subset D$, we have $z_0 \in D$. Furthermore we have that for any $z \in D$,

$$\begin{aligned} g(z) - g(z_0) &= \mu_s \|u(s) - z\|^2 - \mu_s \|u(s) - z_0\|^2 \\ &= \mu_s (\|u(s) - z\|^2 - \|u(s) - z_0\|^2) \\ &= \mu_s \big(\|u(s)\|^2 - 2\langle u(s), z\rangle + \|z\|^2 - (\|u(s)\|^2 - 2\langle u(s), z_0\rangle + \|z_0\|^2)\big) \\ &= \mu_s (-2\langle u(s), z\rangle + \|z\|^2 + 2\langle u(s), z_0\rangle - \|z_0\|^2) \\ &= -2\langle z_0, z\rangle + \|z\|^2 + 2\langle z_0, z_0\rangle - \|z_0\|^2 \\ &= -2\langle z_0, z\rangle + \|z\|^2 + \|z_0\|^2 \\ &= \|z - z_0\|^2. \end{aligned}$$

Then we have that

$$g(z) = g(z_0) + \|z - z_0\|^2, \quad \forall z \in D.$$

This implies that z_0 is a unique point in D such that

$$g(z_0) = \min\{g(z) : z \in D\}.$$

This completes the proof. □

Using mean vectors, we can also prove an attractive point and fixed point theorem for commutative semigroups of mappings without continuity in a Hilbert space.

Theorem 4.2. *Let H be a Hilbert space and let C be a nonempty subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself. Let $\{T_s x : s \in S\}$ be bounded for some $x \in C$ and let μ be a mean on $C(S)$. Suppose that*

$$\mu_s \|T_s x - T_t y\|^2 \leq \mu_s \|T_s x - y\|^2, \quad \forall y \in C,\ t \in S. \tag{4.1}$$

Then $A(\mathcal{S})$ is nonempty. In addition, if C is closed and convex, then $F(\mathcal{S})$ is nonempty.

Proof. Since $\{T_s x\}$ is bounded, we have from (3.1) that there exists a unique point $z_0 \in \overline{co}\{T_s x : s \in S\}$ such that

$$\mu_s \langle T_s x, y\rangle = \langle z_0, y\rangle, \quad \forall y \in H. \tag{4.2}$$

Using this z_0, we have from (2.3), (4.2) and (4.1) that for any $v \in C$ and $t \in S$,

$$\begin{aligned} \langle (z_0 - v) + (z_0 - T_t v), v - T_t v\rangle &= \mu_s \langle (T_s x - v) + (T_s x - T_t v), v - T_t v\rangle \\ &= \mu_s (\|T_s x - T_t v\|^2 - \|T_s x - v\|^2) \\ &= \mu_s \|T_s x - T_t v\|^2 - \mu_s \|T_s x - v\|^2 \\ &\leq 0. \end{aligned} \tag{4.3}$$

Using (2.3) again, we have that

$$\langle (z_0 - v) + (z_0 - T_t v), v - T_t v\rangle = \|z_0 - T_t v\|^2 - \|z_0 - v\|^2.$$

We have from (4.3) that for all $v \in C$ and $t \in S$,

$$\|z_0 - T_t v\|^2 - \|z_0 - v\|^2 \leq 0$$

and hence $\|z_0 - T_t v\| \leq \|z_0 - v\|$. Therefore we have $z_0 \in A(\mathcal{S})$. Moreover, if C is closed and convex, we have from $\{T_s x\} \subset C$ that

$$z_0 \in \overline{co}\{T_s x : s \in S\} \subset C.$$

Since $z_0 \in A(\mathcal{S})$ and $z_0 \subset C$, we have that

$$\|T_t z_0 - z_0\| \leq \|z_0 - z_0\| = 0, \quad \forall t \in S$$

and hence $z_0 \in F(\mathcal{S})$. This completes the proof. □

Using Theorem 4.2, we can prove an attractive point theorem for generalized hybrid mappings obtained by Takahashi and Takeuchi [19] in a Hilbert space.

Theorem 4.3. *Let H be a Hilbert space, let C be a nonempty subset of H and let T be a generalized hybrid mapping of C into itself. Suppose that there exists an element $z \in C$ such that $\{T^n z\}$ is bounded. Then $A(T)$ is nonempty. In addition, if C is closed and convex, then $F(T)$ is nonempty.*

Proof. Since T is generalized hybrid, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. We have that for any $y \in C$ and $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} \alpha\|T^{n+1}z - Ty\|^2 &+ (1-\alpha)\|T^n z - Ty\|^2 \\ &\leq \beta\|T^{n+1}z - y\|^2 + (1-\beta)\|T^n z - y\|^2 \end{aligned}$$

for any $y \in C$. Since $\{T^n z\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. Then we have

$$\mu_n(\alpha\|T^{n+1}z - Ty\|^2 + (1-\alpha)\|T^n z - Ty\|^2) \\ \leq \mu_n(\beta\|T^{n+1}z - y\|^2 + (1-\beta)\|T^n z - y\|^2).$$

So we obtain

$$\alpha\mu_n\|T^{n+1}z - Ty\|^2 + (1-\alpha)\mu_n\|T^n z - Ty\|^2 \\ \leq \beta\mu_n\|T^{n+1}z - y\|^2 + (1-\beta)\mu_n\|T^n z - y\|^2$$

and hence

$$\alpha\mu_n\|T^n z - Ty\|^2 + (1-\alpha)\mu_n\|T^n z - Ty\|^2 \\ \leq \beta\mu_n\|T^n z - y\|^2 + (1-\beta)\mu_n\|T^n z - y\|^2.$$

This implies

$$\mu_n\|T^n z - Ty\|^2 \leq \mu_n\|T^n z - y\|^2$$

for all $y \in C$. If $S = \mathbb{N} \cup \{0\}$, we have from Theorem 4.2 that $A(T)$ is nonempty. Moreover, if C is closed and convex, then we have from Theorem 4.2 that $F(T)$ is nonempty. This completes the proof. □

Using Theorem 4.2, we have the attractive point theorem for commutative semigroups of nonexpansive mappings in a Hilbert space which was proved by Atsushiba and Takakashi [1].

Theorem 4.4. *Let H be a Hilbert space and let C be a nonempty subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a nonexpansive semigroup on C such that $\{T_s z : s \in S\}$ is bounded for some $z \in C$. Then $A(\mathcal{S})$ is nonempty. In addition, if C is closed and convex, then $F(\mathcal{S})$ is nonempty.*

Proof. Since $\mathcal{S} = \{T_s : s \in S\}$ is a nonexpansive semigroup on C, we have that

$$\|T_{t+s}x - T_t y\|^2 \leq \|T_s x - y\|^2$$

for all $x, y \in C$ and $s, t \in S$. Since $\{T_s z\}$ is bounded, we can apply an invariant mean μ to both sides of the inequality. Then we have that for any $y \in C$ and $t \in S$,

$$\mu_s\|T_{t+s}z - T_t y\|^2 \leq \mu_s\|T_s z - y\|^2$$

and hence

$$\mu_s\|T_s z - T_t y\|^2 \leq \mu_s\|T_s z - y\|^2.$$

We have from Theorem 4.2 that $A(\mathcal{S})$ is nonempty. Moreover, if C is closed and convex, then we have from Theorem 4.2 that $F(\mathcal{S})$ is nonempty. This completes the proof. □

5. Nonlinear ergodic theorems

In this section, we prove a mean convergence theorem for commutative semigroups of mappings without continuity in a Hilbert space. Before proving it, we need the following lemmas. We first prove the following result by using ideas of [11] and [20].

Lemma 5.1. *Let H be a Hilbert space and let C be a nonempty subset of H. Let S be a commutative semitopological semigroup with identity and let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $A(\mathcal{S}) \neq \emptyset$. Then, there exists the metric projection P of H onto $A(\mathcal{S})$. Furthermore, for any $x \in C$, $\lim_s PT_s x$ exists in $A(\mathcal{S})$, where $\lim_s PT_s x = q$ means $\lim_s \|PT_s x - q\| = 0$.*

Proof. We have that $A(\mathcal{S})$ is closed and convex. So, there exists the metric projection P of H onto $A(\mathcal{S})$. For an invariant mean μ on $C(S)$, there exists $q \in A(\mathcal{S})$ such that

$$\mu_t \langle PT_t x, y\rangle = \langle q, y\rangle, \quad \forall y \in H.$$

Then we have that for any $s \in S$,

$$\mu_t \langle PT_{t+s} x, y\rangle = \mu_t \langle PT_t x, y\rangle = \langle q, y\rangle, \quad \forall y \in H.$$

Thus we have that

$$q \in \overline{co}\{PT_{t+s}x : t \in S\}, \quad \forall s \in S. \tag{5.1}$$

From the property of the metric projection P, we know that

$$0 \leq \langle v - Pv, Pv - u\rangle, \quad \forall v \in H,\ u \in A(\mathcal{S}). \tag{5.2}$$

We have from (5.2) and (2.2) that

$$\begin{aligned} 0 &\leq 2\langle v - Pv, Pv - u\rangle \\ &= \|v-u\|^2 + \|Pv - Pv\|^2 - \|v - Pv\|^2 - \|Pv - u\|^2 \\ &= \|v-u\|^2 - \|v - Pv\|^2 - \|Pv - u\|^2. \end{aligned}$$

Hence we have that

$$\|Pv - u\|^2 \leq \|v-u\|^2 - \|v - Pv\|^2, \quad \forall v \in H,\ u \in A(\mathcal{S}). \tag{5.3}$$

Since $\|T_s z - u\|^2 \leq \|z-u\|^2$ for all $s \in S$, $u \in A(\mathcal{S})$ and $z \in C$, it follows that

$$\|T_{t+s}x - PT_{t+s}x\|^2 \leq \|T_{t+s}x - PT_s x\|^2 \leq \|T_s x - PT_s x\|^2. \tag{5.4}$$

Hence we have from (5.4) and Theorem 3.3 that

$$\|T_s x - PT_s x\|^2 \to \inf_{w\in S} \|T_w x - PT_w x\|^2, \quad \text{as } s \to \infty. \tag{5.5}$$

Putting $u = PT_s x$ and $v = T_{t+s}x$ in (5.3), we have that

$$\begin{aligned} \|PT_{t+s}x - PT_s x\|^2 &\leq \|T_{t+s}x - PT_s x\|^2 - \|T_{t+s}x - PT_{t+s}x\|^2 \\ &\leq \|T_s x - PT_s x\|^2 - \|T_{t+s}x - PT_{t+s}x\|^2 \\ &\leq \|T_s x - PT_s x\|^2 - \inf_{w\in S}\|T_w x - PT_w x\|^2. \end{aligned}$$

Using (5.1), we have that

$$\|q - PT_s x\|^2 \leq \|T_s x - PT_s x\|^2 - \inf_{w\in S}\|T_w x - PT_w x\|^2, \quad \forall s \in S.$$

Thus we have from (5.5) that

$$\|PT_sx - q\| \to 0, \quad \text{as } s \to \infty.$$

Therefore $\{PT_sx\}$ converges strongly to $q \in A(\mathcal{S})$. This completes the proof. □

Using the idea of [14], we have the following result.

Lemma 5.2. *Let H be a Hilbert space and let C be a nonempty subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself. Suppose that $\{T_sx : s \in S\}$ is bounded for some $x \in C$ and*

$$\mu_s\|T_sx - T_ty\|^2 \le \mu_s\|T_sx - y\|^2, \quad \forall y \in C,\ t \in S$$

for some invariant mean μ on $C(S)$. Then $\cap_s\overline{co}\{T_{t+s}x : t \in S\} \cap A(\mathcal{S})$ consists of one point z_0. Furthermore, $z_0 = \lim_s P_{A(\mathcal{S})}T_sx$, where $P_{A(\mathcal{S})}$ is the metric projection of H onto $A(\mathcal{S})$. In addition, if C is closed and convex, then $\cap_s\overline{co}\{T_{t+s}x : t \in S\} \cap F(\mathcal{S})$ consists of one point z_0.

Proof. From Theorem 4.1, a unique point $z_0 \in H$ such that

$$\mu_s\|T_sx - z_0\|^2 = \min\{\mu_s\|T_sx - y\|^2 :\ y \in H\}$$

is the mean vector of $\{T_sx : s \in S\}$ for the invariant mean μ, that is, a unique point $z_0 \in H$ such that

$$\mu_s\langle T_sx, y\rangle = \langle z_0, y\rangle, \quad \forall y \in H.$$

As in the proof of Theorem 4.2, we have $z_0 \in A(\mathcal{S})$. Furthermore, we have that

$$\mu_s\|T_sx - z_0\|^2 = \min\{\mu_s\|T_sx - y\|^2 :\ y \in A(\mathcal{S})\}.$$

Let us show that $z_0 \in \cap_s\overline{co}\{T_{t+s}x : t \in S\}$. If not, there exists some $s_0 \in S$ such that $z_0 \notin \overline{co}\{T_{t+s_0}x : t \in S\}$. By the separation theorem, there exists $y_0 \in H$ such that

$$\langle z_0, y_0\rangle < \inf\left\{\langle z, y_0\rangle : z \in \overline{co}\{T_{t+s_0}x : t \in S\}\right\}.$$

Using the property of an invariant mean, we have that

$$\begin{aligned}\langle z_0, y_0\rangle &< \inf\left\{\langle z, y_0\rangle : z \in \overline{co}\{T_{t+s_0}x : t \in S\}\right\}\\ &\le \inf\{\langle T_{t+s_0}x, y_0\rangle : t \in S\}\\ &\le \mu_t\langle T_{t+s_0}x, y_0\rangle\\ &= \mu_t\langle T_tx, y_0\rangle\\ &= \langle z_0, y_0\rangle.\end{aligned}$$

This is a contradiction. Thus we have that $z_0 \in \cap_s\overline{co}\{T_{t+s}x : t \in S\}$. Next we show that $\cap_s\overline{co}\{T_{t+s}x : s \in S\} \cap A(\mathcal{S})$ consists of one point z_0. Assume that $z_1 \in \cap_s\overline{co}\{T_{t+s}x : t \in S\} \cap A(\mathcal{S})$. Since $z_1 \in A(\mathcal{S})$, we have that

$$\|T_{t+s}x - z_1\|^2 \le \|T_sx - z_1\|^2, \quad \forall s, t \in S.$$

Then $\lim_s\|T_sx - z_1\|^2$ exists. In general, since $\lim_s\|T_sx - z\|^2$ exists for every $z \in A(\mathcal{S})$, we define a function $g : A(\mathcal{S}) \to \mathbb{R}$ as follows:

$$g(z) = \lim_s\|T_sx - z\|^2, \quad \forall z \in A(\mathcal{S}).$$

Since

$$\|z_0 - z_1\|^2 = \|T_s x - z_1\|^2 - \|T_s x - z_0\|^2 - 2\langle z_0 - z_1, T_s x - z_0\rangle$$

for every $s \in S$, we have

$$\begin{aligned}\|z_0 - z_1\|^2 + 2\lim_s\langle z_0 - z_1, T_s x - z_0\rangle &= \lim_s \|T_s x - z_1\|^2 - \lim_s \|T_s x - z_0\|^2 \\ &\geq 0.\end{aligned}$$

Let $\epsilon > 0$. Then we have

$$2\lim_s\langle z_0 - z_1, T_s x - z_0\rangle > -\|z_0 - z_1\|^2 - \epsilon.$$

Hence there exists $s_0 \in S$ such that

$$2\langle z_0 - z_1, T_{s+s_0} x - z_0\rangle > -\|z_0 - z_1\|^2 - \epsilon$$

for every $s \in S$. Since $z_1 \in \cap_s \overline{co}\{T_{t+s}x : t \in S\}$, we have

$$2\langle z_0 - z_1, z_1 - z_0\rangle \geq -\|z_0 - z_1\|^2 - \epsilon.$$

This inequality implies that $\|z_0 - z_1\|^2 \leq \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $z_0 = z_1$. Therefore

$$\{z_0\} = \cap_s \overline{co}\{T_{t+s}x : s \in S\} \cap A(\mathcal{S}).$$

We next show that $z_0 = \lim_s P_{A(\mathcal{S})} T_s x$. Since

$$\|T_{t+s}x - z\| \leq \|T_s x - z\|, \quad \forall s, t \in S,\ z \in A(\mathcal{S}),$$

we have from Lemma 5.1 that $\{P_{A(\mathcal{S})} T_s x\}$ converges strongly to some $u \in A(\mathcal{S})$. Since $P_{A(\mathcal{S})} T_s x \in A(\mathcal{S})$ for all $s \in S$, we have

$$\|P_{A(\mathcal{S})} T_s x - T_{t+s}x\| \leq \|P_{A(\mathcal{S})} T_s x - T_s x\|$$

for all $s, t \in S$. Furthermore, we have from the property of $P_{A(\mathcal{S})}$ that

$$\|P_{A(\mathcal{S})} T_s x - T_s x\| \leq \|z - T_s x\|$$

for all $z \in A(\mathcal{S})$. Thus

$$\|P_{A(\mathcal{S})} T_s x - T_{t+s}x\|^2 \leq \|P_{A(\mathcal{S})} T_s x - T_s x\|^2 \leq \|z - T_s x\|^2$$

for all $s, t \in S$ and $z \in A(\mathcal{S})$. Then we have that

$$\begin{aligned} g(P_{A(\mathcal{S})} T_s x) &= \lim_t \|P_{A(\mathcal{S})} T_s x - T_{t+s}x\|^2 \\ &\leq \|P_{A(T)} T_s x - T_s x\|^2 \\ &\leq \|z - T_s x\|^2. \end{aligned}$$

Since g is continuous and $P_{A(\mathcal{S})} T_s x \to u \in A(\mathcal{S})$, we have that

$$g(u) \leq \lim_s \|z - T_s x\|^2 = g(z), \quad \forall z \in A(\mathcal{S}).$$

Since z_0 is a unique minimizer of g in $A(\mathcal{S})$, we have $u = z_0$. Therefore

$$z_0 = \lim_s P_{A(\mathcal{S})} T_s x.$$

Moreover, if C is closed and convex, then we know that

$$z_0 \in \cap_s \overline{co}\{T_{t+s}x : t \in S\} \cap F(\mathcal{S}).$$

Since $\cap_s \overline{co}\{T_{t+s}x : s \in \mathcal{S}\} \cap A(\mathcal{S})$ consists of one point z_0, we have

$$\cap_s \overline{co}\{T_{t+s}x : s \in S\} \cap F(\mathcal{S}) = \{z_0\}.$$

This completes the proof. □

Lemma 5.3. *Let H be a Hilbert space and let C be a nonempty subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself. Suppose that $\{T_s x : s \in S\}$ is bounded for some $x \in C$ and*

$$\mu_s \|T_s x - T_t y\|^2 \le \mu_s \|T_s x - y\|^2, \quad \forall y \in C,\ t \in S \tag{5.6}$$

for all invariant means μ on $C(S)$. Let $\{\mu_\alpha\}$ be a net of means on $C(S)$ such that for each $f \in C(S)$ and $s \in S$, $\mu_\alpha(f) - \mu_\alpha(l_s f) \to 0$. If a subnet $\{T_{\mu_{\alpha_\beta}} x\}$ of $\{T_{\mu_\alpha} x\}$ converges weakly to a point $u \in H$, then $u \in A(\mathcal{S})$. In addition, if C is closed and convex, then $u \in F(\mathcal{S})$.

Proof. Since $\{\mu_\alpha\}$ is a net of means on $C(S)$, it has a cluster point μ in the weak* topology. We show that μ is an invariant mean on $C(S)$. In fact, since the set

$$\{\lambda \in C(S)^* : \lambda(e) = \|\lambda\| = 1\}$$

is closed in the weak* topology, it follows that μ is a mean on $C(S)$. Furthermore, for any $\varepsilon > 0$, $f \in C(S)$ and $s \in S$, there exists α_0 such that

$$|\mu_\alpha(f) - \mu_\alpha(l_s f)| \le \frac{\varepsilon}{3}, \quad \forall \alpha \ge \alpha_0.$$

Since μ is a cluster point of $\{\mu_\alpha\}$, we can choose $\beta \ge \alpha_0$ such that

$$|\mu_\beta(f) - \mu(f)| \le \frac{\varepsilon}{3} \quad \text{and} \quad |\mu_\beta(l_s f) - \mu(l_s f)| \le \frac{\varepsilon}{3}.$$

Hence we have

$$\begin{aligned} |\mu(f) - \mu(l_s f)| &\le |\mu(f) - \mu_\beta(f)| \\ &\quad + |\mu_\beta(f) - \mu_\beta(l_s f)| + |\mu_\beta(l_s f) - \mu(l_s f)| \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\mu(f) = \mu(l_s f), \quad \forall f \in C(S),\ s \in S.$$

Suppose that a subnet $\{T_{\mu_{\alpha_\beta}} x\}$ of $\{T_{\mu_\alpha} x\}$ converges weakly to some $u \in H$. If λ is a cluster point of $\{\mu_{\alpha_\beta}\}$ in the weak* topology, then λ is a cluster point of $\{\mu_\alpha\}$, too. Then λ is an invariant mean on $C(S)$. Without loss of generality, we may assume that $\mu_{\alpha_\beta} \rightharpoonup \lambda$ in the weak* topology. Furthermore, we have from $T_{\mu_{\alpha_\beta}} x \rightharpoonup u$ that

$$\lambda_s \langle T_s x, y\rangle = \lim_\beta\, (\mu_{\alpha_\beta})_s \langle T_s x, y\rangle = \lim_\beta\, \langle T_{\mu_{\alpha_\beta}} x, y\rangle = \langle u, y\rangle, \quad \forall y \in H.$$

On the other hand, we have from (2.2) that for $y \in C$ and $s, t \in S$,

$$2\langle T_s x - T_t y, y - T_t y\rangle - \|T_t y - y\|^2 = \|T_s x - T_t y\|^2 - \|T_s x - y\|^2.$$

Applying μ_{α_β} to both sides of the inequality, we have that

$$2(\mu_{\alpha_\beta})_s \langle T_s x - T_t y, y - T_t y\rangle - \|T_t y - y\|^2$$

$$= (\mu_{\alpha_\beta})_s\|T_sx - T_ty\|^2 - (\mu_{\alpha_\beta})_s\|T_sx - y\|^2.$$

Since $\mu_{\alpha_\beta} \rightharpoonup \lambda$, we have that

$$2\lambda_s\langle T_sx - T_ty, y - T_ty\rangle - \|T_ty - y\|^2 = \lambda_s\|T_sx - T_ty\|^2 - \lambda_s\|T_sx - y\|^2.$$

We have from (5.6) that

$$2\langle u - T_ty, y - T_ty\rangle - \|T_ty - y\|^2 = \lambda_s\|T_sx - T_ty\|^2 - \lambda_s\|T_sx - y\|^2 \leq 0.$$

Since $2\langle u - T_ty, y - T_ty\rangle - \|T_ty - y\|^2 = \|u - T_ty\|^2 - \|u - y\|^2$, we have that

$$\|u - T_ty\|^2 \leq \|u - y\|^2, \quad y \in C,\ t \in S. \tag{5.7}$$

This implies that $u \in A(T_t)$. Therefore $u \in A(\mathcal{S})$.

In particular, if C is closed and convex, then we have that u is an element of C. Putting $y = u$ in (5.7), we have $T_tu = u$. Therefore $u \in F(\mathcal{S}) = \cap\{F(T_t) : t \in S\}$. This completes the proof. □

Now we can prove the following mean convergence theorem for commutative families of mappings without continuity in a Hilbert space.

Theorem 5.4. *Let H be a Hilbert space and let C be a nonempty subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $A(\mathcal{S}) \neq \emptyset$. Suppose that*

$$\mu_s\|T_sx - T_ty\|^2 \leq \mu_s\|T_sx - y\|^2, \quad \forall x, y \in C,\ t \in S \tag{5.8}$$

for all invariant means μ on $C(S)$. Let $\{\mu_\alpha\}$ be a net of means on $C(S)$ such that for each $f \in C(S)$ and $s \in S$, $\mu_\alpha(f) - \mu_\alpha(l_sf) \to 0$. Then, $\{T_{\mu_\alpha}x\}$ converges weakly to a point $u \in A(\mathcal{S})$, where $u = \lim_s P_{A(\mathcal{S})}T_sx$. In addition, if C is closed and convex, then $u \in F(\mathcal{S})$, where $u = \lim_s P_{F(\mathcal{S})}T_sx$.

Proof. Since $A(\mathcal{S}) \neq \emptyset$, we have that $\{T_sx : s \in S\}$ is bounded for all $x \in C$. Fix $x \in C$. We have from (5.8) that for any invariant mean μ on $C(S)$,

$$\mu_s\|T_sx - T_ty\|^2 \leq \mu_s\|T_sx - y\|^2, \quad \forall y \in C\ t \in S.$$

We also know from Lemma 5.2 that $\cap_t\overline{co}\{T_{t+s}x : s \in S\} \cap A(\mathcal{S})$ consists of one point z_0 and $z_0 = \lim_s P_{A(\mathcal{S})}T_sx$, where $P_{A(\mathcal{S})}$ is the metric projection of H onto $A(\mathcal{S})$. To prove $T_\alpha x \rightharpoonup z_0 \in A(\mathcal{S})$, it is sufficient to show that if a subnet $\{T_{\alpha_\beta}x\}$ of $\{T_\alpha x\}$ converges weakly to a point $v \in H$, then $v \in A(\mathcal{S})$ and

$$v \in \cap_t\overline{co}\{T_{t+s}x : s \in S\}.$$

From Lemma 5.3, we have that $v \in A(\mathcal{S})$. Since $\{T_{\mu_{\alpha_\beta}}x\} \rightharpoonup v$, we also know that

$$\lambda_s\langle T_sx, y\rangle = \langle v, y\rangle, \forall y \in H$$

for some invariant mean λ on $C(S)$. Then $v \in \cap_t\overline{co}\{T_{t+s}x : s \in S\}$. Therefore $\{T_{\mu_\alpha}x\}$ converges weakly to z_0 of $A(\mathcal{S})$. Moreover, if C is closed and convex, then $z_0 \in C$ and hence $z_0 \in F(\mathcal{S})$. Therefore $\{T_{\mu_\alpha}x\}$ converges weakly to $z_0 \in F(\mathcal{S})$. To show $z_0 = \lim_s P_{F(\mathcal{S})}T_sx$, we may follow the proof of Lemma 5.1. This completes the proof. □

Using Theorem 5.4, we can prove the following mean convergence theorem obtained by Takahashi and Takeuchi [19].

Theorem 5.5. *Let H be a Hilbert space, let C be a nonempty subset of H and let T be a generalized hybrid mapping of C into itself such that $A(T)$ is nonempty. Then for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to $z_0 \in A(T)$, where $z_0 = \lim_{n\to\infty} P_{A(T)} T^n x$. In addition, if C is closed and convex, then $\{S_n x\}$ converges weakly to $z_0 \in F(T)$, where $z_0 = \lim_{n\to\infty} P_{F(T)} T^n x$.

Proof. Since $T : C \to C$ is generalized hybrid, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{5.9}$$

for all $x, y \in C$. Since $A(T)$ is nonempty, $\{T^n x\}$ is bounded for any $x \in C$. We know from the proof of Theorem 4.3 that for any Banach limits μ on l^∞,

$$\mu_n\|T^n x - Ty\|^2 \le \mu_n\|T^n x - y\|^2, \quad \forall y \in C.$$

Let $S = \{0\} \cup \mathbb{N}$. For any $f = (x_0, x_1, x_2, \dots) \in B(S)$, define

$$\mu_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} x_k, \quad \forall n \in \mathbb{N}.$$

Then $\{\mu_n : n \in \mathbb{N}\}$ is an asymptotically invariant sequence of means on $B(S)$; see [15, p.78]. Furthermore, we have that for any $x \in E$ and $n \in \mathbb{N}$,

$$T_{\mu_n} x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x.$$

Therefore, we have the desired result from Theorem 5.4. □

Using Theorem 5.4, we have a mean convergence theorem for commutative semigroups of nonexpansive mappings in a Hilbert space which was proved by Atsushiba and Takakashi [1].

Theorem 5.6. *Let H be a Hilbert space and let C be a nonempty subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a nonexpansive semigroup on C such that such that $A(\mathcal{S})$ is nonempty. Let $\{\mu_\alpha\}$ be a net of means on $C(S)$ such that for each $f \in C(S)$ and $s \in S$, $\mu_\alpha(f) - \mu_\alpha(l_s f) \to 0$. Then, $\{T_\alpha x\}$ converges weakly to a point $u \in A(\mathcal{S})$, where $u = \lim_s P_{A(\mathcal{S})} T_s x$. In addition, if C is closed and convex, then $u \in F(\mathcal{S})$, where $u = \lim_s P_{F(\mathcal{S})} T_s x$.*

Proof. Since $\mathcal{S} = \{T_s : s \in S\}$ is a nonexpansive semigroup on C, we have that

$$\|T_{t+s} x - T_t y\|^2 \le \|T_s x - y\|^2$$

for all $x, y \in C$ and $s, t \in S$. Since $\{T_s z\}$ is bounded, we can apply an invariant mean μ to both sides of the inequality. Then we have that for any $y \in C$ and $t \in S$,

$$\mu_s\|T_{t+s} z - T_t y\|^2 \le \mu_s\|T_s z - y\|^2$$

and hence

$$\mu_s\|T_s z - T_t y\|^2 \le \mu_s\|T_s z - y\|^2.$$

Therefore, we have the desired result from Theorem 5.4. □

Let H be a Hilbert space and let C be a nonempty subset of H. Let $S = \mathbb{R}^+ = \{t \in \mathbb{R} : 0 \le t < \infty\}$. Then a family $\mathcal{S} = \{S(t) : t \in \mathbb{R}^+\}$ of mappings of C into itself is called a *one-parameter nonexpansive semigroup* on C if $\mathcal{S}$ satisfies the following:

(1) $S(t+s)x = S(t)S(s)x, \quad \forall x \in C,\ t, s \in \mathbb{R}^+$;
(2) $S(0)x = x, \quad \forall x \in C$;
(3) for each $x \in C$, the mapping $t \mapsto S(t)x$ from $\mathbb{R}^+$ into C is continuous;
(2) for each $t \in \mathbb{R}^+$, $S(t)$ is nonexpansive.

Using Theorem 5.6, we have the following nonlinear ergodic theorem.

Theorem 5.7. *Let H be a Hilbert space and let C be a nonempty subset of H. Let $\mathcal{S} = \{S(t) : t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on C such that $A(\mathcal{S}) \neq \emptyset$. Then for any $x \in C$,*

$$S_\lambda x = \frac{1}{\lambda}\int_0^\lambda S(t)x dt$$

converges weakly to $z_0 \in A(\mathcal{S})$ as $\lambda \to \infty$, where $z_0 = \lim_{t\to\infty} P_{A(\mathcal{S})}S(t)x$. In addition, if C is closed and convex, then $z_0 = \lim_{t\to\infty} P_{F(\mathcal{S})}S(t)x$.

Proof. Let $S = \mathbb{R}^+$. For any $f \in C(\mathbb{R}^+)$, define

$$\mu_\lambda(f) = \frac{1}{\lambda}\int_0^\lambda f(t)dt, \quad \forall \lambda \in (0, \infty).$$

Then $\{\mu_\lambda : \lambda \in (0,\infty)\}$ is an asymptotically invariant net of means on $C(\mathbb{R}^+)$; see [15, p.80]. Furthermore, we have that for any $x \in E$ and $\lambda \in (0, \infty)$,

$$T_{\mu_\lambda} x = \frac{1}{\lambda}\int_0^\lambda S(t)x dt.$$

Therefore, we have the desired result from Theorem 5.6. □

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