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COUPLED FIXED POINT THEOREMS IN UNIFORM SPACES AND APPLICATIONS

TRAN VAN AN, KIEU PHUONG CHI, AND LE KHANH HUNG

Abstract. In this paper, we prove some coupled fixed point theorems for generalized contractive mappings in uniform spaces and apply them to study the existences-uniqueness problem for a class of nonlinear integral equations with unbounded deviations.

1. Introduction

It is known that the fixed point theory plays a crucial role not only in the existence theory of differential equations, integral equations, functional equations, partial differential equations, random differential equations but also in computer science and economics, and the modern fixed point theory is attached usually to the nonlinear analysis. In this paper we present some results concerning the coupled fixed point theorems in uniform spaces as natural extensions of metric coupled fixed point theorems that have recently exposed by many authors (see [4], [5] and the references given therein). Our goal is to explore not only the results themselves but also to apply them in solving nonlinear integral equations. Note that the known results in metric spaces are not applicable to the problems in the following section 4 (see [5], [9] and the references given therein).

2. Preliminaries

Let X be a uniform space. The uniform topology on X is generated by a family of uniform continuous pseudometrics on X (see [7]). In this paper, by $(X,\mathcal{P})$ we mean a Hausdorff uniform space whose uniformity is generated by a saturated family of pseudometrics $\mathcal{P}=\{d_\alpha(x,y):\alpha\in I\}$, where I is an index set. Note that, $(X,\mathcal{P})$ is Hausdorff if and only if $d_\alpha(x,y)=0$ for all $\alpha\in I$ implies $x=y$.

Definition 2.1 ([1])**.** Let $(X,\mathcal{P})$ be a Hausdorff uniform space.

1) A sequence $\{x_n\}\subset X$ is Cauchy if $d_\alpha(x_n,x_m)\to 0$ as $m,n\to\infty$ for every $\alpha\in I$.

2) X is said to be sequentially complete if every Cauchy sequence $\{x_n\}$ in X converges to $x\in X$.

Definition 2.2 ([1])**.** Let $j:I\to I$ be an arbitrary mapping of the index I into itself. The iterations of j can be defined inductively by

$$j^0(\alpha)=\alpha, j^k(\alpha)=j\big(j^{k-1}(\alpha)\big), k=1,2,\dots$$

The following concept was introduced by V. Lakshmikantham.

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Definition 2.3 ([6])**.** Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any $x, y \in X$

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Now, we introduce a class of functions which plays a crucial role in the fixed point theory. Sometimes, they are called to be control functions.

Let $\Phi = \{\varphi_\alpha : \alpha \in I\}$ be a family of functions (which one call Φ-contractive) with the properties

i) $\varphi_\alpha : [0, +\infty) \to [0, +\infty)$ is monotone non-decreasing and continuous.

ii) $0 < \varphi_\alpha(t) < t$ for all $t > 0$ and $\varphi_\alpha(0) = 0$.

Remark 2.4. 1) If E is a locally convex space with a saturated family of seminorms $\{p_\alpha\}_{\alpha \in I}$, then the associated family of pseudometrics $\{\rho_\alpha\}_{\alpha \in I}$ defined by $\rho_\alpha(x, y) = p_\alpha(x - y)$ for every $x, y \in E$ and $\alpha \in I$. The uniform topology, which is generated by this family of pseudometrics, coincides with the original topology of the space E. Therefore, as a corollary of our results, we obtain the fixed point theorems in the locally convex spaces.

2) Let us point out that here the mapping j arises in a natural way. It is generated by the deviating arguments. This allows us to obtain an application in solving nonlinear integral equations with unbounded deviations. We note that if $j(\alpha) = \alpha$ for every $\alpha \in I$ then some fixed point theorems in uniform spaces easily follow from the fixed point theorems in metric spaces with variably contractive conditions (see [10]).

3. Coupled fixed point theorems in uniform spaces

We begin this section at giving a new coupled fixed point theorem in ordered uniform spaces.

Theorem 3.1. *Let $(X, \leq)$ be a partially ordered set and $\mathcal{P} = \{d_\alpha(x, y) : \alpha \in I\}$ be a family of pseudometrics on X such that $(X, \mathcal{P})$ is a Hausdorff sequentially complete uniform space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Suppose that*

1) *For every $\alpha \in I$ there exists $\varphi_\alpha \in \Phi$ such that*

$$d_\alpha\big(F(x, y), F(u, v)\big) \leq \varphi_\alpha\Big(\frac{d_{j(\alpha)}(x, u) + d_{j(\alpha)}(y, v)}{2}\Big), \tag{3.1}$$

for all $x \leq u, y \geq v$;

2) *For each $\alpha \in I$, there exists $\overline{\varphi}_\alpha \in \Phi$ such that* $\sup\{\varphi_{j^n(\alpha)}(t) : n = 0, 1, \ldots\} \leq \overline{\varphi}_\alpha(t)$ *and* $\dfrac{\overline{\varphi}_\alpha(t)}{t}$ *is non-decreasing;*

3) *There are $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0), y_0 \geq F(y_0, x_0)$ and*

$$d_{j^n(\alpha)}\big(x_0, F(x_0, y_0)\big) + d_{j^n(\alpha)}\big(y_0, F(y_0, x_0)\big) < 2p(\alpha) < \infty,$$

for every $\alpha \in I, n \in \mathbb{N}$.

Also, assume either a) F is continuous; or,

b) *X has the following property*
   i) *If a non-decreasing sequence $\{x_n\}$ in X converges to x then $x_n \leq x$ for all $n \in \mathbb{N}$.*
   ii) *If a non-increasing sequence $\{y_n\}$ in X converges to y then $y_n \geq y$ for all $n \in \mathbb{N}$.*

Then F has a coupled fixed point, that is, there exist $x, y \in X$ such that $x = F(x,y), y = F(y,x)$.

Proof. Let $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0), y_0 \geq F(y_0, x_0)$. Put $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Then $x_0 \leq x_1$ and $y_0 \geq y_1$. Again, set $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$. Since F has the mixed monotone property, then we have $x_1 \leq x_2$ and $y_1 \geq y_2$. Continuing this way, we get two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n)$ and

$$x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots, \quad y_0 \geq y_1 \geq y_2 \geq \cdots \geq y_n \geq y_{n+1} \geq \cdots$$

Now, for each $n = 0, 1, 2, \ldots$ and $\alpha \in I$, we put

$$\delta_n^\alpha = d_\alpha(x_n, x_{n+1}) + d_\alpha(y_n, y_{n+1}).$$

By the assumption 3), we have

$$\delta_0^{j^n(\alpha)} = d_{j^n(\alpha)}\big(x_0, F(x_0, y_0)\big) + d_{j^n(\alpha)}\big(y_0, F(y_0, x_0)\big) < 2p(\alpha) < \infty. \tag{3.2}$$

Now we claim that $\delta_n^\alpha \leq 2\varphi_\alpha\Big(\frac{\delta_{n-1}^{j(\alpha)}}{2}\Big)$ for every $\alpha \in I, n \in \mathbb{N}$. Indeed, in view of the condition 1) and since $x_{n-1} \leq x_n$ and $y_{n-1} \geq y_n$, we obtain

$$\begin{aligned} d_\alpha(x_n, x_{n+1}) &= d_\alpha\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big) \\ &\leq \varphi_\alpha\Big(\frac{d_{j(\alpha)}(x_{n-1}, x_n) + d_{j(\alpha)}(y_{n-1}, y_n)}{2}\Big) \\ &= \varphi_\alpha\Big(\frac{\delta_{n-1}^{j(\alpha)}}{2}\Big). \end{aligned} \tag{3.3}$$

By the same argument, we have

$$d_\alpha(y_n, y_{n+1}) \leq \varphi_\alpha\Big(\frac{\delta_{n-1}^{j(\alpha)}}{2}\Big). \tag{3.4}$$

Combining (3.3) and (3.4), we deduce that

$$\delta_n^\alpha \leq 2\varphi_\alpha\Big(\frac{\delta_{n-1}^{j(\alpha)}}{2}\Big) \text{ for all } n \in \mathbb{N}, \alpha \in I. \tag{3.5}$$

Since φ_α is a monotone non-decreasing function, then by (3.5) and (3.2), we have

$$\begin{aligned}\frac{\delta_n^\alpha}{2} \le \varphi_\alpha\Big(\frac{\delta_{n-1}^{j(\alpha)}}{2}\Big) &\le \varphi_\alpha\Big(\varphi_{j(\alpha)}\Big(\frac{\delta_{n-2}^{j^2(\alpha)}}{2}\Big)\Big)\\ &\le \varphi_\alpha\Big(\varphi_{j(\alpha)}\Big(\cdots\varphi_{j^{n-1}(\alpha)}\Big(\frac{\delta_0^{j^n(\alpha)}}{2}\Big)\cdots\Big)\Big)\\ &\le \overline{\varphi}_\alpha^n\Big(\frac{\delta_0^{j^n(\alpha)}}{2}\Big) \le \overline{\varphi}_\alpha^n\big(p(\alpha)\big).\end{aligned}$$

Put $\overline{\varphi}_\alpha^n\big(p(\alpha)\big) = b_n^\alpha$ for each $\alpha \in I$ and $n \in \mathbb{N}$. Then $\delta_n^\alpha \le 2b_n^\alpha$ for every $\alpha \in I, n \in \mathbb{N}$. Using the triangle inequality, we get

$$\begin{aligned}d_\alpha(x_n, x_{n+p}) + d_\alpha(y_n, y_{n+p}) &\le \sum_{i=0}^{p-1}\big[d_\alpha(x_{n+i}, x_{n+i+1}) + d_\alpha(y_{n+i}, y_{n+i+1})\big]\\ &= \sum_{i=0}^{p-1}\delta_{n+i}^\alpha \le 2\sum_{i=0}^{p-1} b_{n+i}^\alpha.\end{aligned}$$

Since $p(\alpha) > 0$ and $\overline{\varphi}_\alpha(t) < t$ for every $t > 0$, we obtain

$$\overline{\varphi}_\alpha^n\big(p(\alpha)\big) = \overline{\varphi}_\alpha\big(\overline{\varphi}_\alpha^{n-1}(p(\alpha))\big) < \overline{\varphi}_\alpha^{n-1}\big(p(\alpha)\big) < \cdots < \overline{\varphi}_\alpha\big(p(\alpha)\big) < p(\alpha). \tag{3.6}$$

Because $\frac{\overline{\varphi}_\alpha(t)}{t}$ is a monotone non-decreasing function, then by (3.6), we have

$$\frac{b_{n+1}^\alpha}{b_n^\alpha} = \frac{\overline{\varphi}_\alpha\big(\overline{\varphi}_\alpha^n(p(\alpha))\big)}{\overline{\varphi}_\alpha^n\big(p(\alpha)\big)} \le \frac{\overline{\varphi}_\alpha\big(p(\alpha)\big)}{p(\alpha)} < 1. \tag{3.7}$$

This implies that the series $\sum_{m=0}^\infty b_m^\alpha$ is convergent. Hence $\sum_{i=0}^{p-1} b_{n+i}^\alpha \to 0$ as $n \to \infty$ for all p. Thus it follows that $d_\alpha(x_n, x_{n+p}) \to 0$ and $d_\alpha(y_n, y_{n+p}) \to 0$ as $n \to \infty$ for all p, that is $\{x_n\}, \{y_n\}$ are Cauchy sequences. Because X is sequentially complete, there exist $x, y \in X$ such that $\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y$.

Now, we consider the following cases

Case 1: Let F be continuous. Then we have $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(x, y), y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F(y, x)$. It follows that x, y is a coupled fixed point of F.

Case 2: Let X have the property as in the assumption b), $\{x_n\}$ be a non-decreasing sequence such that $x_n \to x$, and $\{y_n\}$ be a non-increasing sequence such that $y_n \to y$. Then we have $x_n \le x$ and $y_n \ge y$ for all $n \in \mathbb{N}$. Using the triangle inequality and the contractive condition, we get

$$\begin{aligned}d_\alpha\big(x, F(x,y)\big) &\le d_\alpha(x, x_{n+1}) + d_\alpha\big(x_{n+1}, F(x,y)\big)\\ &= d_\alpha(x, x_{n+1}) + d_\alpha\big(F(x_n, y_n), F(x,y)\big)\\ &\le d_\alpha(x, x_{n+1}) + \varphi_\alpha\Big(\frac{d_{j(\alpha)}(x_n, x) + d_{j(\alpha)}(y_n, y)}{2}\Big)\\ &\le d_\alpha(x, x_{n+1}) + \frac{d_{j(\alpha)}(x_n, x) + d_{j(\alpha)}(y_n, y)}{2}.\end{aligned}$$

Letting $n \to \infty$, we arrive at $d_\alpha\big(x, F(x,y)\big) = 0$ for all $\alpha \in I$. This implies that $x = F(x,y)$. By the same way, we have $y = F(y,x)$. The proof is completed. $\square$

It is our interest to provide additional conditions to ensure that the coupled fixed point in Theorem 3.1 is in fact unique. Therefore, we have to add the properties for a partial order on $X \times X$ and the mapping $j : I \to I$.

Definition 3.2 ([1])**.** A uniform space $(X, \mathcal{P})$ is said to be j-bounded if for every $\alpha \in I$ and $x, y \in X$ there exists $q(x,y,\alpha)$ such that

$$d_{j^n(\alpha)}(x,y) \le q(x,y,\alpha) < \infty, \text{ for all } n \in \mathbb{N}.$$

Let $(X, \le)$ be a partially ordered set. Then, we consider the partial order on $X \times X$ that defined by

$$\text{for } (x,y),(u,v) \in X \times X, (x,y) \le (u,v) \Leftrightarrow x \le u, y \ge v.$$

Theorem 3.3. *Suppose that the conditions of Theorem 3.1 are fulfilled. If X is j-bounded and for every $(x,y),(z,t) \in X \times X$ there exists $(u,v) \in X \times X$ which is comparable to them, then F has a unique coupled fixed point.*

Proof. By Theorem 3.1, we conclude that the set of coupled fixed points of F is nonempty. Assume that $(x,y),(z,t)$ are coupled fixed points of F, that is $x = F(x,y), y = F(y,x)$ and $z = F(z,t), t = F(t,z)$. We shall prove that $x = z$ and $y = t$. By assumption, there exists $(u,v) \in X \times X$ which is comparable to (x,y) and (z,t). Put $u_0 = u, v_0 = v$. By induction, we construct the sequences $\{u_n\}, \{v_n\}$ defined by $u_{n+1} = F(u_n, v_n), v_{n+1} = F(v_n, u_n)$ for $n = 0, 1, 2, \ldots$ Suppose that $(x,y) \le (u,v) = (u_0, v_0)$. Since F has the mixed monotone property, we have $u_1 = F(u_0, v_0) \ge F(u,y) \ge F(x,y) = x$; $v_1 = F(v_0, u_0) \le F(y,x) = y$. Hence $(u_1, v_1) \ge (x,y)$. By the same argument, we infer that $(u_n, v_n) \ge (x,y)$ for every $n = 0, 1, 2, \ldots$. Then

$$\begin{aligned} d_\alpha(x, u_{n+1}) &= d_\alpha\big(F(x,y), F(u_n, v_n)\big) \\ &\le \varphi_\alpha\Big(\frac{d_{j(\alpha)}(x, u_n) + d_{j(\alpha)}(y, v_n)}{2}\Big), \end{aligned}$$

and

$$\begin{aligned} d_\alpha(y, v_{n+1}) &= d_\alpha\big(F(y,x), F(v_n, u_n)\big) \\ &= d_\alpha\big(F(v_n, u_n), F(y,x)\big) \\ &\le \varphi_\alpha\Big(\frac{d_{j(\alpha)}(v_n, y) + d_{j(\alpha)}(u_n, x)}{2}\Big). \end{aligned}$$

It follows that

$$\frac{d_\alpha(x, u_{n+1}) + d_\alpha(y, v_{n+1})}{2} \le \varphi_\alpha\Big(\frac{d_{j(\alpha)}(x, u_n) + d_{j(\alpha)}(y, v_n)}{2}\Big),$$

for every $n = 0, 1, 2, \ldots$. Since X is j-bounded, we have

$$\begin{aligned}
&\frac{d_\alpha(x, u_{n+1}) + d_\alpha(y, v_{n+1})}{2}\\
&\quad\le \varphi_\alpha\Big(\varphi_{j(\alpha)}\Big(\frac{d_{j^2(\alpha)}(x, u_{n-1}) + d_{j^2(\alpha)}(y, v_{n-1})}{2}\Big)\Big)\\
&\quad\le \varphi_\alpha\Big(\varphi_{j(\alpha)}\Big(\ldots\varphi_{j^{n-1}(\alpha)}\Big(\frac{d_{j^n(\alpha)}(x, u_1) + d_{j^n(\alpha)}(y, v_1)}{2}\Big)\ldots\Big)\Big)\\
&\quad\le \overline{\varphi}_\alpha^n\Big(\frac{d_{j^n(\alpha)}(x, u_1) + d_{j^n(\alpha)}(y, v_1)}{2}\Big)\\
&\quad\le \overline{\varphi}_\alpha^n\Big(\frac{q(x, u_1, \alpha) + q(y, v_1, \alpha)}{2}\Big).
\end{aligned}$$

Put $c_n^\alpha = \overline{\varphi}_\alpha^n\Big(\frac{q(x, u_1, \alpha) + q(y, v_1, \alpha)}{2}\Big)$. By the same computation as in the proof of Theorem 3.1, we can deduce that $\sum_{n=0}^\infty c_n^\alpha$ is convergent. This yields $c_n^\alpha \to 0$ as $n \to \infty$. It follows that $d_\alpha(x, u_{n+1}) + d_\alpha(y, v_{n+1}) \to 0$ as $n \to \infty$ for every $\alpha \in I$. Thus we have $x = \lim_{n\to\infty} u_n, y = \lim_{n\to\infty} v_n$.

Similarly, we get $z = \lim_{n\to\infty} u_n, t = \lim_{n\to\infty} v_n$. Hence $x = z$ and $y = t$. □

Corollary 3.4. *In addition to hypotheses of Theorem 3.3, if x_0 and y_0 are comparable then F has a unique fixed point, that is, there exists $x \in X$ such that $F(x, x) = x$.*

Proof. Since x_0, y_0 are comparable, we have $x_0 \ge y_0$ or $x_0 \le y_0$. Assume that $x_0 \ge y_0$. Then, by the mixed monotone property of F, we have

$$x_1 = F(x_0, y_0) \ge F(y_0, y_0) \ge F(y_0, x_0) = y_1,$$

and by induction we get the sequences $\{x_n\}$ and $\{y_n\}$ satisfying $x_n \ge y_n$ for all $n \ge 0$. Now, since $x = \lim_{n\to\infty} x_{n+1}, y = \lim_{n\to\infty} y_{n+1}$, by the continuity of d_α, we have

$$d_\alpha(x, y) = \lim_{n\to\infty} d_\alpha(x_{n+1}, y_{n+1}). \tag{3.8}$$

On the other hand

$$\begin{aligned}
d_\alpha(x_{n+1}, y_{n+1}) &= d_\alpha\big(F(x_n, y_n), F(y_n, x_n)\big)\\
&\le \varphi_\alpha\Big(\frac{d_{j(\alpha)}(x_n, y_n) + d_{j(\alpha)}(y_n, x_n)}{2}\Big) = \varphi_\alpha\big(d_{j(\alpha)}(x_n, y_n)\big)\\
&\le \varphi_\alpha\big(\varphi_{j(\alpha)}(d_{j^2(\alpha)}(x_{n-1}, y_{n-1}))\big)\\
&\le \varphi_\alpha\big(\varphi_{j(\alpha)}\big(\ldots\varphi_{j^{n-1}(\alpha)}\big(d_{j^n(\alpha)}(x_1, y_1)\big)\ldots\big)\big)\\
&\le \overline{\varphi}_\alpha^n\big(d_{j^n(\alpha)}(x_1, y_1)\big) \le \overline{\varphi}_\alpha^n\big(q(x_1, y_1, \alpha)\big).
\end{aligned} \tag{3.9}$$

By the same argument as in the proof of Theorem 3.3, we infer that the series $\sum_{n=0}^\infty \overline{\varphi}_\alpha^n\big(q(x_1, y_1, \alpha)\big)$ is convergent. Thus

$$\lim_{n\to\infty} \overline{\varphi}_\alpha^n\big(q(x_1, y_1, \alpha)\big) = 0. \tag{3.10}$$

From (3.8), (3.9) and (3.10), we have $d_\alpha(x, y) = \lim_{n\to\infty} d_\alpha(x_{n+1}, y_{n+1}) \le \lim_{n\to\infty} \overline{\varphi}_\alpha^n\big(q(x_1, y_1, \alpha)\big) = 0$. Hence, $d_\alpha(x, y) = 0$ for all $\alpha \in I$. This implies that $x = y$ or $F(x, x) = x$.

In the case $x_0 \le y_0$, the proof is similar. □

The following examples illustrate for our theorems.

Example 3.5. Let $X = \mathbb{R}^\infty = \big\{x = \{x_n\} : x_n \in \mathbb{R}, n = 1, 2, \dots\big\}$ and $P_n : X \to \mathbb{R}$ be a mapping defined by $P_n(x) = P_n\big(\{x_n\}\big) = x_n$ for each $n = 1, 2, \dots$. Let $I = \mathbb{N}^* \times \mathbb{R}^+$ be the index set. For every $(n, r) \in I$ consider the map $d_{(n,r)} : X \times X \to \mathbb{R}$ defined by

$$d_{(n,r)}(x, y) = r\big|P_n(x) - P_n(y)\big|, \text{ for every } x, y \in X.$$

Then $\{d_{(n,r)} : (n, r) \in I\}$ is a family of pseudometrics on X generating the uniform structure on X.

Consider the partially ordered relation " $\leq$ " on X which defined by $x \leq y \Leftrightarrow x_n \leq y_n$ for every $n = 1, 2, \dots$

Let $F : X \times X \to X$ be a map defined by

$$F(x, y) = \Big\{1, 1 + \Big(1 - \frac{1}{2}\Big)(x_2 - y_2), 1 + \Big(1 - \frac{1}{3}\Big)(x_3 - y_3), \dots\Big\} \tag{3.11}$$
$$\text{for every } x = \{x_n\}, y = \{y_n\} \in X.$$

We now claim that F satisfies all conditions of Theorem 3.1. Indeed, for every $(n, r) \in I$ we consider the map $\varphi_{(n,r)} : [0, +\infty) \to [0, +\infty)$ defined by $\varphi_{(n,r)}(t) = \frac{2(n-1)}{2n-1}t$ for every $t \geq 0$, and denote by $j : I \to I$ the map defined by $j(n, r) = \big(n, 2r(1 - \frac{1}{2n})\big)$ for every $(n, r) \in I$. It is easy to see that

$$\varphi_{j^k(n,r)}(t) = \frac{2(n-1)}{2n-1}t = \varphi_{(n,r)}(t) \text{ for every } k = 0, 1, 2, \dots$$

Denote

$$\overline{\varphi}_{(n,r)}(t) = \frac{2(n-1)}{2n-1}t, \text{ for every } t \geq 0 \text{ and } (n, r) \in I.$$

Then we have

$$\sup\{\varphi_{j^k(n,r)}(t) : k = 0, 1, 2, \dots\} \leq \overline{\varphi}_{(n,r)}(t), \text{ for every } t \geq 0,$$

and $\frac{\overline{\varphi}_{(n,r)}(t)}{t} = \frac{2(n-1)}{2n-1}$ is monotone non-decreasing.

Next, we show that F has the mixed monotone property. Indeed, if $x^1, x^2, y \in X$ and $x^1 \leq x^2$ then $x^1_n \leq x^2_n$ for every $n = 1, 2, \dots$ It follows that $x^1_n - y_n \leq x^2_n - y_n$, for every $n = 1, 2, \dots$. This implies that

$$\Big(1 - \frac{1}{n}\Big)(x^1_n - y_n) \leq \Big(1 - \frac{1}{n}\Big)(x^2_n - y_n), \ n = 1, 2, \dots$$

or

$$P_n\big(F(x^1, y)\big) \leq P_n\big(F(x^2, y)\big).$$

Thus $F(x^1, y) \leq F(x^2, y)$.

Now, if $x, y^1, y^2 \in X$ and $y^1 \leq y^2$ then $y^1_n \leq y^2_n$ for every $n = 1, 2, \dots$ It follows that $x_n - y^1_n \geq x_n - y^2_n$. This implies that $\big(1 - \frac{1}{n}\big)(x_n - y^1_n) \geq \big(1 - \frac{1}{n}\big)(x_n - y^2_n)$, for every $n = 1, 2, \dots$, that is $F(x, y^1) \geq F(x, y^2)$. Hence F has the mixed monotone property.

Now, we show that F satisfies the contractive condition (3.1) with φ_α and j above mentioned. Indeed, let $x \le u, y \ge v$ then for every $(n,r) \in I$ we have

$$\begin{aligned} d_{(n,r)}\big(F(x,y),F(u,v)\big) &= r\big|P_n\big(F(x,y)\big) - P_n\big(F(u,v)\big)\big| \\ &= r\Big|\Big(1-\frac{1}{n}\Big)(x_n - y_n) - \Big(1-\frac{1}{n}\Big)(u_n - v_n)\Big| \\ &= r\Big(1-\frac{1}{n}\Big)\big|(x_n-y_n)-(u_n-v_n)\big| \\ &= r\Big(1-\frac{1}{n}\Big)(u_n - x_n + y_n - v_n), \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} d_{j(n,r)}(x,u) &= d_{\left(n,2r(1-\frac{1}{2n})\right)}(x,u) \\ &= 2r\Big(1-\frac{1}{2n}\Big)|x_n - u_n| \\ &= 2r\Big(1-\frac{1}{2n}\Big)(u_n - x_n), \end{aligned}$$

and

$$\begin{aligned} d_{j(n,r)}(y,v) &= d_{\left(n,2r(1-\frac{1}{2n})\right)}(y,v) \\ &= 2r\Big(1-\frac{1}{2n}\Big)|y_n - v_n| \\ &= 2r\Big(1-\frac{1}{2n}\Big)(y_n - v_n). \end{aligned}$$

This implies that

$$\begin{aligned} \varphi_{(n,r)}\Big(\frac{d_{j(n,r)}(x,u)+d_{j(n,r)}(y,v)}{2}\Big) &= \varphi_{(n,r)}\Big(r\Big(1-\frac{1}{2n}\Big)(u_n - x_n + y_n - v_n)\Big) \\ &= \frac{2(n-1)}{2n-1}r\Big(1-\frac{1}{2n}\Big)(u_n - x_n + y_n - v_n) \\ &= \frac{n-1}{n}r(u_n - x_n + y_n - v_n). \end{aligned} \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$d_{(n,r)}\big(F(x,y),F(u,v)\big) \le \varphi_{(n,r)}\Big(\frac{d_{j(n,r)}(x,u)+d_{j(n,r)}(y,v)}{2}\Big).$$

Now, put $x^0 = y^0 = (1,1,\ldots)$ then by a simple computation we have $x^0 = y^0 = F(x^0,y^0) = F(y^0,x^0)$ and $d_{j^k(n,r)}\big(x^0,F(x^0,y^0)\big) + d_{j^k(n,r)}\big(y^0,F(y^0,x^0)\big) = 0 < \infty$.

Finally, it is easy to see from (3.11) that F is continuous. Hence, the conditions of Theorem 3.1 are fulfilled for F and F has at least of a coupled fixed point.

Example 3.6. Let $X = \mathbb{R}^\infty = \big\{x = \{x_n\} : x_n \in \mathbb{R}, n = 1,2,\ldots\big\}$ and $P_n : X \to \mathbb{R}$ be a mapping defined by $P_n(x) = P_n\big(\{x_n\}\big) = x_n$ for each $n = 1,2,\ldots$. Let $I = \mathbb{N}^* \times \mathbb{R}^+$ be the index set. For every $(n,r) \in I$ consider the map $d_{(n,r)} : X \times X \to \mathbb{R}$ defined by

$$d_{(n,r)}(x,y) = r\big|P_n(x) - P_n(y)\big|, \text{ for every } x,y \in X.$$

Then $\{d_{(n,r)} : (n,r) \in I\}$ is a family of pseudometrics on X generating the uniform structure on X.

Consider the partially ordered relation " $\leq$ " on X which defined by $x \leq y \Leftrightarrow x_n \leq y_n$ for every $n = 1, 2, \ldots$

Let $F : X \times X \to X$ be a map defined by

$$F(x,y) = \Big\{1, 1 + \Big(1 - \frac{1}{2}\Big).\frac{x_2 - y_2}{2}, 1 + \Big(1 - \frac{1}{3}\Big).\frac{x_3 - y_3}{2}, \ldots\Big\}.$$

Denote $\varphi_{(n,r)}(t) = \dfrac{2(n-1)}{2n-1}t$ for every $t \geq 0$ and define $j : I \to I$ by $j(n,r) = \big(n, r(1 - \frac{1}{2n})\big)$ for every $(n,r) \in I$. By the same computation as in Example 3.5, we can show that F satisfies the conditions of Theorem 3.1 with $\overline{\varphi}_{(n,r)}(t) = \dfrac{2(n-1)}{2n-1}t$.

Now, we check that X is j-bounded. Indeed, for each $(x,y) \in X$ we have

$$\begin{aligned} d_{j^k(n,r)}(x,y) &= d_{\left(n, r(1-\frac{1}{2n})^k\right)}(x,y) \\ &= r\Big(1 - \frac{1}{2n}\Big)^k \big|P_n(x) - P_n(y)\big| \\ &\leq r\big|P_n(x) - P_n(y)\big| = q\big(x, y, (n,r)\big) < \infty. \end{aligned}$$

This proves that X is j-bounded. It is easy to see that if $(x,y), (z,t) \in X \times X$ then there exists $(u,v) \in X$ is comparable to them. Thus F satisfies Theorem 3.3. Hence, F has a unique coupled fixed point, and it is $x = y = \{1, 1, \ldots\}$.

Remark 3.7. Note that we cannot omit the j-bounded property of X in the Theorem 3.3. This is illustrated by the following example.

Example 3.8. By using Example 3.5, it is not difficult to see that X is not j-bounded. Indeed, for every $(n,r) \in I$ we have $j(n,r) = \big(n, 2r(1 - \frac{1}{2n})\big)$, $j^2(n,r) = j\big(n, 2r(1 - \frac{1}{2n})\big) = \big(n, 2^2 r(1 - \frac{1}{2n})^2\big)$, and by induction we get

$$j^k(n,r) = \Big(n, 2^k r\Big(1 - \frac{1}{2n}\Big)^k\Big), \text{ for every } k = 1, 2, \ldots$$

Thus, for any $x, y \in X$, we have

$$\begin{aligned} d_{j^k(n,r)}(x,y) &= d_{\left(n, 2^k r(1-\frac{1}{2n})^k\right)}(x,y) \\ &= 2^k r\Big(1 - \frac{1}{2n}\Big)^k \big|P_n(x) - P_n(y)\big| \\ &= r\Big(\frac{2n-1}{n}\Big)^k \big|P_n(x) - P_n(y)\big|. \end{aligned}$$

Since $\lim\limits_{k\to\infty} (\frac{2n-1}{n})^k = \infty$ for each $n > 1$, we can conclude that there is no $q\big(x, y, (n,r)\big) < +\infty$ such that $d_{j^k(n,r)}(x,y) < q\big(x, y, (n,r)\big)$ for every $k = 0, 1, 2, \ldots$. This proves that X is not j-bounded.

In fact, F have more than one coupled fixed point. For example, we consider

$$x = \{1, x_2, 1, 1, \ldots\}, y = \{1, y_2, 1, 1, \ldots\},$$

with $x_2 + y_2 = 2$. It is easy to check that x, y are coupled fixed points of F.

4. Application to nonlinear intergral equations

As an application of the coupled fixed point theorems proved in the previous section, in this section we will investigate the existence of a unique solution to nonlinear integral equations.

Let us consider the following integral equations

$$x(t) = h(t) + \int_0^{\Delta(t)} \big[K_1(t,s) + K_2(t,s)\big]\big(f(s,x(s)) + g(s,x(s))\big)ds, \tag{4.1}$$

where $K_1, K_2 \in C\big([0,+\infty)\times[0,+\infty),\mathbb{R}\big)$, $f, g \in C\big([0,+\infty)\times\mathbb{R},\mathbb{R}\big)$, and an unknown function $x(t) \in C\big([0,+\infty),\mathbb{R}\big)$. The deviation $\Delta : [0,+\infty) \to [0,+\infty)$ is a continuous function, in general case, unbounded. Note that, since deviation $\Delta : [0,+\infty) \to [0,+\infty)$ is unbounded, we can not apply the known coupled fixed point theorems in metric space (see [4], [5], [9]) for the above integral equations.

Adopting the assumptions in [3] and [9], we assume that the functions K_1, K_2, f, g fulfill the following conditions

Assumption 4.1. A) $K_1(t,s) \geq 0$ and $K_2(t,s) \leq 0$ for all $t, s \geq 0$;

B) For each compact subset $K \subset \mathbb{R}$, there exist the positive numbers λ, μ and $\varphi_K \in \Phi$ such that for all $x, y \in \mathbb{R}$, $x \geq y$ and for all $t \in K$,

$$0 \leq f(t,x) - f(t,y) \leq \lambda\varphi_K\Big(\frac{x-y}{2}\Big)$$

and

$$-\mu\varphi_K\Big(\frac{x-y}{2}\Big) \leq g(t,x) - g(t,y) \leq 0$$

and

$$\max(\lambda,\mu)\sup_{t\in K}\int_0^{\Delta(t)} \big(K_1(t,s) - K_2(t,s)\big)ds \leq \frac{1}{2};$$

C) For each compact subset $K \subset \mathbb{R}$, there exists a compact set $\overline{K} \subset \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$\Delta^n(K) \subset \overline{K};$$

D) For each compact subset $K \subset \mathbb{R}$, there exists $\overline{\varphi}_K \in \Phi$ such that $\dfrac{\overline{\varphi}_K(t)}{t}$ is non-decreasing and

$$\varphi_{\Delta^n(K)}(t) \leq \overline{\varphi}_K(t)$$

for all $n \in \mathbb{N}$ and for all $t \geq 0$.

Definition 4.2. An element $(\alpha,\beta) \in C\big([0,+\infty),\mathbb{R}\big) \times C\big([0,+\infty),\mathbb{R}\big)$ is a coupled lower and upper solution of the integral equation (4.1) if for any $t \in [0,+\infty)$ we have $\alpha(t) \leq \beta(t)$ and

$$\begin{aligned}\alpha(t) \;\leq\; & h(t) + \int_0^{\Delta(t)} K_1(t,s)\big(f(s,\alpha(s)) + g(s,\beta(s))\big)ds \\ & + \int_0^{\Delta(t)} K_2(t,s)\big(f(s,\beta(s)) + g(s,\alpha(s))\big)ds,\end{aligned}$$

and

$$\begin{aligned}\beta(t) \quad &\geq \quad h(t) + \int_0^{\Delta(t)} K_1(t,s)\big(f(s,\beta(s)) + g(s,\alpha(s))\big)ds \\ &\quad + \int_0^{\Delta(t)} K_2(t,s)\big(f(s,\alpha(s)) + g(s,\beta(s))\big)ds.\end{aligned}$$

Theorem 4.3. *Consider the integral equation (4.1) with* $K_1, K_2 \in C\big([0,+\infty) \times [0,+\infty), \mathbb{R}\big)$ *and* $f, g \in C\big([0,+\infty) \times \mathbb{R}, \mathbb{R}\big)$ *and* $h \in C\big([0,+\infty), \mathbb{R}\big)$ *and suppose that Assumption 4.1 is fulfilled. Then the existence of a coupled lower and upper solution for (4.1) provides the existence of a unique solution of (4.1) in* $C\big([0,+\infty), \mathbb{R}\big)$.

Proof. Denote $\mathbb{R}^+ = [0,+\infty)$. Let $X = C(\mathbb{R}^+, \mathbb{R})$. Then X is a partially ordered set with the partially ordered relation on X defined

$$\text{for } x, y \in X, \quad x \leq y \Leftrightarrow x(t) \leq y(t), \ \text{ for all } \ t \in \mathbb{R}^+.$$

For each compact subset $K \subset \mathbb{R}$, we define

$$p_K(f) = \sup\{|f(t)| : t \in K\}, \ \text{for every } f \in C(\mathbb{R}^+, \mathbb{R}).$$

It is known that the family of seminorms $\{p_K\}$, where K runs over all compact subsets of $\mathbb{R}$, defines a locally convex Hausdorff topology of the space. Hence, X is a Hausdorff sequentially uniform space whose uniformity is generated by the family of pseudometrics $\{d_K\}$ (K runs over all compact subsets of $\mathbb{R}$) defined by

$$d_K(f,g) = p_K(f-g) = \sup\{|f(t) - g(t)| : t \in K\}.$$

Let us next define the map $j : I \to I$, where the index set I consists of all compact subsets of $\mathbb{R}^+$, by the following way: For an arbitrary compact set $K \subset \mathbb{R}^+$ we put $j(K) := [0, \max_{t \in K} \Delta(t)]$, and $j^n(K) = j\big(j^{n-1}(K)\big)$, for every $n \in \mathbb{N}$. Then, since $\Delta : [0,+\infty) \to [0,+\infty)$ is a continuous function the sets $j(K), j^2(K), j^3(K), \ldots$ are also compact.

Suppose $\{u_n\}$ is a monotone non-decreasing sequence in X that converges to $u \in X$. Then for every $t \in \mathbb{R}^+$, we have $u_1(t) \leq u_2(t) \leq \cdots \leq u_n(t) \leq \ldots$, and $u_n(t) \to u(t)$ as $n \to \infty$. This implies that $u_n(t) \leq u(t)$ for every $t \in \mathbb{R}^+$, and $n \in \mathbb{N}$. Hence $u_n \leq u$, for all $n \in \mathbb{N}$.

Similarly, we can verify that if $\{v_n\}$ is a monotone non-increasing sequence in X that converges to $v \in X$, then $v \leq v_n$ for all n. Hence, the condition (b) in Theorem 3.1 holds.

Now we define on $X \times X$ the following partially ordered relation: for every $(x,y), \ (u,v) \in X \times X$

$$(x,y) \leq (u,v) \Leftrightarrow x(t) \leq u(t) \ \text{ and } \ y(t) \geq v(t), \ \text{for every } t \in \mathbb{R}^+.$$

Observe that for every $x, y \in X$, by the uniform topology of X, we easily see that the functions $\max\{x(t), y(t)\}$, $\min\{x(t), y(t)\}$ for each $t \in \mathbb{R}^+$ are the upper and lower bounds of x, y, respectively in X. This follows that for every $(x,y), (u,v) \in X \times X$, there exists $\big(\max\{x,u\}, \min\{y,v\}\big) \in X \times X$ which is comparable to (x,y) and (u,v).

Define $F : X \times X \to X$ by

$$\begin{aligned} F(x,y)(t) &= \int_0^{\Delta(t)} K_1(t,s)\big(f(s,x(s)) + g(s,y(s))\big)ds \\ &\quad + \int_0^{\Delta(t)} K_2(t,s)\big(f(s,y(s)) + g(s,x(s))\big)ds + h(t) \end{aligned}$$

for all $t \in \mathbb{R}^+$.

Next, we show that F has the mixed monotone property. Indeed, for $x_1, x_2 \in C(\mathbb{R}^+, \mathbb{R})$ and $x_1 \leq x_2$, that is $x_1(t) \leq x_2(t)$ for every $t \in \mathbb{R}^+$, by Assumption 4.1 we have

$$\begin{aligned} F(x_1,y)(t) - F(x_2,y)(t) &= \int_0^{\Delta(t)} K_1(t,s)\big[f(s,x_1(s)) + g(s,y(s))\big]ds \\ &\quad + \int_0^{\Delta(t)} K_2(t,s)\big[f(s,y(s)) + g(s,x_1(s))\big]ds + h(t) \\ &\quad - \int_0^{\Delta(t)} K_1(t,s)\big[f(s,x_2(s)) + g(s,y(s))\big]ds \\ &\quad - \int_0^{\Delta(t)} K_2(t,s)\big[f(s,y(s)) + g(s,x_2(s))\big]ds - h(t) \\ &= \int_0^{\Delta(t)} K_1(t,s)\big[f(s,x_1(s)) - f(s,x_2(s))\big]ds \\ &\quad - \int_0^{\Delta(t)} K_2(t,s)\big[g(s,x_2(s)) - g(s,x_1(s))\big]ds \leq 0, \end{aligned}$$

for every $t \in \mathbb{R}^+$. This yields $F(x_1,y)(t) \leq F(x_2,y)(t)$ for every $t \in \mathbb{R}^+$, that is $F(x_1,y) \leq F(x_2,y)$.

By the same computation, we arrive at $F(x,y_1) \leq F(x,y_2)$ if $y_1, y_2 \in C(\mathbb{R}^+, \mathbb{R})$ and $y_1 \geq y_2$. Hence, F has the mixed monotone property.

Now, for each compact subset K of $\mathbb{R}$ and for $x \geq u$ and $y \leq v$, that is $x(t) \geq u(t)$ and $y(t) \leq v(t)$ for every $t \in \mathbb{R}^+$, we have

$$\begin{aligned} &d_K\big(F(x,y), F(u,v)\big) = \sup_{t\in K} \big|F(x,y)(t) - F(u,v)(t)\big| \\ &= \sup_{t\in K} \Big| \int_0^{\Delta(t)} K_1(t,s)\big(f(s,x(s)) + g(s,y(s))\big)ds \\ &\quad + \int_0^{\Delta(t)} K_2(t,s)\big(f(s,y(s)) + g(s,x(s))\big)ds + h(t) \\ &\quad - \int_0^{\Delta(t)} K_1(t,s)\big(f(s,u(s)) + g(s,v(s))\big)ds \\ &\quad - \int_0^{\Delta(t)} K_2(t,s)\big(f(s,v(s)) + g(s,u(s))\big)ds - h(t)\Big| \end{aligned}$$

$$
\begin{aligned}
&= \sup_{t\in K}\Big|\int_0^{\Delta(t)} K_1(t,s)\big[\big(f(s,x(s))-f(s,u(s))\big)+\big(g(s,y(s))-g(s,v(s))\big)\big]ds\\
&\quad+\int_0^{\Delta(t)} K_2(t,s)\big[\big(f(s,y(s))-f(s,v(s))\big)+\big(g(s,x(s))-g(s,u(s))\big)\big]ds\Big|\\
&= \sup_{t\in K}\Big|\int_0^{\Delta(t)} K_1(t,s)\big[\big(f(s,x(s))-f(s,u(s))\big)-\big(g(s,v(s))-g(s,y(s))\big)\big]ds\\
&\quad-\int_0^{\Delta(t)} K_2(t,s)\big[\big(f(s,v(s))-f(s,y(s))\big)-\big(g(s,x(s))-g(s,u(s))\big)\big]ds\Big|\\
&\le \sup_{t\in K}\Big|\int_0^{\Delta(t)} K_1(t,s)\Big[\lambda\varphi_K\Big(\frac{x(s)-u(s)}{2}\Big)+\mu\varphi_K\Big(\frac{v(s)-y(s)}{2}\Big)\Big]ds\\
&\qquad-\int_0^{\Delta(t)} K_2(t,s)\Big[\lambda\varphi_K\Big(\frac{v(s)-y(s)}{2}\Big)+\mu\varphi_K\Big(\frac{x(s)-u(s)}{2}\Big)\Big]ds\Big|\\
&\le \max\{\lambda,\mu\}\sup_{t\in K}\int_0^{\Delta(t)}\big[K_1(t,s)-K_2(t,s)\big]\Big[\varphi_K\Big(\frac{x(s)-u(s)}{2}\Big)\\
&\qquad+\varphi_K\Big(\frac{v(s)-y(s)}{2}\Big)\Big]ds\\
&\le \max\{\lambda,\mu\}\sup_{t\in K}\int_0^{\Delta(t)}\big[K_1(t,s)-K_2(t,s)\big]ds\Big[\varphi_K\Big(\sup_{s\in[0,\max_{t\in K}\Delta(t)]}\frac{|x(s)-u(s)|}{2}\Big)\\
&\qquad+\varphi_K\Big(\sup_{s\in[0,\max_{t\in K}\Delta(t)]}\frac{|v(s)-y(s)|}{2}\Big)\Big]\\
&= \max\{\lambda,\mu\}\sup_{t\in K}\int_0^{\Delta(t)}\big[K_1(t,s)-K_2(t,s)\big]ds\Big[\varphi_K\Big(\sup_{s\in j(K)}\frac{|x(s)-u(s)|}{2}\Big)\\
&\qquad+\varphi_K\Big(\sup_{s\in j(K)}\frac{|v(s)-y(s)|}{2}\Big)\Big]\\
&\le \frac12\Big[\varphi_K\Big(\frac{d_{j(K)}(x,u)}{2}\Big)+\varphi_K\Big(\frac{d_{j(K)}(y,v)}{2}\Big)\Big]\\
&\le \frac12\Big[\varphi_K\Big(\frac{d_{j(K)}(x,u)+d_{j(K)}(y,v)}{2}\Big)+\varphi_K\Big(\frac{d_{j(K)}(y,v)+d_{j(K)}(x,u)}{2}\Big)\Big]\\
&= \varphi_K\Big(\frac{d_{j(K)}(x,u)+d_{j(K)}(y,v)}{2}\Big).
\end{aligned}
$$

Now, let us $(\alpha,\beta)\in X\times X$ be a coupled lower and upper solution of the integral equation of (4.1). Then, we have

$$\alpha(t)\le F(\alpha,\beta)(t)\quad\text{and}\quad \beta(t)\ge F(\beta,\alpha)(t)\ \text{ for all } t\in\mathbb{R}^+,$$

that is $\alpha\le F(\alpha,\beta)$ and $\beta\ge F(\beta,\alpha)$. Moreover, for each compact subset $K\subset\mathbb{R}$, by the continuity and assumption, we have

$$
\begin{aligned}
d_{j^n(K)}\big(\alpha,F(\alpha,\beta)\big)+d_{j^n(K)}\big(\beta,F(\beta,\alpha)\big) \;\le\;& d_{[0,\max_{s\in\overline{K}}\Delta(s)]}\big(\alpha,F(\alpha,\beta)\big)\\
&+d_{[0,\max_{s\in\overline{K}}\Delta(s)]}\big(\beta,F(\beta,\alpha)\big)<+\infty.
\end{aligned}
$$

Using again assumption (C), we have

$$\begin{aligned} d_{j^n(K)}(x,y) &= \sup_{t\in j^n(K)} \big|x(t)-y(t)\big| \le \sup_{t\in[0,\max_{s\in\overline{K}}\Delta(s)]} \big|x(t)-y(t)\big| \\ &= d_{[0,\max_{s\in\overline{K}}\Delta(s)]}(x,y) < +\infty, \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore X is j-bounded.

Finally, applying Theorem 3.3, we can conclude that F has a unique coupled fixed point (x, y). Since $\alpha \le \beta$ and Corollary 3.4, we have $x = y$, that is $x(t) = y(t)$ for every $t \in \mathbb{R}^+$. Hence $F(x, x) = x$ and x is the unique solution of the equation (4.1). □

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Tran Van An
Department of Mathematics, Vinh University, 182 Le Duan, Vinh City, Vietnam
E-mail address: andhv@yahoo.com

Kieu Phuong Chi
Department of Mathematics, Vinh University, 182 Le Duan, Vinh City, Vietnam
E-mail address: chidhv@gmail.com

Le Khanh Hung
High school for Gifted students, Vinh University, 182 Le Duan, Vinh City, Vietnam
E-mail address: lekhanhhungdhv@gmail.com