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AN ITERATION METHOD FOR ZEROS OF ACCRETIVE OPERATORS IN BANACH SPACES

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Abstract. We prove weak and strong convergence for Mann type iteration of resolvent of accretive operator A to approach some zero of A. First, the approximation fixed point of the resolvent J_r is showed. Further, in a reflexive Banach space E which satisfies the Opial's condition or in a uniformly convex Banach space E which has Fréchet differentiable norm (or its dual E^* has the Kadec-Klee property), the weak convergence theorems are obtained. Finally, we prove strong convergence theorems of the iteration in Banach space E.

1. Introduction

Let E be a real Banach space with the norm $\|\cdot\|$, E^* be the dual space of E and K be a nonempty closed convex subset of E. Let J denote the *normalized duality mapping* from E into 2^{E^*} given by

$$J(x) = \{f \in E^*, \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\}, \quad \forall\, x \in E,$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing. It is well known (see, for example, [20]) that E is smooth if and only if J is single-valued. In the sequel, we shall denote the single-valued normalized duality mapping by j.

Let K be a nonempty closed convex subset of a Banach space E. A mapping $T : K \to K$ is nonexpansive if $\|Tx - Ty\| \le \|x - y\| \quad \forall x, y \in K$. Mann [11] introduced the following iteration for T in a Hilbert space:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \ge 0, \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. This Mann's iteration process has extensively been studied over the last twenty years for constructions of fixed points of nonlinear mappings and of solutions of nonlinear operator equations involving monotone, accretive and pseudocontractive operators.

A mapping $A : D(A) \subset E \to 2^E$ is said to be *accretive* if for all $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle u - v, j(x - y)\rangle \ge 0, \quad \forall u \in Ax,\ v \in Ay.$$

If A is accretive, then for all $x, y \in D(A)$ and $u \in Ax,\ v \in Ay$ and $t > 0$, we have

$$\|x - y\| \le \|(x - y) + t(u - v)\|.$$

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Note that, if E is a Hilbert space, then the accretive mapping is also called monotone.

An operator A is said to be *m-accretive* if it is accretive and $R(I + rA)$ (: the range of $(I + rA)$) is E for all $r > 0$ and A is said to satisfy the *range condition* if

$$\overline{D(A)} \subset R(I + rA), \quad \forall r > 0,$$

where I is the identity mapping of E and $\overline{D(A)}$ denotes the closure of the domain of A.

In fact, theory of monotone operator theory is very important in nonlinear analysis and is connected with theory of differential equations. It is well known (see [23]) that many physically significant problems can be modeled by the initial-value problems of the form

$$\begin{cases} x'(t) + Ax(t) = 0, \\ x(0) = x_0, \end{cases}$$

where A is an accretive operator in an appropriate Banach space. Typical examples where such evolution equations occur can be found in the heat and wave equations or Schrodinger equations.

Let J_r is the resolvent of A, $J_r = (I + rA)^{-1}, \forall r > 0$. The convergence of the iteration for the resolvent J_r has been studied by many mathematical workers to find zeros of accretive operators. For example, Bruck [1] introduced an iteration process and proved the convergence of the process to a zero of a maximal monotone operator in the setting of Hilbert spaces. Reich [14] extended this result to uniformly smooth Banach spaces provided that the operator is m-accretive. In 2000, Kamimura-Takahashi [10, Theorem 6], in uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial's condition, obtained several weakly convergent theorems of the following iteration for an m-accretive operator A: for $x_1 \in K$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n. \tag{1.2}$$

Recently, still in the framework of uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial's condition, for an m-accretive operator A, Benavides-Acedoand-Xu [5, Theorem 3.7,3.8] obtained that the weak convergence of the iteration scheme (1.2) to some zero of A. Other investigation for zeros of accretive operators can be found in [2, 4, 6, 8, 7, 12, 13, 15, 16, 17, 18, 21].

Motivated by Benavides-Acedoand-Xu [5] and Kamimura-Takahashi [10], for generical accretive operator A, we will research the weak and strong convergence of Mann type iterative scheme (1.2). Firstly, either in a reflexive Banach space E which satisfies the Opial's condition or in a uniformly convex Banach space E which has Fréchet differentiable norm (or its dual E^* has the Kadec-Klee property), the weak convergence of $\{x_n\}$ to some $p \in A^{-1}0$ is showed as $n \to \infty$. Secondly, We also obtain that $\{x_n\}$ strongly converges to some $p \in A^{-1}0$ in Banach space E if some compact condition is arrived.

2. Preliminaries

Throughout this paper, we shall denote denote $F(T) = \{x \in E; Tx = x\}$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (respectively $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (respectively weak, weak*) convergence of the sequence $\{x_n\}$ to x. Let $A : D(A) \subset E \to 2^E$ be an accretive operator and $A^{-1}0 = \{x \in D(A); 0 \in Ax\}$. We use J_r and A_r to denote the resolvent and Yosida's approximation of A, respectively. Namely,

$$J_r = (I + rA)^{-1} \text{ and } A_r = \frac{I - J_r}{r},\ r > 0.$$

For J_r and A_r, the following is well known (see, [20, pp.129-144]):

(i) $A_r x \in AJ_r x$ for all $x \in R(I + rA)$;
(ii) $\|A_r x\| \le |Ax| = \inf\{\|y\|; y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$;
(iii) $J_r : R(I + rA) \to D(A)$ is nonexpansive (i.e. $\|J_r x - J_r y\| \le \|x - y\|$ for all $x, y \in R(I + rA)$);
(iv) $A^{-1}0 = F(J_r) = \{x \in D(J_r); J_r x = x\}$;
(v) (The Resolvent Identity) For $r > 0$ and $t > 0$ and $x \in E$,

$$J_r x = J_t\Big(\frac{t}{r}x + \Big(1 - \frac{t}{r}\Big)J_r x\Big). \tag{2.1}$$

The norm of a Banach space E is said *Fréchet differentiable* if, for any $x \in S(E)$, the unit sphere of E, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists uniformly for $y \in S(E)$. The modulus of convexity of E is defined by

$$\delta_E(\varepsilon) = \inf\Big\{1 - \frac{\|x + y\|}{2};\ \|x\| \le 1,\ \|y\| \le 1,\ \|x - y\| \ge \varepsilon\Big\}$$

for each $\varepsilon \in [0, 2]$. A Banach space E is said to be *uniformly convex* if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. If E is uniformly convex, then

$$\|\lambda x + (1 - \lambda)y\| \le r\Big[1 - 2\lambda(1 - \lambda)\delta_E\Big(\frac{\varepsilon}{R}\Big)\Big] \tag{2.2}$$

for every $x, y \in E$ with $\|x\| \le r,\ \|y\| \le r,\ 0 < \varepsilon < r \le R$ and $\|x - y\| \ge \varepsilon$ and $\lambda \in [0, 1]$ (see, for example, [20, pp.93-98]). In a uniform convex Banach space, Reich [14] proved the following result which also can be found in Tan-Xu [22, Lemma 4, Theorem 1].

Lemma 2.1 (Reich [14, Proposition]). *Let C be a closed convex subset of a uniform convex Banach space E, and let $\{T_n; n \ge 1\}$ be a sequence of nonexpansive self-mappings of C with $F = \bigcap_{n=1}^{\infty} F(T_n) \ne \emptyset$. If $x_1 \in C$ and $x_{n+1} = T_n x_n$ for $n \ge 1$, then for all $f_1, f_2 \in F$ and $t \in (0, 1)$,*

(i) $\lim_{n\to\infty} \|tx_n - (1 - t)f_1 - f_2\|$ *exists;*
(ii) *If the norm of E is also Fréchet differentiable, then* $\lim_{n\to\infty} \langle x_n, j(f_1 - f_2)\rangle$ *exists.*

A Banach space E satisfies *Opial's condition* if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ $(n \to \infty)$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \forall y \in E \text{ with } x \neq y.$$

A Banach space E has the *Kadec-Klee property* if every sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $x_n \to x$. We know that the dual of a reflexive Banach space with a Fréchet differentiable norm has the Kadec-Klee property (see [9]). But there exists a uniformly convex Banach space which have neither a Fréchet differentiable norm nor the Opial property but its dual has the Kadec-Klee property [9, Example 3.1].

In the sequel, we also need the following lemmas.

Lemma 2.2 (Browder [3])**.** *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E. Suppose $T : C \to E$ is nonexpansive. Then the mapping $I - T$ is demiclosed at zero, i.e.*

$$x_n \rightharpoonup x, x_n - Tx_n \to 0 \quad \text{implies} \quad x = Tx.$$

Lemma 2.3 ([9, Lemma 3.2])**.** *Let E be a uniformly convex Banach space such that its dual E^* has the Kadec-Klee property. Suppose $\{x_n\}$is a bounded sequence in E and $f_1, f_2 \in \omega_w(x_n)$, where $\omega_w(x_n)$ denotes the weak limit set of $\{x_n\}$. If $\lim_{n\to\infty} \|tx_n + (1-t)f_1 - f_2\|$ exists for all $t \in [0,1]$, then $f_1 = f_2$.*

Lemma 2.4 (T. Suzuki [19, Lemma 2])**.** *Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a Banach space E and $\beta_n \in [0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n x_n + (1-\beta_n) y_n$ for all integers $n \geq 1$ and*

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

3. Weakly convergence theorems

Theorem 3.1. *Let E be a Banach space and $A : D(A) \subset E \to 2^E$ be an accretive operator that satisfies the range condition. Assumed that K is a nonempty closed convex subset of E such that $\overline{D(A)} \subset K \subset \bigcap_{r>0} R(I + rA)$. For $x_0 \in K$, define*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) J_{r_n} x_n, \quad n \geq 0. \tag{3.1}$$

If $0 \in R(A)$ and $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0, +\infty)$, then

(i) *$\lim_{n\to\infty} \|x_n - p\|$ exists and hence $\{x_n\}$ is bounded;*

(ii) *If $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} r_n = \infty$, then for each $r > 0$,*

$$\lim_{n\to\infty} \|x_n - J_r x_n\| = 0;$$

(iii) *If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ and $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty} \frac{r_n}{r_{n+1}} = 1$, then for each $r > 0$,*

$$\lim_{n\to\infty} \|x_n - J_r x_n\| = 0;$$

(iv) *If E is uniformly convex and $\limsup_{n\to\infty} \alpha_n < 1$ and $\liminf_{n\to\infty} r_n > 0$, then for each $r > 0$,*

$$\lim_{n\to\infty} \|x_n - J_r x_n\| = 0.$$

Proof. (i) Since $0 \in R(A)$, we can take $p \in A^{-1}0 = F(J_r)$ for all $r > 0$. Then we have

$$\begin{aligned}\|x_{n+1} - p\| =&\|\alpha_n(x_n - p) + (1-\alpha_n)(J_{r_n}x_n - p)\| \\ \le&\alpha_n\|x_n - p\| + (1-\alpha_n)\|J_{r_n}x_n - p\| \\ \le&\|x_n - p\| \\ &\vdots \\ \le&\|x_0 - p\|.\end{aligned}$$

Therefore, $\{\|x_n - p\|\}$ is non-increasing and bounded below, and that (i) is proved.

(ii) From (i), we get the boundedness of $\{J_{r_n}x_n\}$ since

$$\|J_{r_n}x_n\| \le \|J_{r_n}x_n - p\| + \|p\| \le \|x_n - p\| + \|p\|.$$

Using the condition $\lim_{n\to\infty} \alpha_n = 0$, we obtain that

$$\lim_{n\to\infty} \|x_{n+1} - J_{r_n}x_n\| = \lim_{n\to\infty} \alpha_n\|x_n - T_{n+1}x_n\| = 0. \tag{3.2}$$

For each $r > 0$, we also have

$$\begin{aligned}\|J_{r_n}x_n - J_rJ_{r_n}x_n\| =&\|(I - J_r)J_{r_n}x_n\| = r\|A_rJ_{r_n}y_n\| \\ \le&r|AJ_{r_n}x_n| \le r\|A_{r_n}x_n\| \\ =&r\frac{\|x_n - J_{r_n}x_n\|}{r_n} \to 0 (n\to\infty).\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \|J_{r_n}x_n - J_rJ_{r_n}x_n\| = 0. \tag{3.3}$$

Hence, for each $r > 0$,

$$\begin{aligned}\|x_{n+1} - J_rx_{n+1}\| &\le \|x_{n+1} - J_{r_n}x_n\| + \|J_{r_n}x_n - J_rJ_{r_n}x_n\| + \|J_rJ_{r_n}x_n - J_rx_{n+1}\| \\ &\le 2\|x_{n+1} - J_{r_n}x_n\| + \|J_{r_n}x_n - J_rJ_{r_n}x_n\|.\end{aligned}$$

Combining (3.2) and (3.3), we obtain that for each $r > 0$,

$$\lim_{n\to\infty} \|x_n - J_rx_n\| = 0.$$

(iii) From the resolvent identity (2.1), we have

$$J_{r_{n+1}}x_{n+1} = J_{r_n}\Big(\frac{r_n}{r_{n+1}}x_{n+1} + \Big(1 - \frac{r_n}{r_{n+1}}\Big)J_{r_{n+1}}x_{n+1}\Big).$$

Therefore, for a constant $M > 0$ with $M \ge \max\{\|J_{r_n}x_n\|, \|x_n\|\}$,

$$\begin{aligned}\|J_{r_{n+1}}x_{n+1} - J_{r_n}x_n\| \le&\Big\|\frac{r_n}{r_{n+1}}(x_{n+1} - x_n) + \Big(1 - \frac{r_n}{r_{n+1}}\Big)(J_{r_{n+1}}x_{n+1} - x_n)\Big\| \\ \le&\|x_{n+1} - x_n\| + \Big|1 - \frac{r_n}{r_{n+1}}\Big|\|J_{r_{n+1}}x_{n+1} - x_{n+1}\| \\ \le&\|x_{n+1} - x_n\| + 2M\Big|1 - \frac{r_n}{r_{n+1}}\Big|.\end{aligned}$$

Hence, from $\lim_{n\to\infty} \frac{r_n}{r_{n+1}} = 1$, we have

$$\limsup_{n\to\infty}(\|J_{r_{n+1}}x_{n+1} - J_{r_n}x_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

An application of Lemma 2.4 to yield

$$\lim_{n\to\infty} \|x_n - J_{r_n}x_n\| = 0. \tag{3.4}$$

Since $0 < \liminf_{k\to\infty} r_n$, then there exists $\varepsilon > 0$ and a positive integer $N > 0$ such that $\forall n > N$, $r_n \geq \varepsilon$. Thus for each $r > 0$, using the resolvent identity (2.1) again, we also have

$$J_{r_n}x_n = J_r\Big(\frac{r}{r_n}x_n + \Big(1 - \frac{r}{r_n}\Big)J_{r_n}x_n\Big),$$

and hence

$$\begin{aligned}\|J_{r_n}x_n - J_rx_n\| \leq& \Big\|\frac{r}{r_n}x_n + \Big(1 - \frac{r}{r_n}\Big)J_{r_n}x_n - x_n\Big\|\\ =&\Big|1 - \frac{r}{r_n}\Big|\|J_{r_n}x_n - x_n\|\\ \leq&\Big(1 + \frac{r}{r_n}\Big)\|x_n - J_{r_n}x_n\|\\ \leq&\Big(1 + \frac{r}{\varepsilon}\Big)\|x_n - J_{r_n}x_n\|.\end{aligned}$$

It follows from (3.4) that

$$\lim_{n\to\infty} \|J_{r_n}x_n - J_rx_n\| = 0. \tag{3.5}$$

Since $\|x_n - J_rx_n\| \leq \|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - J_rx_n\|$, combining (3.4) and (3.5), the desired result is reached.

(iv) Without loss of generality, we may assume that $\lim_{n\to\infty} \|x_n - p\| \neq 0$ for some $p \in A^{-1}0$. Since A is accretive and E is uniformly convex, we have from (2.2) ($\lambda = \frac{1}{2}$, $R = \|x_0 - p\|$ and $r = \|x_n - p\|$),

$$\begin{aligned}\|J_{r_n}x_n - p\| \leq&\Big\|J_{r_n}x_n - p + \frac{r_n}{2}(A_{r_n}x_n - 0)\Big\|\\ =&\Big\|J_{r_n}x_n - p + \frac{1}{2}(x_n - J_{r_n}x_n)\Big\|\\ =&\Big\|\frac{1}{2}(x_n + J_{r_n}x_n) - p\Big\|\\ \leq&\|x_n - p\|\Big[1 - \frac{1}{2}\delta_E\Big(\frac{\|x_n - J_{r_n}x_n\|}{\|x_0 - p\|}\Big)\Big].\end{aligned}$$

Since $\|x_{n+1} - p\| \leq \alpha_n\|x_n - p\| + (1 - \alpha_n)\|J_{r_n}x_n - p\|$, we have

$$\begin{aligned}\frac{1}{2}(1 - \alpha_n)\|x_n - p\|\delta_E\Big(\frac{\|x_n - J_{r_n}x_n\|}{\|x_0 - p\|}\Big) \leq&\alpha_n\|x_n - p\| + (1 - \alpha_n)\|x_n - p\|\\ &- \|x_{n+1} - p\|\\ =&\|x_n - p\| - \|x_{n+1} - p\|.\end{aligned}$$

By $\limsup_{n\to\infty} \alpha_n < 1$ and $\lim_{n\to\infty} \|x_n - p\| \neq 0$, we obtain

$$\lim_{n\to\infty} \delta_E\Big(\frac{\|x_n - J_{r_n}x_n\|}{\|x_0 - p\|}\Big) = 0.$$

This implies

$$\lim_{n\to\infty} \|x_n - J_{r_n}x_n\| = 0. \tag{3.6}$$

Consequently, combining (3.5) and (3.6), we obtain the desired result. □

Theorem 3.2. *Let E be a reflexive Banach space which satisfies Opial's condition and $A : D(A) \subset E \to 2^E$ be an accretive operator that satisfies the range condition and $0 \in R(A)$. Assumed that K is a nonempty closed convex subset of E such that $\overline{D(A)} \subset K \subset \bigcap_{r>0} R(I + rA)$. If $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, +\infty)$ satisfy one of the conditions (a) and (b),*

(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} r_n = \infty$*;*
(b) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$ *and* $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} \frac{r_n}{r_{n+1}} = 1$.

Then as $n \to \infty$, $\{x_n\}$, defined by (3.1) weakly converges to some zero x^* of A.

Proof. It follows from Theorem 3.1 (i) and (ii) and (iii) that $\{x_n\}$ is bounded and for each $r > 0$,

$$\lim_{n\to\infty} \|x_n - J_r x_n\| = 0.$$

Then $\{x_n\}$ is weakly sequentially compact by the reflexivity of E, and hence we may assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. We claim that $x^* \in A^{-1}0 = F(J_r)$. Indeed, suppose $x^* \neq J_r x^*$, then from the Opial's property of E, we obtain that

$$\begin{aligned}\limsup_{k\to\infty} \|x_{n_k} - x^*\| &< \limsup_{k\to\infty} \|x_{n_k} - J_r x^*\| \\ &\le \limsup_{k\to\infty} (\|x_{n_k} - J_r x_{n_k}\| + \|J_r x_{n_k} - J_r x^*\|) \\ &\le \limsup_{k\to\infty} \|x_{n_k} - x^*\|.\end{aligned}$$

This is a contradiction, thus $x^* = J_r x^*$. The claim is proved.

Now we prove $\{x_n\}$ converges weakly to x^*. Supposed that $\{x_n\}$ doesn't converge weakly to x^*, then there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which weakly converges to some $y \neq x^*$, $y \in K$. We also have $y \in A^{-1}0$. Because $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in A^{-1}0$ by Theorem 3.1 (i) and E satisfies the Opial's condition, then we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - x^*\| &= \lim_{k\to\infty} \|x_{n_k} - x^*\| < \lim_{k\to\infty} \|x_{n_k} - y\| \\ &= \lim_{j\to\infty} \|x_{n_j} - y\| < \lim_{j\to\infty} \|x_{n_j} - x^*\| \\ &= \lim_{n\to\infty} \|x_n - x^*\|.\end{aligned}$$

Which is a contradiction, we must have $y = x^*$. In a summary, we have proved that the set $\{x_n\}$ is weakly sequentially compact and each cluster point in the weak topology equals to x^*. Hence, $\{x_n\}$ converges weakly to $x^* \in A^{-1}0$. The proof is complete. □

Using the same methods as Theorem 3.2, we can easily obtain the following results.

Corollary 3.3. *Let E be a reflexive Banach space which satisfies Opial's condition and $A : D(A) \subset E \to 2^E$ be an m-accretive operator with $0 \in R(A)$. If $\{\alpha_n\}$ and $\{r_n\}$ are as Theorem 3.2. Then as $n \to \infty$, $\{x_n\}$, defined by (3.1) weakly converges to some zero x^* of A.*

Proof. Since A is m-accretive, A is accretive and satisfies the range condition $\overline{D(A)} \subset E = R(I + rA)$ for all $r > 0$. Putting $K = E$, the desired result is reached. □

Corollary 3.4. *Let E be a reflexive Banach space which satisfies Opial's condition and $A : D(A) \subset E \to 2^E$ be an accretive operator that satisfies the range condition and $0 \in R(A)$. Assumed that $D(A)$ is convex and $\{\alpha_n\}$ and $\{r_n\}$ are as Theorem 3.2. Then as $n \to \infty$, $\{x_n\}$, defined by (3.1) weakly converges to some zero x^* of A.*

Proof. Putting $K = \overline{D(A)}$ and following Theorem 3.2, we can obtain the desired conclusion. □

Theorem 3.5. *Let E be a uniformly convex Banach space which satisfies the Opial's conditionand $A : D(A) \subset E \to 2^E$ be an accretive operator that satisfies the range condition and $0 \in R(A)$. Assumed that K is a nonempty closed convex subset of E such that $\overline{D(A)} \subset K \subset \bigcap_{r>0} R(I + rA)$. If $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, +\infty)$ satisfy $\limsup_{n\to\infty} \alpha_n < 1$ and $\liminf_{n\to\infty} r_n > 0$. Then as $n \to \infty$, $\{x_n\}$, defined by (3.1) weakly converges to some zero x^* of A.*

Theorem 3.6. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and $A : D(A) \subset E \to 2^E$ be an accretive operator that satisfies the range condition and $0 \in R(A)$. Assumed that K is a nonempty closed convex subset of E such that $\overline{D(A)} \subset K \subset \bigcap_{r>0} R(I + rA)$. If $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, +\infty)$ satisfy $\limsup_{n\to\infty} \alpha_n < 1$ and $\liminf_{n\to\infty} r_n > 0$. Then as $n \to \infty$, $\{x_n\}$, defined by (3.1) weakly converges to some zero x^* of A.*

Proof. Theorem 3.1 guarantees $\{x_n\}$ is bounded and for each $r > 0$,

$$\lim_{n\to\infty} \|x_n - J_r x_n\| = 0.$$

Similar to Theorem 3.2, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to some $x^* \in K$. By Lemma 2.2, we have $x^* \in F(J_r) = A^{-1}0$.

Now we prove $\{x_n\}$ converges weakly to x^*. Supposed that $\{x_n\}$ doesn't converge weakly to x^*, then there exists another subsequence $\{x_{n_l}\}$ of $\{x_n\}$ which weakly converges to some $y \in K$. We also have $y \in F(J_r) = A^{-1}0$. Next we show $x^* = y$.

Set $T_n = \alpha_n I + (1-\alpha_n) J_{r_n}$, then it is clear that $\{T_n\}$ is a sequence of nonexpansive mappings of K with $F = \bigcap_{n=0}^{\infty} F(T_n) = \bigcap_{n=0}^{\infty} F(J_{r_n}) = A^{-1}0 \neq \emptyset$ and $x_{n+1} = T_n x_n$. Therefore, Lemma 2.1(ii) assures that $\lim_{n\to\infty} \langle x_n, j(x^* - y)\rangle$ exists. Hence, we have

$$\lim_{n\to\infty} \langle x_n, j(x^* - y)\rangle = \lim_{k\to\infty} \langle x_{n_k}, j(x^* - y)\rangle = \langle x^*, j(x^* - y)\rangle,$$

and

$$\lim_{n\to\infty}\langle x_n, j(x^*-y)\rangle = \lim_{l\to\infty}\langle x_{n_l}, j(x^*-y)\rangle = \langle y, j(x^*-y)\rangle.$$

Consequently,

$$\langle x^*, j(x^*-y)\rangle = \langle y, j(x^*-y)\rangle,$$

that is $\|x^*-y\|=0$. We must have $y=x^*$. Thus $\{x_n\}$ converges weakly to $x^*\in A^{-1}0$. The proof is complete. □

Theorem 3.7. *Let E be a uniformly convex Banach space and its dual E^* have the Kadec-Klee property and $A : D(A)\subset E\to 2^E$ be an accretive operator that satisfies the range condition and $0\in R(A)$. Assumed that K is a nonempty closed convex subset of E such that $\overline{D(A)}\subset K\subset \bigcap_{r>0} R(I+rA)$. If $\{\alpha_n\}\subset(0,1)$ and $\{r_n\}\subset(0,+\infty)$ satisfy $\limsup_{n\to\infty}\alpha_n<1$ and $\liminf_{n\to\infty} r_n>0$. Then as $n\to\infty$, $\{x_n\}$, defined by (3.1) weakly converges to some zero x^* of A.*

Proof. As in the proof of Theorem 3.6, we can reach the following objectives:

(1) there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to some $x^*\in A^{-1}0$;

(2) the nonexpansive mappings sequence $\{T_n\}$ satisfies the conditions of Lemma 2.1.

Now we prove $\{x_n\}$ converges weakly to x^*. Supposed that $\{x_n\}$ doesn't converge weakly to x^*, then there exists another subsequence $\{x_{n_l}\}$ of $\{x_n\}$ which weakly converges to some $y\in K$. We also have $y\in A^{-1}0$.

Next we show $x^*=y$. In fact, from Lemma 2.1(i), we have $\lim_{n\to\infty}\|tx_n-(1-t)x^*-y\|$ exists. Using Lemma 2.3 we obtain $y=x^*$. Thus $\{x_n\}$ converges weakly to $x^*\in A^{-1}0$. □

Using the same argumentation technique as Corollary 3.3 and 3.4, we can easily obtain the following results.

Corollary 3.8. *Let E be a uniformly convex Banach space which either satisfies the Opial's condition or has Fréchet differentiable norm or its dual E^* have the Kadec-Klee property. Suppose that $A : D(A)\subset E\to 2^E$ be an accretive operator that satisfies the range condition and $0\in R(A)$. If $D(A)$ is convex and $\{\alpha_n\}$ and $\{r_n\}$ satisfy $\limsup_{n\to\infty}\alpha_n<1$ and $\liminf_{n\to\infty} r_n>0$. Then as $n\to\infty$, $\{x_n\}$, defined by (3.1) weakly converges to some zero x^* of A.*

Corollary 3.9. *Let E be a uniformly convex Banach space which either satisfies the Opial's condition or has Fréchet differentiable norm or its dual E^* have the Kadec-Klee property. Suppose that $A : D(A)\subset E\to 2^E$ be an m-accretive operator with $0\in R(A)$. If $\{\alpha_n\}$ and $\{r_n\}$ satisfy $\limsup_{n\to\infty}\alpha_n<1$ and $\liminf_{n\to\infty} r_n>0$. Then as $n\to\infty$, $\{x_n\}$, defined by (3.1) weakly converges to some zero x^* of A.*

4. Strongly convergence theorems

Theorem 4.1. *Let E be a Banach space and $A : D(A)\subset E\to 2^E$ be an accretive operator that satisfies the range condition and $0\in R(A)$. Assumed that K is a* *nonempty compact convex subset of E such that $\overline{D(A)} \subset K \subset \bigcap_{r>0} R(I + rA)$ and for $x_0 \in K$, iteratively define*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0.$$

If $\{\alpha_n\}$ and $\{r_n\}$ satisfy one of the following two conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} r_n = \infty$*;*
(b) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ *and* $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} \frac{r_n}{r_{n+1}} = 1$.

Then as $n \to \infty$, $\{x_n\}$ strongly converges to some zero x^* of A.

Proof. By Theorem 3.1 (i) and the compactness of K, we see that $\{x_n\}$ admits a strongly convergent subsequence $\{x_{n_k}\}$ whose limit we shall denote by x^*. Then, again by Theorem 3.1 (ii) or (iii), we have $x^* \in A^{-1}0 = F(J_r)$. As $\forall p \in A^{-1}0$, $\lim_{n\to\infty} \|x_n - p\|$ exists by Theorem 3.1 (i), x^* is actually the strong limit of the sequence $\{x_n\}$ itself. □

From the proof of Theorem 4.1, we can get the following Corollary.

Corollary 4.2. *Let $E, K, A, \{x_n\}, \{\alpha_n\}, \{r_n\}$ be as Theorem 4.1. Then as $n \to \infty$, $\{x_n\}$ strongly converges to some zero x^* of A if and only if there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in A^{-1}0$ $(k \to \infty)$.*

Similar to the argumentation of Theorem 4.1, we can get the following. Since the proof is a repeating work, we omit it.

Theorem 4.3. *Let E be a uniformly convex Banach space and $A : D(A) \subset E \to 2^E$ be an accretive operator that satisfies the range condition and $0 \in R(A)$. Assumed that K is a nonempty compact convex subset of E such that $\overline{D(A)} \subset K \subset \bigcap_{r>0} R(I + rA)$ and for $x_0 \in K$, iteratively define*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0.$$

If $\{\alpha_n\}$ and $\{r_n\}$ satisfy $\limsup_{n\to\infty} \alpha_n < 1$ and $\liminf_{n\to\infty} r_n > 0$. Then as $n \to \infty$, $\{x_n\}$ strongly converges to some zero x^* of A.

Corollary 4.4. *Let E be a Banach space and $A : D(A) \subset E \to 2^E$ be an accretive operator that satisfies the range condition and $0 \in R(A)$. Assumed that $\overline{D(A)}$ is a compact convex subset of E and for $x_0 \in K$, iteratively define*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0.$$

(1) *If $\{\alpha_n\}$ and $\{r_n\}$ satisfy one of the following two conditions:*
  (a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} r_n = \infty$*;*
  (b) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ *and* $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} \frac{r_n}{r_{n+1}} = 1$.

  Then as $n \to \infty$, $\{x_n\}$ strongly converges to some zero x^* of A;

(2) *If E is a uniformly convex Banach space and $\{\alpha_n\}$ and $\{r_n\}$ satisfy $\limsup_{n\to\infty} \alpha_n < 1$ and $\liminf_{n\to\infty} r_n > 0$. Then as $n \to \infty$, $\{x_n\}$ strongly converges to some zero x^* of A.*

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