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SOME PROPERTIES OF THE DEMAND CORRESPONDENCE IN THE CONSUMER THEORY

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Abstract. We consider some properties of the demand correspondence in the consumer theory such as nonemptiness of values, singlevaluedness, closedness. These results rely on a study of the passage from the utility function to the inverse utility function and on new notions of generalized concavity. They have obvious economic interpretations. Closedness for instance amounts to a stability of the demand when prices are subject to changes.

1. Introduction

We revisit the consumer theory, focusing our study on the properties of the inverse utility function and of the demand correspondence associated with an utility function u. In general the inverse utility function v is nonsmooth and the demand D is a multimap (a multivalued map or correspondence). Still, some generalized convexity properties can be detected. These properties can be used for an interpretation of the demand set $D(p)$ associated to a price p in terms of the normal cone to the sublevel set of the inverse utility function v. The present paper is devoted to such an interpretation and to the (mild) assumptions required to get it.

As several authors have pointed it out ([2], [3], [4], [6], [26]), the normal cone to the sublevel set of a quasiconvex function is an appropriate concept of generalized subdifferential of the function.

We leave to the companion paper [28] the study of subdifferentials and superdifferentials of u and v and the interpretation of the demand correspondence as a generalized derivative of the inverse utility function v. Such an interpretation requires a more sophisticated analysis than the one we present here.

Besides the interpretation of the demand set as a base of the opposite of the normal cone to the sublevel set of v, we deal with semicontinuity properties and generalized convexity properties of v in a very mild sense.

We give criteria for nonemptiness and single-valuedness of the demand set. We also briefly consider closedness properties and continuity properties of the demand correspondence. Obviously, they are of economical interest: if the demand set shrinks abruptly with a slight change of prices, part of the industrial production may be damaged.

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2. Preliminaries: from the utility function to the inverse utility function

We use simple concepts of nonsmooth analysis and variational analysis. Given a multimap (or correspondence) $F : X \rightrightarrows Y$ between two sets, we identify F with its graph $\mathrm{gph}F \subset X \times Y$ given by $\mathrm{gph}F := \{(x,y) : y \in F(x)\}$. The graph of the inverse F^{-1} of F (given by $F^{-1}(y) := \{x \in X : y \in F(x)\}$) is then identified with $F^{-1} := \{(y,x) \in Y \times X : (x,y) \in F\}$. The closure (resp. interior) of a subset S of a normed vector space X is denoted by $\mathrm{cl}(S)$ (resp. $\mathrm{int}(S)$). If C is a subset of X and if S is a subset of C we denote by $\mathrm{int}_C S$ the interior of S relatively to C, i.e., the set of $x \in S$ such that there exists $r > 0$ for which $B(x,r) \cap C \subset S$, $B(x,r)$ being the open ball with center x and radius r in X.

In the sequel, we represent the set of goods by a closed convex cone X_+ of a reflexive Banach space X. That means that we do not exclude the case of a continuum of goods. However, the reader looking for simplicity may suppose X_+ is the nonnegative orthant of $\mathbb{R}^n$. The set of *prices* is the subset

$$Y_+ := \{y \in X^* : \forall x \in X_+,\ y(x) \geq 0\} := (-X_+)^0$$

of the (topological) dual $Y := X^*$ of X. Whereas it is classical to assume that all goods have a positive price, occasionally we accept that some goods (such as air) have a null price and even that some goods have a negative price (such as waste, polluted materials). We normalize the prices by assuming that the income of the consumer is 1. Then, the *budget set* for a price $p \in Y_+$ is the set

$$B(p) := \{x \in X_+ : p.x \leq 1\},$$

where $p.x := \langle p, x \rangle := p(x)$ is the value of p at x. We assume that the preferences of the consumer are defined by a function $u : X_+ \to \overline{\mathbb{R}} := \mathbb{R}$ called an *utility function*, $x \in X_+$ being preferred to $x' \in X_+$ if, and only if $u(x) > u(x')$. The preferences of the modern consumer are determined by the *nuisance function* $w := -u$ the consumer wishes to minimize under his/her budget constraint (as chemical products, fat, sugar are prevalent in food). This function occurs in a symmetric presentation of the duality of the consumer theory. The *indirect utility function* v is defined by

$$v(p) := \sup\{u(x) : x \in B(p)\} \qquad p \in Y. \tag{2.1}$$

If u is not defined over X it may be convenient to extend u to X by setting $u(x) := -\infty$ for $x \in X \backslash X_+$, so that $v(p)$ is the supremum of u on X. One has a characterization of the *sublevel sets* of v as

$$[v \leq r] = \{p \in Y : px \leq 1 \Rightarrow u(x) \leq r\} = \{p \in Y : [u > r] \subset [p > 1]\}. \tag{2.2}$$

It follows that v is *quasiconvex,* i.e. that its sublevel sets are convex: given $r \in \mathbb{R}$, $p_0, p_1 \in [v \leq r]$ and $t \in [0,1]$, for $p := (1-t)p_0 + tp_1$ one has $[u > r] \subset [p_0 > 1] \cap [p_1 > 1] \subset [p > 1]$, so that $p \in [v \leq r]$. In fact, v is *evenly quasiconvex* in the sense that for all $r \in \mathbb{R}$ its sublevel set $[v \leq r]$ is the intersection over $x \in U(r) := [u > r]$ of the family Y_x of open half-spaces $Y_x := \{p : p.x > 1\}$. When we consider that prices must belong to Y_+, we use the fact that Y_+ is itself evenly convex as the intersection of the half-spaces $Y_{x,r} := \{p : p.x > -r\}$ for $x \in X_+$, $r \in \mathbb{P} :=]0, +\infty[$.

3. Semicontinuity properties of the inverse utility function

In general, v is not lower semicontinuous (l.s.c., in brief). The following criterion offers a sufficient lower semicontinuity condition.

Lemma 3.1. *If u is continuous along rays, or, more generally, if u is lower radially l.s.c. in the sense that for all $x \in X_+$ one has $u(x) \leq \liminf_{t \to 1_-} u(tx)$, then v is l.s.c. on Y_+ endowed either with its weak* topology or with its strong topology.*

The assumption means that for all $x \in X_+$ the consumer will be almost as satisfied in buying a slightly less quantity tx rather than x: in other terms, the consumer is supposed to be non capricious. Note that for a function u which is nondecreasing along rays, u is lower radially l.s.c. if and only if it is l.s.c. along rays.

Proof. Given $p \in Y_+$, $r \in \mathbb{R}$ and a sequence (or a net) $(p_n) \to p$ for the weak* topology such that $v(p_n) \leq r$ for all n, let us show that $v(p) \leq r$. If, on the contrary, $v(p) > r$, there exists some $x \in B(p) := [p \leq 1] \cap X_+$ such that $u(x) > r$. Suppose first that $p(x) = 0$. Then we have $p_n(x) \leq 1$ for n large enough, hence $v(p_n) \geq u(x) > r$, a contradiction. Thus $p(x) > 0$. Let $t_n := p_n(x)/p(x)$, so that $(t_n) \to 1$. Let us show that $N := \{n \in \mathbb{N} : t_n \leq 1\}$ is empty. For $n \in N$ one has $p_n(x) = t_n p(x) \leq 1$ hence $x \in B(p_n)$ and $v(p_n) \geq u(x) > r$, a contradiction with our assumption. Thus $t_n > 1$ for all $n \in \mathbb{N}$ and $(t_n^{-1}) \to 1_-$. Now, since $p_n(t_n^{-1}x) = p(x) \leq 1$, we have $u(t_n^{-1}x) \leq v(p_n) \leq r$. Then $u(x) \leq r$ by our assumption and again we reach a contradiction. □

When v is convex, the preceding lemma ensures that v is continuous on the interior of its domain whenever u is lower radially l.s.c.; without convexity, such a conclusion can be reached on the interior of Y_+ by combining the preceding lemma with the next proposition.

Proposition 3.2. *If u is weakly upper semicontinuous (u.s.c.) on X_+, then v is u.s.c. on the interior of Y_+.*

Proof. Let $p \in \text{int}Y_+$ and let $r > 0$ be such that $p + rB_Y \subset Y_+$, where B_Y is the closed unit ball of Y. First (for the sake of completeness, the fact being well known), let us show that $C := C_p := \{x \in X_+ : p.x = 1\}$ is a weakly compact base of X_+. Clearly, C is closed, convex and generates X_+ in the sense that $X_+ = \mathbb{R}_+ C$ since for all $x \in X_+ \backslash \{0\}$ we have $p.x > 0$ (as $p \in \text{int}Y_+$), hence $x = tw$ for $t := p.x$ and $w := x/t \in C$. For all $y \in Y$, setting $s := \|y\|$, $q := rs^{-1}y$, we have $p - q \in Y_+$, hence, for all $b \in C$ we get $(p - q).b \geq 0$ and

$$y.b = r^{-1}sq.b \leq r^{-1}sp.b = r^{-1}s.$$

The uniform boundedness theorem ensures that C is bounded, hence weakly compact, X being reflexive.

Now let $r \in \mathbb{R}$ and let $(p_n) \to p$ in Y (for the norm topology) be such that $v(p_n) \geq r$ for all n. For all $r' < r$, by definition of v, one can find $x_n \in X_+$ such that $u(x_n) > r'$ and $p_n.x_n \leq 1$. Let $b_n \in C$ and $t_n \in \mathbb{R}_+$ be such that $x_n := t_n b_n$. Taking subsequences, we may assume that (b_n) weakly converges to some $b \in C$. Then $(p_n.b_n) \to p.b = 1$ and $t_n p_n.b_n \leq 1$, so that (t_n) is bounded. Taking a further subsequence if necessary, we may assume that (x_n) weakly converges to some $x \in X_+$ and our upper semicontinuity assumption ensures that $u(x) \geq r'$. Since $p.x \leq \lim_n p_n.x_n \leq 1$, we get $v(p) \geq u(x) \geq r'$. Since r' is arbitrarily close to r, we obtain $v(p) \geq r$ and v is u.s.c. at p. □

The following characterization of inverse utility functions contains a slight supplement (assertion (b)) to [10] and [23, Thm 2.2] already detected in the comments of [23, Thm 2.2]. It sheds some light on the property considered in assertion (a) by showing it is not so exceptional. We present a simple proof for the convenience of the reader. Here $\mathrm{bd}(C)$ denotes the boundary of a subset C and $\overline{v}$ denotes the lower semicontinuous hull of the function v. It is given by $\overline{v}(p) = \liminf_{q \to p} v(q)$. We say that v is *nonincreasing* (or antitone) if $v(p) \leq v(q)$ when p, $q \in Y_+$ satisfy $p - q \in Y_+$. If $-u$ is nonincreasing, we say that u is *nondecreasing* (or homotone).

Proposition 3.3. *For a function $v : Y_+ \to \overline{\mathbb{R}}$, the following assertions are equivalent:*

(a) *v is nonincreasing, evenly quasiconvex and for all $p \in \mathrm{bd}(Y_+)$ one has $v(p) \leq \inf_{0<s<1} \overline{v}(sp)$;*

(b) *v is nonincreasing, evenly quasiconvex and for all $p \in Y_+$ one has $v(p) \leq \inf_{0<s<1} \overline{v}(sp)$;*

(c) *v is the inverse utility function of some utility function $u : X_+ \to \overline{\mathbb{R}}$.*

Moreover, one can take for u the function v^C given by

$$v^C(x) := \inf\{v(p) : p \in B^{-1}(x)\}.$$

In fact, v^C is the greatest utility function u whose inverse utility function is v : for all $x \in X_+$ and all $p \in B^{-1}(x)$ we have $v(p) = \sup\{u(x') : x' \in B(p)\} \geq u(x)$, whence $v^C(x) \geq u(x)$.

Proof. The implication (b)⇒(a) is obvious.

(c)⇒(b) Given an utility function u and its inverse utility function v, we already observed that v is evenly quasiconvex; it is clearly nonincreasing since for $p, q \in Y_+$ satisfying $q \in Y_+ + p$ we have $B(q) \subset B(p)$. For all $s \in]0, 1[$, $p \in Y_+$ and all $x \in B(p)$ we can find a neighborhood V of p such that $sq.x < 1$ for all $q \in V$. It follows that $v(sq) \geq u(x)$ for all $q \in V$, hence $\overline{v}(sp) \geq \inf_{q \in V} v(sq) \geq u(x)$. Taking the infimum on $s \in]0, 1[$, we obtain $\inf_{s \in]0,1[} \overline{v}(sp) \geq u(x)$ and $\inf_{s \in]0,1[} \overline{v}(sp) \geq \sup\{u(x) : x \in B(p)\} = v(p)$.

(a)⇒(c) Let us prove that v is the indirect utility function associated with $u := v^C$, i.e. that

$$\forall p \in Y_+ \qquad v(p) = \sup\{v^C(x) : x \in B(p)\}.$$

Fixing $p \in Y_+$, for all $x \in B(p)$ we have $p \in B^{-1}(x)$, hence $v^C(x) \leq v(p)$ so that $\sup\{v^C(x) : x \in B(p)\} \leq v(p)$. Let us show that for all $r < v(p)$ we can find $x_r \in B(p)$ such that $r \leq v^C(x_r)$. It suffices to find some $\overline{x}_r \in X$, $t > 0$ such that

$$(3.1) \qquad \forall q \in S(r) := [v \leq r] \qquad q.\overline{x}_r > t \geq p.\overline{x}_r.$$

In fact, since $S(r) + Y_+ \subset S(r)$, v being nonincreasing, relation (3.1) implies $\overline{x}_r \in X_+$, so that $x_r := \overline{x}_r/t \in B(p)$ and $[v \leq r] \subset Y_+ \backslash B^{-1}(x_r)$ or, equivalently, $v(q) > r$ for all $q \in B^{-1}(x_r)$, hence $v^C(x_r) \geq r$. Since (3.1) is satisfied with $t := 0$, $\overline{x}_r := 0$ when $S(r)$ is empty, we may suppose that $S(r)$ is nonempty. Let us first consider the case $r < \overline{v}(p)$. Then, we can find an open ball $U := B(p, \rho)$ centered at p such that $r < v(q)$ for all $q \in U$, so that $U \cap S(r) = \varnothing$. The Hahn-Banach separation theorem then yields $\overline{x}_r \in X\backslash\{0\}$ and $t' \in \mathbb{R}$ satisfying $q.\overline{x}_r \geq t' > p'.\overline{x}_r$ for all $p' \in U$ and all $q \in S(r)$ which is convex. Then, as noted above, $\overline{x}_r \in X_+$ and $t := t' - p.\overline{x}_r \geq \sup\{(p'-p).\overline{x}_r : p' \in U\} = \rho \|\overline{x}_r\| > 0$ and (3.1) holds. Now, let us consider the case $r \in [\overline{v}(p), v(p))$. Since $p \notin S(r)$ and $S(r)$ is evenly convex, we can find $\overline{x}_r \in X\backslash\{0\}$ and $t \in \mathbb{R}$ satisfying (3.1). Again, we have $\overline{x}_r \in X_+$, hence $p.\overline{x}_r \geq 0$. If $p \in \mathrm{int}Y_+$, since $\overline{x}_r$ is non null, this inequality is strict, so that $t > 0$. When $p \in \mathrm{bd}(Y_+) := Y_+\backslash\mathrm{int}Y_+$, condition (a) ensures that

$$v(2p) \leq \inf_{s\in]0,1[} \overline{v}(2sp) \leq \overline{v}(p) \leq r.$$

Then $2p \in S(r)$ and (3.1) entails the inequality $2p.\overline{x}_r > p.\overline{x}_r$ and $t \geq p.\overline{x}_r > 0$. □

Note that the passage from v to v^C is similar to the passage from an utility function u to the associated indirect utility function v since $-v^C(x) = \sup\{-v(p) : p.x \leq 1\}$ for all $x \in X_+$.

Corollary 3.4. *If $v : Y_+ \to \overline{\mathbb{R}}$ is nonincreasing, quasiconvex and lower semicontinuous, then v is the inverse utility function of v^C. In such a case, v^C is lower semicontinous on the interior of X_+. If moreover v is continuous along rays, then v^C is upper semicontinuous.*

Proof. Since v is quasiconvex and lower semicontinuous, it is evenly quasiconvex. Since $\overline{v} = v$ and v is nonincreasing, the inequality $v(p) \leq \inf_{0<s<1} \overline{v}(sp)$ holds.

Since the nuisance function $-v^C$ is given by $-v^C(x) := \sup\{-v(p) : p \in B^{-1}(x)\}$ and since $-v$ is upper semicontinuous and quasiconcave, hence weakly upper semicontinuous, Proposition 3.2 shows that $-v^C$ is upper semicontinuous on the interior of X_+.

If v is continuous along rays, Lemma 3.1 ensures that $-v^C$ is l.s.c., and then v^C is u.s.c.. □

It will be convenient to introduce a piece of terminology. A function $u : X_+ \to \mathbb{R}$ will be called a *Diewert function* if for all $x \in X_+$ one has

$$u(x) = v^C(x) := \inf\{v(p) : p \in B^{-1}(x)\}$$

where B and v are defined as above: $B^{-1}(x) := \{p \in Y_+ : p.x \leq 1\}$, $v(p) := \sup\{u(x) : x \in B(p)\}$. For any utility function u the function v^C is a Diewert function since v is the inverse utility function associated with v^C. Characterizations of Diewert functions have been given in [10] and [23] (see also [25]) while a first sufficient condition appeared in [16]. Changing signs in the passages from u to v and from v to v^C, we deduce from Proposition 3.3 the following characterization.

Proposition 3.5. *A function u on X_+ is a Diewert function if, and only if it is nondecreasing, evenly quasiconcave (i.e. $-u$ is evenly quasiconvex) and for all $x \in X_+$ one has $u(x) \geq \sup_{s\in]0,1[} \limsup_{w\to x} u(sw)$.*

Proposition 3.6. *The passage from an utility function to its inverse utility function is a bijection between the set of Diewert functions on X_+ onto the set of functions on Y_+ whose opposites are Diewert functions. This passage preserves suprema.*

4. The demand map and normal cones

Let us turn to a study of the demand correspondence $D(\cdot)$. For a price $p \in Y_+$ the demand set $D(p)$ is defined by

$$D(p) := \{x \in X_+ : u(x) = v(p)\} = \{x \in X_+ : u(x) \geq u(x') \ \forall x' \in B(p)\} \qquad p \in Y_+.$$

Let us first give a criterion ensuring the nonemptiness of $D(p)$.

Proposition 4.1. *Suppose u is weakly u.s.c. on X_+. Then, for every $p \in \mathrm{int} Y_+$, the demand set $D(p)$ is nonempty and weakly compact.*

If $p \in Y_+ \backslash \mathrm{int} Y_+$ and if one has $v(p) > \limsup_{x \in B(p), \|x\| \to \infty} u(x)$ then $D(p)$ is nonempty.

Proof. For $p \in \mathrm{int} Y_+$ we have seen in the proof of Proposition 3.2 that the base

$$C_p := \{x \in X_+ : p.x = 1\}$$

of X_+ is weakly compact. Thus the budget set $B(p) = [0,1]C_p$ is weakly compact too, so that the Weierstrass theorem ensures that u attains its maximum on $B(p)$: $D(p)$ is nonempty. Since u is weakly u.s.c., $D(p)$ is weakly closed. Since $D(p)$ is contained in the weakly compact set $B(p)$, it is weakly compact.

Now, let $p \in Y_+ \backslash \mathrm{int} Y_+$ and let $s < v(p)$, $s > \limsup_{x \in B(p), \|x\| \to \infty} u(x)$. Then $K := \{x \in B(p) : u(x) \geq s\}$ is bounded and nonempty. Since u is weakly u.s.c. K is weakly compact and again the Weierstrass theorem ensures that u attains its maximum on K. Since $\sup u(B(p)) = \sup u(K)$, u attains its maximum on $B(p)$. □

One says that the *non satiety condition* is satisfied if for any $(x,p) \in X_+ \times Y_+$ such that $p.x < 1$ there exists some $x' \in X_+$ such that $u(x') > u(x)$ and $p.x' \leq 1$. That means that for any such pair (x,p) one has $u(x) < v(p)$. The following characterization is obvious.

Lemma 4.2. *The* non satiety condition *is satisfied if, and only if for all $p \in Y_+$, one has $D(p) \subset H(p) := \{x \in X : p.x = 1\}$.*

Note that such a property is satisfied whenever u has no local maximizer on X_+, in particular when u is (strictly) increasing along rays.

Let us introduce, for $x \in X_+$, the set of *qualified prices at x* and the set of *proper prices at x* by

$$Q(x) := \{p \in Y_+ : \ p.x \leq 1, \ v(q) \geq v(p) \ \forall q \in B^{-1}(x)\}$$
$$P(x) := \{p \in Y_+ : p.x = 1, \ v(q) \geq v(p) \ \forall q \in B^{-1}(x)\},$$

so that $P(x) = Q(x) \cap H^{-1}(x)$, where $H^{-1}(x) := \{y \in Y : y.x = 1\}$. These sets of prices are sets of sensitive prices: passing from a price $p \in Q(x)$ to another price q, the maximum utility the consumer can get while buying the basket of goods x cannot decrease. Thus, the sets $P(x)$ and $Q(x)$ can be interpreted as the sets of worst prices in terms of satisfaction of the consumer wanting to buy a combination x of goods or another combination for which the budget constraint is satisfied. Changing u into $-u$, or v into $-v$, we see that the nature of $Q(\cdot)$ is similar to the one of $D(\cdot)$.

Proposition 4.3. *For all* $(x, p) \in X_+ \times Y_+$ *one has* $x \in D(p) \Rightarrow p \in Q(x)$*. If the* non satiety condition *is satisfied, then one has* $x \in D(p) \Rightarrow p \in P(x)$*.*

Proof. Let $x \in D(p)$. Then, for all $q \in B^{-1}(x)$, one has $v(q) \geq u(x) = v(p)$, hence $p \in Q(x)$. Moreover, the preceding lemma shows that $p.x = 1$ when the non satiety condition is satisfied. □

The reverse implication holds when u is a Diewert function since then the roles of u and $-v$ are symmetric.

Proposition 4.4. *If* u *is a Diewert function, then for all* $(x, p) \in X_+ \times Y_+$ *the equivalence* $p \in Q(x) \Leftrightarrow x \in D(p)$ *holds. If moreover the* non satiety condition *is satisfied,* P *is the inverse multimap of* D*.*

This result shows that for a Diewert utility function the demand correspondence is determined by the inverse utility function v.

Proof. Let $x \in X_+ \backslash D(p)$. Then there exists $x' \in B(p)$ satisfying $u(x') > u(x)$ and we have $v(p) \geq u(x') > u(x)$. Thus, when u is a Diewert function, we can find $q \in B^{-1}(x)$ such that $u(x') > v(q)$. Then, we get $v(p) \geq u(x') > v(q)$ and $p \notin Q(x)$. □

In order to give an interpretation of $P(x)$ in terms of normal cones, let us recall that the *normal cone* $N(C, p)$ in the sense of convex analysis to a subset C of Y at $p \in \mathrm{cl}(C)$ is the set

$$N(C, p) = \{x \in X : \forall q \in C \ \langle x, q - p \rangle \leq 0\}. \tag{4.1}$$

When C is convex, it is also the polar cone

$$(T(C, p))^0 := \{x \in X : \forall y \in T(C, p) \ \langle x, y \rangle \leq 0\}$$

of the *tangent cone* $T(C, p) := \mathrm{cl}(\mathbb{R}_+(C - p))$ to C at p. For a subset C of Y and $p \in Y$, one defines the *strict normal cone* to C at p by

$$N^<(C, p) := \{z \in X : q(z) < p(z) \ \forall q \in C \backslash \{p\}\}$$

and a similar definition can be given for a subset of X. A first use of this notion describes $Q(x)$.

Proposition 4.5. *Let* $x \in D(p)$ *for some* $p \in Y_+$*. Setting* $[u > u(x)] := \{x' \in X_+ : u(x') > u(x)\}$*, one has*

$$-p \in N^<([u > u(x)], x).$$

Proof. For all $x' \in X_+$ one has $u(x') \leq u(x)$ whenever x' satisfies $p.x' \leq 1$. Equivalently, for $x' \in X_+$ one has $p.x' > 1$ whenever x' satisfies $u(x') > u(x)$. Since $p.x \leq 1$, we see that $-p.(x' - x) < 0$. □

In order to describe $D(p)$, let us introduce for $p \in Y_+$ the *strict sublevel set* of v by

$$S^<(p) := [v < v(p)] := \{q \in Y_+ : v(q) < v(p)\}.$$

Now, for $x \in X_+$, $p \in B^{-1}(x)$, one has $p \in Q(x)$ if, and only if $S^<(p) \subset \{q \in Y_+ : q(x) > 1\}$. Thus, for $x \in X_+$, $p \in Y_+$,

$$x \in (-N^<(S^<(p), p)) \cap H(p) \Rightarrow p \in Q(x) \Rightarrow x \in -N^<(S^<(p), p) \tag{4.2}$$

or, for $x \in X_+$, $p \in Y_+$

$$p \in P(x) \Leftrightarrow x \in (-N^<(S^<(p), p)) \cap H(p). \tag{4.3}$$

Thus, when u is a Diewert function and the non satiety condition is satisfied one has

$$D(p) = (-N^<(S^<(p), p)) \cap H(p). \tag{4.4}$$

Let us note the following observation bearing on a comparison with the normal cone to the sublevel set of v.

Lemma 4.6. (a) *When the sublevel set* $S(p) := [v \leq v(p)]$ *is contained in the closure of* $S^<(p)$*, one has* $N^<(S^<(p), p) \subset N(S(p), p)$.

(b) *In particular, when the interior* $\mathrm{int}_{Y_+} S(p)$ *of* $S(p)$ *relatively to* Y_+ *is nonempty and contained in the closure of* $S^<(p)$*, one has* $N^<(S^<(p), p) \subset N(S(p), p)$.

(c) *When* $S^<(p)$ *is contained in* $\mathrm{int}S(p)$ *or when* $S^<(p) \subset \mathrm{int}_{Y_+} S(p)$ *and* X_+ *is pointed, one has* $N(S(p), p) \subset N^<(S^<(p), p) \cup \{0\}$.

(d) *When* $S^<(p) \subset \mathrm{int}S(p) \subset \mathrm{cl}(S^<(p))$ *and* $S^<(p)$ *is nonempty one has* $N(S(p), p) = N^<(S^<(p), p) \cup \{0\}$.

Proof. Assertion (a) follows from a passage to the limit. Assertion (b) is a consequence of the fact that when the interior $\mathrm{int}_{Y_+} S(p)$ of $S(p)$ relatively to Y_+ is nonempty and contained in the closure of $S^<(p)$, one has $S(p) \subset \mathrm{cl}(\mathrm{int}_{Y_+} S(p)) \subset \mathrm{cl}(S^<(p))$. Let us prove assertion (c). Let $x \in N(S(p), p) \backslash \{0\}$. By assumption, for every $q \in S^<(p)$, we have $q \in \mathrm{int}S(p)$, hence, for $q' \in Y$ close to q we have $q' \in S(p)$ and $(q' - p).x \leq 0$. Since x is non null, the Hahn-Banach Theorem ensures that there exists some $y \in Y$ such that $y.x > 0$. Then, for $t > 0$ small enough we have $(q + ty - p).x \leq 0$, hence $(q - p).x < 0$ and $x \in N^<(S^<(p), p)$. When $S^<(p) \subset \mathrm{int}_{Y_+} S(p)$ and X_+ is pointed, the Hahn-Banach Theorem ensures that one can pick $y \in Y_+$ such that $y.x > 0$ since $x \notin -X_+$. Then, for $t > 0$ small enough, we still have $(q + ty - p).x \leq 0$.

Assertion (d) is the conjunction of assertions (b) and (c). □

Remark. (a) In fact, the inclusion $N(S(p), p) \subset N^<(S^<(p), p) \cup \{0\}$ holds whenever $S^<(p)$ is contained in the algebraic interior

$$S(p)^i := \{q \in Y : \forall y \in Y \ \exists \varepsilon > 0 : q + [0, \varepsilon]y \subset S(p)\}$$

of $S(p)$. To see that, given $x \in N(S(p), p) \backslash \{0\}$ and $q \in S^<(p)$, we pick $y \in Y$ such that $y.x = 1$ and $\varepsilon > 0$ such that $q + [0, \varepsilon]y \subset S(p)$; then we have $(q - p).x + \varepsilon = (q + \varepsilon y - p).x \leq 0$, hence $(q - p).x < 0$.

(b) Let us note that the inclusion $S^<(p) \subset \mathrm{int}S(p)$ (resp. $S^<(p) \subset S(p)^i$) is satisfied if v is u.s.c. (resp. radially u.s.c. i.e. if its restriction to line segments is u.s.c.). □

The next definition is obtained as a splitting of a notion introduced in [14] under the name of geometric pseudoconvexity. It can be given for a function defined on a general topological space.

Definition 4.7. A real-valued function f is said to be *quasi-solid* (resp. *fully quasi-solid*) if, for every $r > \inf f$ (resp. $r \in \mathbb{R}$), the interior of the sublevel set $[f \leq r]$ is contained in the strict sublevel set $[f < r]$.

A real-valued function f is said to be flatfree (resp. fully flatfree) if for every $r > \inf f$ (resp. $r \in \mathbb{R}$) the sublevel set $[f \leq r]$ is contained in the closure of the strict sublevel set $[f < r]$.

Note that f is flatfree if, and only if all local minimizers of f are global minimizers; it is fully flatfree if f has no local minimizer. Similarly, f is quasi-solid if, and only if its local maximizers are global minimizers; it is fully quasi-solid if, and only if f has no local maximizer. When f is finitely valued with $\inf f = -\infty$ and $\sup f = +\infty$, f is flatfree if, and only if, $-f$ is quasi-solid. These properties which can be studied for any function on any topological space can be viewed as generalized convexity properties. Any convex function f is flatfree; it is also the case if f is *semi-strictly quasiconvex* in the sense that f is quasiconvex and that for every x_0, x_1 such that $f(x_1) < f(x_0)$ and every $t \in]0,1[$ one has $f(x_t) < f(x_0)$ for $x_t := (1-t)x_0 + tx_1$. We also observe that a semi-strictly quasiconvex function f is also quasi-solid: given $r > \inf f$ and $x_0 \in \operatorname{int}[f \leq r]$, picking $x_1 \in [f < r]$, we have $x_t := (1-t)x_0 + tx_1 \in [f \leq r]$ for some $t < 0$ and since $x_0 = (1-s)x_t + sx_1$ for $s := -t(1-t)^{-1} \in]0,1[$, when $f(x_1) < f(x_t)$ we get $f(x_0) < f(x_t) \leq r$, and when $f(x_t) \leq f(x_1)$ we have $f(x_0) \leq \max(f(x_t), f(x_1)) < r$. If f is quasi-solid and if the sublevel sets of f are strictly convex in the sense of the next definition, then f is *strictly quasiconvex*: given distinct points x_0, x_1, setting $r := \max(f(x_0), f(x_1))$, for all $t \in]0,1[$ one has $x_t \in \operatorname{int}[f \leq r] \subset [f < r]$. Conversely, if f is strictly quasiconvex and upper semicontinuous, then, for every $r > \inf f$ the sublevel set $[f \leq r]$ is strictly convex: given distinct $x_0, x_1 \in [f \leq r]$, for $t \in]0,1[$ we have $f(x_t) < \max(f(x_0), f(x_1)) \leq r$ and, since f is upper semicontinuous, we have $x_t \in \operatorname{int}[f \leq r]$.

Proposition 4.8. *If v is decreasing along rays, then v is* flatfree*,* quasi-solid *and for all $p \in Y_+$ one has $(-N(S(p),p)) \cap H(p) \subset D^C(p) \subset -N(S(p),p)$, where D^C is the demand correspondence associated with the utility function v^C. If moreover the non satiety condition is satisfied by v^C one has*

$$D^C(p) = (-N(S(p),p)) \cap H(p). \tag{4.5}$$

In particular, if u is a Diewert function, if the non satiety condition is satisfied, and if for q near p the set $D(q)$ is nonempty, then one has

$$D(p) = (-N(S(p),p)) \cap H(p). \tag{4.6}$$

Proof. If v is decreasing along rays, taking a sequence $(t_n) \to 1$ with $t_n > 1$ for all n, one has $t_n p \in S^<(p)$ for all n, so that $p \in \operatorname{cl}(S^<(p))$ and v is flatfree.

Now, let $r > \inf v$ and let $p \in \operatorname{int}[v \leq r]$. One cannot have $v(p) = r$ because otherwise, for $t \in]0,1[$ close enough to 1 one would have $v(tp) > v(p)$ and $tp \in S(p)$ by continuity of $r \mapsto rp$, a contradiction. Thus v is quasi-solid. The announced inclusions stem from Lemma 4.6.

If the non satiety condition is satisfied, if u is a Diewert function and if for $t > 1$ there exist some $x_t \in D(tp)$ one has $v(tp) < v(p)$: otherwise one would have $u(x_t) = v(tp) \geq v(p)$ and $tp.x_t = 1$ by the non satiety condition, hence $p.x_t < 1$, so that one could find some $x'_t \in B(p)$ with $u(x'_t) > u(x_t) \geq v(p)$, a contradiction. Thus $r \mapsto v(rp)$ is decreasing on a neighborhood of 1 and the preceding assertion applies. □

Corollary 4.9. *If u is increasing along rays, and if for q near p the set $D(q)$ is nonempty, then v is decreasing along rays and equality (4.6) holds.*

Proof. It suffices to observe that the assumptions imply the non satiety condition and ensure that $r \mapsto v(rp)$ is decreasing on a neighborhood of 1; in fact, given $s < t$ close to 1, picking $x_t \in D(tp)$ one has $sp.x_t < 1$, so that $v(sp) > u(x_t) = v(tp)$. □

Another assumption involving the following definition will ensure the inclusion $N(S(p), p) \subset N^<(S^<(p), p) \cup \{0\}$ and even the relation $N(S(p), p) = N^<(S(p), p) \cup \{0\}$, hence the equality

$$D(p) = (-N(S(p), p)) \cap H(p)$$

when u is a Diewert function and the non satiety condition holds. This definition is classical for the unit ball of a normed vector space.

Definition 4.10. A subset C of a n.v.s. Y with nonempty interior $\text{int}C$ is said to be strictly convex at $y \in C$ if it is convex and if for all $w \in C\backslash\{y\}$ there exists some $t \in]0,1[$ such that $(1-t)w + ty \in \text{int}C$. It is strictly convex if it is strictly convex at each point of C.

Note that if C is strictly convex, then, for all $t \in]0,1[$ and distinct points w, y in C one has $(1-t)w + ty \in \text{int}C$. Moreover, for any distinct points w, y in C one can find $r > 0$ such that $B((1-t)w + ty, 2rt) \subset C$ for all $t \in]0, 1/2[$. Thus, C is strictly convex if, and only if it is *strictly rotund* in the sense that any of its boundary points is an extremal point. Recall that $b \in C$ is an *extremal point* of a convex set C if one cannot find $t \in]0,1[$, $w, y \in C$ such that $b = (1-t)w + ty$. That notion is classical. The following proposition clarifies the relationships of strict convexity with strict normality in the sense that $N(C, y)\backslash\{0\} = N^<(C, y)$.

Proposition 4.11. *Let C be a convex subset of a n.v.s. Y with nonempty interior. Then C is strictly convex at $y \in C\backslash\text{int}C$ if, and only if for all $w \in C\backslash\{y\}$, $y^* \in N(C, y)\backslash\{0\}$ one has $\langle y^*, w - y\rangle < 0$, if, and only if*

$$N(C, y)\backslash\{0\} = N^<(C, y) := \{y^* \in Y^* : \forall w \in C\backslash\{y\}\ y^*(w) < y^*(y)\}.$$

Proof. Suppose C is strictly convex at $y \in C$. Given $w \in C\backslash\{y\}$, $y^* \in N(C, y)\backslash\{0\}$ one has $z := (1-t)w + ty \in \text{int}C$ for all $t \in]0,1[$, hence $\langle y^*, z' - y\rangle \le 0$ for all z' near z. Since $y^* \neq 0$, one must have $\langle y^*, z - y\rangle < 0$, hence $\langle y^*, w - y\rangle < 0$.

Conversely, suppose C is not strictly convex at $y \in C$: there exists some $w \in C\backslash\{y\}$ such that $]w, y[\cap \text{int}C = \varnothing$, where $]w, y[:= \{(1-t)w + ty : t \in]0,1[\}$. The geometric Hahn-Banach Theorem yields some $y^* \in Y^*\backslash\{0\}$, $r \in \mathbb{R}$ such that

$$\forall y \in \text{int}C,\ \forall z \in]w, y[\qquad \langle y^*, v\rangle < r \le \langle y^*, z\rangle.$$

Since w, y belong to the intersection of the closures of $\text{int}C$ and $]w, y[$, one gets $\langle y^*, w\rangle = r = \langle y^*, y\rangle$, hence $\langle y^*, w - y\rangle = 0$ and, since $C \subset \text{cl}(\text{int}C)$, we have $\langle y^*, v\rangle \le r$ for all $v \in C$, hence $y^* \in N(C, y)\backslash N^<(C, y)$. □

Example. The Poincaré half-space $C := \mathbb{R} \times \mathbb{R}_+$ has a nonempty interior but is not strictly convex. For $y := 0$ one has $N(C, y) = \{0\} \times \mathbb{R}_-$ and $N^<(C, y) = \varnothing$.

5. Uniqueness results

Uniqueness results can be deduced from the preceding results. Here we say that a convex subset C of Y is *smooth at* p if the normal cone to C at p is generated by a single element.

Proposition 5.1. *Suppose* $S(p) := [v \leq v(p)]$ *is smooth at* p*, the non satiety condition holds and* $S(p) \subset \mathrm{cl}(S^{<}(p))$*. Then* $D(p)$ *is either empty or a singleton.*

Proof. Suppose $S(p)$ is smooth at p and let $z \in X$ be such that $N(S(p), p) = \mathbb{R}_+ z$. Let $x_1, x_2 \in D(p)$, so that $p \in P(x_1)$ and $p \in P(x_2)$. Since the non satiety condition holds and $S(p) \subset \mathrm{cl}(S^{<}(p))$, relation (4.3), Lemma 4.6 and Proposition 4.3 yield some $\lambda_1, \lambda_2 \in \mathbb{R}_+$ such that $x_i = -\lambda_i z$ for $i = 1, 2$. Since $x_i \in H(p)$, we get $1 = p.x_i = -\lambda_i p.z$, so that $\lambda_1 = \lambda_2$ and $x_1 = x_2$. □

Corollary 5.2. *The same conclusion holds when* v *is differentiable at* p *with* $v'(p) \neq 0$ *and the non satiety condition is satisfied.*

Proof. Suppose v is differentiable at p with $v'(p) \neq 0$ and the non satiety condition is satisfied. Let $x \in D(p)$. Let $d_0 \in Y$ be such that $v'(p).d_0 < 0$. Then, for $t > 0$ small enough we have $v(p + td_0) < v(p)$, hence, by relation (4.3), $(p + td_0 - p).(-x) < 0$ or $d_0.x > 0$. Now, for any $d \in Y$ such that $v'(p).d \leq 0$ and every $n \in \mathbb{N}\backslash\{0\}$ we have $v'(p)(d + (1/n)d_0) < 0$, hence $(d + (1/n)d_0).x > 0$. Thus, taking limits, $d.x \geq 0$. The Farkas Lemma ensures that $x = -\lambda v'(p)$ for some $\lambda \in \mathbb{R}_+$. Since $x \in H(p)$, we get $1 = p.x = -\lambda v'(p).p$, so that $v'(p).p \neq 0$, $x = v'(p)/(v'(p).p)$ and $D(p)$ is a singleton. □

A different result is as follows.

Proposition 5.3. *If the indirect utility function* v *is quasi-solid and if its sublevel sets with heights* $r > \inf v$ *are strictly convex, then the multimap* $Q(\cdot)$ *is single-valued on* $[u > \inf v]$.

If the nuisance function $w := -u$ *is quasi-solid and if its sublevel sets with heights* $s > \inf w$ *are strictly convex, then the demand multimap* $D(\cdot)$ *is single-valued on* $[v < \sup u]$.

If the nuisance function w *is fully quasi-solid and if its sublevel sets are strictly convex, then* $D(\cdot)$ *is single-valued.*

Proof. Suppose that for some $x \in [u > \inf v]$ the set $Q(x)$ contains two different points p_0, p_1. Let $r := \inf v(B^{-1}(x)) = v(p_0) = v(p_1) \geq u(x) > \inf v$ and let $p := (1/2)(p_0 + p_1)$. Since $[v \leq r]$ is strictly convex, we have $p \in \mathrm{int}[v \leq r]$. Now, v being quasi-solid, we get $p \in [v < r]$. Since $p.x = (1/2)(p_0.x + p_1.x) \leq 1$, i.e. $p \in B^{-1}(x)$, we get a contradiction with the relation $r := \inf v(B^{-1}(x))$.

Given $p \in [v < \sup u]$, suppose $D(p)$ contains two different points, x_0, x_1. Then, for $i = 0, 1$, $r := u(x_i) = \sup u(B(p)) = v(p) < \sup u$. Then, for $x := (1/2)(x_0 + x_1)$, we have $x \in \mathrm{int}[w \leq -r] \subset [w < -r]$ and $u(x) > r$. Since $x \in B(p)$, we get a contradiction with $r := v(p) := \sup u(B(p))$.

The proof of the last assertion is similar, the restriction $-r > \inf w$ being dropped. □

Let us give a criterion in order that v be quasi-solid.

Proposition 5.4. *If the non satiety condition is satisfied, the indirect utility function v is quasi-solid whenever the demand correspondence has nonempty values.*

In particular, if u is weakly u.s.c. on X_+ and if the non satiety condition is satisfied, then v is quasi-solid.

Proof. Let $r > \inf v$ and let $p_0 \in \mathrm{int}[v \leq r]$. Let $\rho > 0$ be such that the ball $B(p_0, \rho)$ with center p_0 and radius ρ is contained in $[v \leq r]$, hence in Y_+, and let $x_0 \in D(p_0)$. Then we have $p_0.x_0 = 1$, hence $x_0 \neq 0$ and we can pick $q \in B(0, \rho)$ such that $q.x_0 < 0$. Then, for $p := p_0 + q$, we have $p.x_0 < 1$. The non satiety condition yields some $x \in B(p)$ such that $u(x) > u(x_0)$. Since we have $p \in [v \leq r]$, we get $r \geq v(p) \geq u(x) > u(x_0) = v(p_0) : p_0 \in [v < r]$. □

Now let us give a criterion in order that v be flatfree.

Proposition 5.5. *If the* non satiety condition *is satisfied, if the interior of Y_+ is nonempty and if u is upper semicontinuous for the weak topology, then the indirect utility function v is flatfree.*

Proof. It suffices to prove that for all $p \in Y_+$, $t > 1$, $q_0 \in \mathrm{int}Y_+$, $q := q_0 + tp$, one has $v(q) < v(p)$ since q can be chosen as close to p as required. Suppose, on the contrary, that $v(q) \geq v(p)$. Then, for all $n \geq 1$ there exists some $x_n \in B(q)$ such that $u(x_n) > v(p) - 1/n$. Since $q_0.x_n \leq q.x_n \leq 1$, the sequence (x_n) is contained in the weakly compact set $B(q_0)$ (see the proof of Lemma 3.2). A subsequence of (x_n) has a weak limit x. Since u is u.s.c. for the weak topology, we have $u(x) \geq v(p)$. Now $tp.x \leq q.x \leq 1$, so that $p.x < 1$. Thus, we get a contradiction with the non satiety condition. □

6. Closure properties of the demand map

The following example shows that the demand corespondence is not lower semicontinuous in general. However, we shall be able to give closure and upper semicontinuity results.

Example. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be the utility function given by $u(x_1, x_2) = \min(2x_1 + \frac{1}{2}x_2, x_1 + x_2, \frac{1}{2}x_1 + 2x_2)$. It is a continuous concave function. One has $\overline{x} := (3, 5) \in D(\frac{1}{8}, \frac{1}{8})$. For $\varepsilon \in (0, \frac{1}{3})$ one has $D(\frac{1}{8} - 3\varepsilon, \frac{1}{8} + 5\varepsilon) = \{(5, 3)\}$, so that one cannot find some $x_\varepsilon \in D(\frac{1}{8} - 3\varepsilon, \frac{1}{8} + 5\varepsilon)$ converging to $\overline{x}$.

The following closure property is a variant of [13, Prop. 3.12], [14, Prop. 2].

Proposition 6.1. *Suppose $f : X \to \mathbb{R}$ is quasiconvex, lower semicontinuous and fully flatfree (resp. flatfree). Then the multimap N_f defined by $N_f(x) := N([f \leq f(x)], x)$ is closed at any $x \in X$ (resp. at any $x \in X$ such that $f(x) > \inf f$).*

Proof. Let $(x_n) \to x$, $(x_n^*) \to x^*$ with $x_n^* \in N_f(x_n)$ for all $n \in \mathbb{N}$. Given $x' \in [f < f(x)]$, we have $f(x') < f(x_n)$ for n large enough since f is lower semicontinuous at x. Thus $\langle x_n^*, x' - x_n \rangle \leq 0$. Taking limits, we get $\langle x^*, x' - x \rangle \leq 0$. Since $[f \leq f(x)]$ is contained in the closure of $[f < f(x)]$, we get $x^* \in N_f(x)$. □

The preceding proof shows that this closure property is even valid when X is endowed with the strong convergence and X^* with the weak* convergence or when X is endowed with the weak convergence and X^* with the strong convergence. But then the result is a sequential property, not a topological property.

Corollary 6.2. *Suppose v is lower semicontinuous and flatfree. Then, the correspondence P is sequentially closed at any $p \in Y_+$ such that $v(p) > \inf v$ and such that $S(p)$ is strictly convex. If moreover u is a Diewert function and if the non satiety assumption is satisfied, then D is sequentially closed at any such p.*

Again, in this statement we can assume Y is endowed with the strong convergence and X with the weak convergence, or the reverse.

Proof. Let $(x_n) \to x$, $(p_n) \to p \in [v > \inf v]$ be such that $p_n \in P(x_n)$ for all n. Since $p_n.x_n = 1$ for all n, we have $p.x = 1$ and $x \in X_+$, $p \in Y_+$. Relation (4.3) ensures that $-x_n \in N^<(S^<(p_n), p_n)$. Since v is l.s.c., we have $v(p_n) > \inf v$ for n large enough. Moreover, since v is flatfree, we have $N^<(S^<(p_n), p_n) \subset N(S(p_n), p_n)$. The proposition entails that $-x \in N(S(p), p)$. Now, $S(p)$ being strictly convex at p, we get $-x \in N^<(S^<(p), p) \cup \{0\}$ and in fact $-x \in N^<(S^<(p), p)$ since $p.x = 1$. Therefore, by relation (4.3), we conclude that $p \in P(x)$.

If u is a Diewert function and if the non satiety assumption is satisfied, whenever $(x_n) \to x$, $(p_n) \to p \in [v > \inf v]$ are such that $x_n \in D(p_n)$ for all n, we have $p_n \in P(x_n)$ for all n and the preceding argument can be used, so that $p \in P(x)$ and $x \in D(p)$. □

The following result is related to the preceding closure property but is different.

Proposition 6.3. *Suppose u is upper semicontinuous and radially lower semicontinuous. Then the demand multimap $D(\cdot)$ is closed when X is endowed with the strong convergence and X^* with the weak* convergence. If moreover $D(\cdot)$ is single-valued and u is upper semicontinuous for the weak topology, then $D(\cdot)$ is continuous on* $\mathrm{int}Y_+$.

Proof. Let $(p_i)_{i\in I} \to p$ for the weak* convergence, $(x_i)_{i\in I} \to x$ with $x_i \in D(p_i)$ for all $i \in I$. Given $x' \in X_+$ such that $p.x' < 1$, we have $p_i.x' \leq 1$ for i large enough, hence $u(x') \leq v(p_i) = u(x_i)$. Since u is upper semicontinuous at x, we get $u(x') \leq u(x)$. Now, for all $x' \in B(p)$ and all $t \in]0,1[$ we have $p.tx' < 1$, hence $u(tx') \leq u(x)$ by what precedes. Since $u(x') \leq \liminf_{t\to 1} u(tx')$, we get $u(x') \leq u(x)$ for all $x' \in B(p)$, which means that $x \in D(p) : D(\cdot)$ is closed. □

A simple variant is as follows.

Proposition 6.4. *Suppose u is upper semicontinuous and v is lower semicontinuous. Then the demand multimap $D(\cdot)$ is closed for the strong convergences on X^* and X.*

Proof. Let $(p_n) \to p$ and let $(x_n) \to x$ with $x_n \in D(p_n)$ for all $n \in \mathbb{N}$. Then, by our assumptions, we have $p.x \leq 1$ and

$$u(x) \geq \limsup_n u(x_n) = \limsup_n v(p_n) \geq \liminf_n v(p_n) \geq v(p),$$

so that $x \in D(p)$. □

Proposition 6.5. *The demand correspondence is compact at any $p \in \mathrm{int}Y_+$ in the sense that any sequence (x_n) satisfying $x_n \in D(p_n)$ for all n and some sequence*

$(p_n) \to p$ has a weakly converging subsequence. If u is weakly upper semicontinuous and v is lower semicontinuous, then D is upper semicontinuous for the strong topology on X^* and the weak topology on X.

If moreover D is single-valued around p, then D is continuous at p when X is endowed with the weak topology and X^* is endowed with the strong topology.

Proof. Let $p \in \text{int}Y_+$ and let $r \in]0,1[$, $q := rp$. Then $q + Y_+$ is a neighborhood of p since $p = q + (1-r)p \in q + \text{int}Y_+$. The first part of the proof of Proposition 3.2 shows that $C_q := \{x \in X_+ : q.x = 1\}$ is a weakly compact base of X_+, so that $B(q) := [0,1]C_q$ is sequentially weakly compact. Given a sequence $(p_n) \to p$, for n large enough we have $p_n \in q + Y_+$, so that $B(p_n) \subset B(q)$. Given $x_n \in D(p_n)$, the sequence (x_n) has a weakly converging subsequence $(x_{k(n)})$. Assuming u is weakly u.s.c. and v is l.s.c., a chain of inequalities similar to the one of the preceding proposition shows that the limit x of $(x_{k(n)})$ belongs to $D(p)$.

Suppose now that D is single-valued around p. Let $(p_n) \to p$ and let $D(p_n) = \{x_n\}$ for n large enough. By what precedes, every subsequence of (x_n) has a further subsequence which weakly converges to the unique element x of $D(p)$. Thus $(D(p_n)) \to D(p)$ and D is continuous at p. □

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