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EXISTENCE OF COUPLED FIXED POINTS VIA MEASURE OF NONCOMPACTNESS AND APPLICATIONS

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Abstract. Using the technique of measures of noncompactness we present some results on the existence of coupled fixed points for a class of operators. Also as an application, we discuss the existence of solutions for a general system of nonlinear functional integral equations. Furthermore, we present an example to show the efficiency of our results.

1. Introduction and preliminaries

Bhaskar and Lakshmikantham [13] introduced the concept of a coupled fixed point and obtained some coupled fixed point theorems for a class of operators and later, many authors generalized their results (see [1,16,19,20]). It is worth to mention that existence theorems of coupled fixed point can be used to investigate the existence of solutions for systems of functional integral equations and boundary value problems (see [13, 14, 16, 18, 21]). In this paper, we prove some existence theorems of coupled fixed point for some classes of operators using the concept of measure of noncompactness, that was first introduced by Kuratowski in [17] and has been successfully applied in the theories of differential and integral equations, see for examples ([3, 4, 5, 10, 12, 17]). For this purpose, we reduce the problem of existence of coupled fixed points to the problem of existence of fixed points, and using Schauder's fixed point theorem, Darbo's fixed point theorem for condensing operators and constructing some measures of noncompactness on product spaces, we prove our main results.
Throughout this paper we assume that E is a Banach space. For a subset X of E, the closure and closed convex hull of X in E are denoted by $\bar{X}$ and $co(X)$, respectively. Also let $\bar{B}_r$ be the closed ball in E centered at zero and with radius r and we write $B(x_0, r)$ to denote the closed ball centered at x_0 with radius r. Moreover, we symbolize by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ the subfamily consisting of all relatively compact subsets of E.

The following definitions will be needed in the sequel.

Definition 1.1 ([8])**.** A mapping $\mu : \mathfrak{M}_E \longrightarrow [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions;

(B_1) The family $Ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0$ is nonempty and $Ker\mu \subseteq \mathfrak{N}_E$.
(B_2) $X \subseteq Y \Rightarrow \mu(X) \leq \mu(Y)$.
(B_3) $\mu(\overline{X}) = \mu(X)$.

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(B_4) $\mu(CoX) = \mu(X)$.
(B_5) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1)$.
(B_6) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subseteq X_n$, $(n \ge 1)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $Ker\mu$ described in (B_1) said to be the kernel of the measure of noncompactness μ. Observe that the intersection set X_∞ from (B_6) is a member of the family $Ker\mu$. In fact, since $\mu(X_\infty) \le \mu(X_n)$ for any n, we infer that $\mu(X_\infty) = 0$. This yields that $X_\infty \in Ker\mu$.

Definition 1.2 ([14])**.** An element $(x,y) \in X \times X$ is called a coupled fixed point of a mapping $G : X \times X \to X$ if $G(x,y) = x$ and $G(y,x) = y$.

The following theorems and examples are basic to all the results of this work.

Theorem 1.3 ([11])**.** *Suppose $\mu_1, \mu_2, \ldots, \mu_n$ are measures of noncompactness in $E_1, E_2, \ldots, E_n$ respectively. Moreover assume that the function $F : [0,\infty)^n \to [0,\infty)$ is convex and $F(x_1, x_2, \ldots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 3, \ldots, n$. Then*

$$\mu(X) = F(\mu_1(X_1), \mu_2(X_2), \ldots, \mu_n(X_n))$$

defines a measure of noncompactness in $E_1 \times E_2 \times \cdots \times E_n$ where X_i denotes the natural projection of X into E_i for $i = 1, 2, \ldots, n$.

Now, as results from Theorem 1.3, we present the following examples.

Example 1.4. Let μ be a measure of noncompactness, considering $F(x,y) = \max\{x,y\}$ for any $(x,y) \in [0,\infty)^2$, then all the conditions of Theorem 1.3 are satisfied. Therefore, $\widetilde{\mu}(X) = \max\{\mu(X_1), \mu(X_2)\}$ is a measure of noncompactness in the space $E \times E$ where X_i, $i = 1, 2$ denote the natural projections of X.

Example 1.5. Let μ be a measure of noncompactness. We define $F(x,y) = x + y$ for any $(x,y) \in [0,\infty)^2$. Then F has all the properties mentioned in Theorem 1.3. Hence $\widetilde{\mu}(X) = \mu(X_1) + \mu(X_2)$ is a measure of noncompactness in the space $E \times E$ where X_i, $i = 1, 2$ denote the natural projections of X.

Theorem 1.6 (Darbo [8])**.** *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $G : \Omega \longrightarrow \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0,1)$ such that*

$$\mu(G(X)) \le k\mu(X) \tag{1.1}$$

for any nonempty set $X \subset \Omega$. Then G has a fixed point.

Theorem 1.7 (Schauder [2])**.** *Let Ω be a closed, convex subset of a normed linear space E. Then every compact, continuous map $G : \Omega \to \Omega$ has at least one fixed point.*

In addition, let $BC(\mathbb{R}_+)$ be the set of all real functions defined, bounded and continuous on $\mathbb{R}_+$. The norm in $BC(\mathbb{R}_+)$ is defined as the standard supremom norm, i.e,

$$\|x\|_\infty = \sup\{|x(t)| : \ t \ge 0\}. \tag{1.2}$$

We will use a measure of noncompactness in the space $BC(\mathbb{R}_+)$ which is stated in ([9,11]). In order to define this measure let us fix a nonempty bounded subset of X of $BC(\mathbb{R}_+)$ and a positive number $L > 0$. For $x \in X$ and $\varepsilon \geq 0$ denote by $\omega^L(x, \varepsilon)$, the modulus of continuity of x on the interval $[0, L]$, i.e,

$$\omega^L(x,\varepsilon) = \sup\{|x(t) - x(s)| : \ t, s \in [0, L]\,,\ |t - s| \leq \varepsilon\}.$$

Moreover, let us put

$$\begin{aligned} \omega^L(X,\varepsilon) &= \sup\left\{\omega^L(x,\varepsilon) : \ x \in X\right\}, \\ \omega_0^L(X) &= \lim_{\varepsilon\to 0} \omega^L(X,\varepsilon), \\ \omega_0(X) &= \lim_{l\to\infty} \omega_0^L(X). \end{aligned}$$

If t is a fixed number from $\mathbb{R}_+$, let us denote

$$X(t) = \{x(t) : \ x \in X\}\,.$$

Finally, consider the function μ defined on $\mathfrak{M}_{BC(\mathbb{R}_+)}$ by the formula

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} diam X(t) \tag{1.3}$$

where

$$diam X(t) = \sup\left\{|x(t) - y(t)| : \ x, y \in X\right\}.$$

It can be shown ([9, 11]) that the function $\mu(X)$ defines a measure of noncompactness in the sense of the above accepted definition. The kernel $\ker\mu$ of this measure contains nonempty and bounded sets X such that functions belonging to X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by functions from X tends to zero at infinity.

2. Main results

Before starting the main results, we always suppose that Ω is a nonempty, bounded, closed, and convex subset of E, moreover

$$\Lambda = \left\{\delta : [0,\infty) \to [0,\infty) : \delta \text{ is increasing map and } \lim_{n\to\infty} \delta^n(t) = 0 \text{ for each t} > 0\right\}.$$

Theorem 2.1. *Let $G : \Omega \times \Omega \longrightarrow \Omega$ be a continuous function such that*

$$\mu(G(X_1 \times X_2) \leq k \max\{\mu(X_1), \mu(X_2)\} \tag{2.1}$$

for any $X_1, X_2 \subset \Omega$, where μ is an arbitrary measure of noncompactness and k is a constant with $0 \leq k < 1$. Then G has at least a coupled fixed point.

Proof. First note that , example 1.4 implies that $\widetilde{\mu}(X) = \max\{\mu(X_1), \mu(X_2)\}$ is a measure of noncompactness in the space $E \times E$ where X_i, $i = 1, 2$ denote the natural projections of X. Also the map $\widetilde{G} : \Omega \times \Omega \longrightarrow \Omega \times \Omega$ where

$$\widetilde{G}(x, y) = (G(x, y), G(y, x)).$$

is clearly continuous on $\Omega \times \Omega$ by its definition. Now we claim that $\widetilde{G}$ satisfies all the conditions of Theorem 1.6. To prove this, let $X \subset \Omega \times \Omega$ be a nonempty subset. Then, by $(\mathbf{B}_2)$ and (2.1) we earn

$$\widetilde{\mu}(\widetilde{G}(X)) \ \leq \ \widetilde{\mu}(G(X_1 \times X2), G(X_2 \times X_1))$$

$$
\begin{aligned}
&= \max\{\mu(G(X_1 \times X_2)), \mu(G(X_2 \times X_1))\} \\
&\leq \max\{k \max\{\mu(X_1), \mu(X_2)\}, k \max\{\mu(X_2), \mu(X_1)\}\} \\
&= k \max\{\mu(X_1), \mu(X_2)\}.
\end{aligned}
$$

Hence

$$\widetilde{\mu}(G(X)) \leq k\widetilde{\mu}(X)$$

Thus, our conclusion follows from Theorem 1.6. □

Corollary 2.2. *Let $G : \Omega \times \Omega \longrightarrow \Omega$ be a continuous function such that*

$$\|G(x, y) - G(u, v)\| \leq k \max\{\|x - u\|, \|y - v\|\}$$

for any $(x, y), (u, v) \in \Omega \times \Omega$ where, $0 \leq k < 1$ be a constant. Then G has a coupled fixed point.

Proof. It is easy to see that the map $\mu : \mathfrak{M}_E \longrightarrow [0, \infty)$ defined by $\mu(X) = diam(X)$ is a measure of noncompactness. Therefore, it is sufficient to prove that the inequality (2.1) is satisfied. To do this, let $X_1, X_2 \subset \Omega$ and $(x, y), (u, v) \in X_1 \times X_2$. Then, we get

$$
\begin{aligned}
\|G(x, y) - G(u, v)\| &\leq k \max\{\|x - u\|, \|y - v\|\} \\
&\leq k \max\{diam(X_1), diam(X_2)\}
\end{aligned}
$$

Thus

$$diam(G(X_1 \times X_2)) \leq k \max\{diam(X_1), diam(X_2)\}.$$

So, by Theorem 2.1 G has a coupled fixed point. □

The following lemma is crucial to our next result.

Lemma 2.3. *Let $\delta \in \Lambda$ and $G : \Omega \longrightarrow \Omega$ be a continuous function satisfying*

$$\mu(G(X)) \leq \delta(\mu(X)) \tag{2.2}$$

for each $X \subset \Omega$, where μ is an arbitrary measure of noncompactness. Then G has at least one fixed point.

Proof. Let $\Omega_0 = \Omega$, we construct a sequence (Ω_n) such that $\Omega_n = Co(G\Omega_{n-1})$, for $n \geq 1$. If there exists an integer $N \geq 0$ such that $\mu(C_N) = 0$ then C_N is relatively compact, therefore, Theorem 1.7 implies that G has a fixed point. Hence we shall assume that $\mu(\Omega_n) \neq 0$ for all $n \geq 0$. It is easy to see that $\delta(t) < t$ for all $t > 0$. In addition, by (2.2) we get

$$
\begin{aligned}
\mu(\Omega_{n+1}) &= \mu(Co(G(\Omega_n))) \\
&= \mu(G(\Omega_n)) \\
&\leq \delta(\mu(\Omega_n)) \\
&\leq \delta^2(\mu(\Omega_{n-1})) \\
&\;\;\vdots \\
&\leq \delta^n(\mu(\Omega)). \qquad (2.3)
\end{aligned}
$$

Therefore, taking limit as $n \to \infty$ in the inequality (2.3), we have $\mu(\Omega_n) \to 0$, as $n \to \infty$. And since $\Omega_{n+1} \subseteq \Omega_n$ and $G(\Omega_n) \subseteq \Omega_n$ for all $n \geq 1$, then by (B_6),

$\Omega_\infty = \bigcap_{n=1}^{n=\infty} \Omega_n$ is a nonempty convex closed set, invariant under G and relatively compact. Then by Theorem 1.7, G has a fixed point. $\square$

Theorem 2.4. *Let μ be an arbitrary measure of noncompactness and $\delta \in \Lambda$. Suppose that mapping $G : \Omega \times \Omega \longrightarrow \Omega$ is a continuous function satisfying*

$$\mu(G(X_1 \times X_2)) \leq \delta\Big(\frac{\mu(X_1) + \mu(X_2)}{2}\Big) \tag{2.4}$$

for all $X_1, X_2 \subset \Omega$. Then G has at least a coupled fixed point.

Proof. We define a mapping $\widetilde{G} : \Omega \times \Omega \longrightarrow \Omega \times \Omega$ by putting

$$\widetilde{G}(x, y) = (G(x, y), G(y, x)).$$

It is obvious that $\widetilde{G}$ is continuous. On the other hand, from Example 1.5, we deduce that the formula

$$\widetilde{\mu}(X) := \mu(X_1) + \mu(X_2)$$

defines a measure of noncompactness on $E \times E$ where X_i, $i = 1, 2$ denote the natural projections of X. Now let $X \subset \Omega \times \Omega$ be any nonempty subset. Then by $(\mathbf{B}_2)$ and (2.4) we obtain

$$\begin{aligned}
\widetilde{\mu}(\widetilde{G}(X)) &\leq \widetilde{\mu}(G(X_1 \times X_2), G(X_2 \times X_1)) \\
&= \mu(G(X_1 \times X_2)) + \mu(G(X_2 \times X_1))) \\
&\leq \delta\Big(\frac{\mu(X_1) + \mu(X_2)}{2}\Big) + \delta\Big(\frac{\mu(X_2) + \mu(X_1)}{2}\Big) \\
&= 2\delta\Big(\frac{\mu(X_1) + \mu(X_2)}{2}\Big) \\
&= 2\delta\Big(\frac{\widetilde{\mu}(X)}{2}\Big).
\end{aligned}$$

Hence

$$\frac{1}{2}\widetilde{\mu}(\widetilde{G}(X)) \leq \delta\Big(\frac{1}{2}\widetilde{\mu}(X)\Big).$$

Taking $\widetilde{\mu}' = \frac{1}{2}\widetilde{\mu}$, we get

$$\widetilde{\mu}'(\widetilde{G}(X)) \leq \delta(\widetilde{\mu}'(X)).$$

Since, $\widetilde{\mu}'$ is also a measure of noncompactness, therefore, all the conditions of Lemma 2.3 are satisfied. Hence G has a coupled fixed point. Let, $0 \leq k < 1$ be a constant and $\delta(t) = kt$ for each $t \in [0, \infty)$. Then, Theorem 2.4 gives the following corollary. $\square$

Corollary 2.5. *Assume that $G : \Omega \times \Omega \longrightarrow \Omega$ be a continuous function such that*

$$\mu(G(X_1 \times X_2)) \leq \frac{k}{2}(\mu(X_1) + \mu(X_2)) \tag{2.5}$$

for each $X_1, X_2 \subset \Omega$ where $0 \leq k < 1$ is a constant. Then G has a coupled fixed point.

Corollary 2.6. *Let $G : \Omega \times \Omega \longrightarrow \Omega$ be a continuous function. In addition, suppose that*

$$\|G(x, y) - G(u, v)\| \leq \delta\Big(\frac{\|x - u\| + \|y - v)\|}{2}\Big)$$

for any $(x, y), (u, v) \in \Omega \times \Omega$ *where* $\delta \in \Lambda$*. Then* G *has a coupled fixed point.*

Proof. As it mentioned before, the function $\mu : \mathfrak{M}_E \longrightarrow [0, \infty)$ defined by $\mu(X) = diam(X)$ is a measure of noncompactness. Now, let $X_1, X_2 \subset \Omega$ and $(x, y), (u, v) \in X_1 \times X_2$. Then

$$\begin{aligned} \|G(x,y) - G(u,v)\| &\leq \delta\Big(\frac{\|x-u\| + \|y-v)\|}{2}\Big) \\ &\leq \delta\Big(\frac{diam(X_1) + diam(X_2)}{2}\Big). \end{aligned}$$

This yields

$$diam(G(X)) \leq \delta\Big(\frac{diam(X_1) + diam(X_2)}{2}\Big).$$

Now, Theorem 2.4 concludes the proof □

Corollary 2.7. *Let* $G : \Omega \times \Omega \longrightarrow \Omega$ *be a continuous function. Assume that, there exists a* $k \in [0, 1)$ *with*

$$\|G(x,y) - G(u,v)\| \leq \frac{k}{2}(\|x-u\| + \|y-v)\|)$$

for any $(x, y), (u, v) \in \Omega \times \Omega$*. Then* G *has a coupled fixed point.*

Proof. Taking $\delta(t) = \frac{k}{2}$ in Corollary2.6. □

3. Existence of solutions for a system of integral equations

In this section, as an application of our results we are going to study the existence of solutions for the following system of integral equations

$$\left\{ \begin{array}{l} x(t) = f(t, x(\xi(t)), y(\xi(t))) + \int_0^{q(t)} h(t, s, x(\eta(s)), y(\eta(s)))ds, \\ y(t) = f(t, y(\xi(t)), x(\xi(t))) + \int_0^{q(t)} h(t, s, y(\eta(s)), x(\eta(s)))ds, \end{array} \right. \tag{3.1}$$

under the following general assumptions.

(i) $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and the function $t \to f(t, 0, 0)$ is a member of the space $BC(\mathbb{R}_+)$;

(ii) there exists $k \in [0, 1)$ such that

$$|f(t,x,y) - f(t,u,v)| \leq \frac{k}{2}(|x-u| + |y-v|), \tag{3.2}$$

for any $t \geq 0$ and for all $x, y, u, v \in \mathbb{R}$;

(iii) the functions ξ, η, $q : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\xi(t) \to \infty$ as $t \to \infty$.

(iv) $h : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist $x_0, y_0 \in \mathbb{R}$ and a positive constant d such that

$$\int_0^{q(t)} |h(t, s, x_0, y_0)|\, ds \leq d \tag{3.3}$$

for all $t \in \mathbb{R}_+$. In addiition,

$$\lim_{t\to\infty} \int_0^{q(t)} |h(t, s, x(\eta(s)), y(\eta(s))) - h(t, s, u(\eta(s)), v(\eta(s))|\, ds = 0, \tag{3.4}$$

$$\int_0^{q(t)} |h(t,s,x(\eta(s)),y(\eta(s))) - h(t,s,u(\eta(s)),v(\eta(s))|\,ds \quad \leq \quad \infty \tag{3.5}$$

for any $t \in \mathbb{R}_+$ and uniformly respect to $x, y, u, v \in BC(\mathbb{R}_+)$.

Then we can formulate our assertion as follows.

Theorem 3.1. *Let the conditions* $(i) - (iv)$ *hold. Then the system of equations* (3.1) *has at least one solution in the space* $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.

Proof. Let us consider the map $G : E \times E \to E$ which is defined by the formula

$$G(x,y)(t) = f(t, x(\xi(t), y(\xi(t))) + \int_0^{q(t)} h(t,s,x(\eta(s), y(\eta(s)))ds \tag{3.6}$$

where $E = BC(\mathbb{R}_+)$ and the norm $\|.\|$ on $E\times E$ is defined by $\|(x,y)\| = \|x\|_\infty + \|y\|_\infty$ for any $(x,y) \in E \times E$. It is easy to see that $G(x,y)$ is continuous on $\mathbb{R}_+$ for all $(x,y) \in E \times E$. Also, as a direct consequence of (3.2) and (3.6) we have

$$\begin{aligned}
|(G(x,y)(t)| \quad &\leq \quad |f(t, x(\xi(t)), y(\xi(t))) - f(t,0,0)| + |f(t,0,0)| \\
&\quad + \int_0^{q(t)} |h(t,s,x(\eta(s)), y(\eta(s)))) - h(t,s,x_0,y_0)|\,ds \\
&\quad + \int_0^{q(t)} |h(t,s,x_0,y_0)|\,ds \\
&\leq \quad \frac{k}{2}(|x(\xi(t))| + |y(\xi(t))|) + d_0 \qquad (3.7)
\end{aligned}$$

where by (i), (3.3) and (3.5) we get

$$d_0 := \sup_{t\in\mathbb{R}_+} |f(t,0,0)| + \sup_{t\in\mathbb{R}_+} \{ \int_0^{q(t)} |h(t,s,x(\eta(s)), y(\eta(s))) - h(t,s,x_0,y_0)|\,ds + d.$$

is finite. Hence, G maps $E \times E$ into E. Moreover, from the inequality (3.7) we get

$$\|G(x,y)\|_\infty \leq \frac{k}{2}(\|x\|_\infty + \|y\|_\infty) + d_0. \tag{3.8}$$

Thus, from the estimate (3.8) we see that $G(\bar{B}_r \times \bar{B}_r) \subset \bar{B}_r$ for $r = \frac{d_0}{1-k}$. Now we show that the map $G : \bar{B}_r \times \bar{B}_r \to \bar{B}_r$ is continuous. In fact for $\varepsilon > 0$, $(u,v) \in \bar{B}_r \times \bar{B}$ and $(x,y) \in \bar{B}_r \times \bar{B}_r$ with $\|(x,y) - (u,v)\| \leq \varepsilon$, we have

$$\begin{aligned}
|(G(x,y)(t) - (G(u,v))(t)| &\leq \frac{k}{2}(|x(\xi(t)) - u(\xi(t))| + |y(\xi(t)) - v(\xi(t))|) \\
&\quad + \int_0^{q(t)} |h(t,s,x(\eta(s)), y(\eta(s)) - h(t,s,u(\eta(s)), v(\eta(s))|\,ds.
\end{aligned} \tag{3.9}$$

On the other hand, using (3.4), there exists $L > 0$ such that

$$\int_0^{q(t)} |h(t,s,x(\eta(s)), y(\eta(s)) - h(t,s,u(\eta(s)), v(\eta(s))|\,ds \leq \varepsilon \tag{3.10}$$

for any $t > L$. Thus, from (3.9) and (3.10) we get

$$|(G(x,y)(t) - (G(u,v))(t)| \leq \Big(\frac{k}{2} + 1\Big)\varepsilon \tag{3.11}$$

for $t > L$. Now in case that $t \in [0, L]$, we have

$$
\begin{aligned}
& |(G(x,y)(t) - (G(u,v))(t)| \\
\leq\ & \frac{k}{2}\varepsilon + \int_0^{q(t)} |h(t,s,x(\eta(s)),y(\eta(s)) - h(t,s,u(\eta(s)),v(\eta(s))|\, ds \\
<\ & \varepsilon + \int_0^{q_L} \omega(\varepsilon) ds \\
<\ & \varepsilon + q_L\omega(\varepsilon), \qquad (3.12)
\end{aligned}
$$

where

$$q_L = \sup\{q(t) : t \in [0, L]\}$$

and

$$
\begin{aligned}
\omega(\varepsilon) = \sup\{&|h(t,s,x(\eta(s)),y(\eta(s)) - h(t,s,u(\eta(s)),v(\eta(s))| : \\
& t,s \in [0,L]\,, x,y,u,v \in [-r,r]\,, \|(x,y)-(u,v)\| \leq \varepsilon\}.
\end{aligned}
$$

Therefore, from the uniform continuity of h on $[0, L] \times [0, q_L] \times [-r, r] \times [-r, r]$, we obtain $\omega(\varepsilon) \to 0$, as $\varepsilon \to 0$. Hence, the inequalities (3.11) and (3.12) prove that G is continuous on $\bar{B}_r \times \bar{B}$. Now we claim that the map G satisfies the condition (2.5) of Corollary 2.5. For proving this, fix $L,\ \varepsilon \in \mathbb{R}_+$ and let us choose nonempty subsets X_1, X_2 of $\bar{B}_r$ and $t,\ t' \in [0, L]$, with $\left|t - t'\right| < \varepsilon$. Also, without loss of generality we can assume that $q(\ t) < q(t')$. Now let $(x, y) \in X_1 \times X_2$. Then we get

$$
\begin{aligned}
& \left|G(x,y)(t) - G(x,y)(t')\right| \\
\leq\ & \frac{k}{2}((\omega^L(x,\omega^L(\xi,\varepsilon))) + (\omega^L(y,\omega^L(\xi,\varepsilon))) + \omega_r^L(f,\varepsilon) \\
& + \int_0^{q(t')} \left|h(t,s,x(\eta(s)),y(\eta(s))) - h(t',s,x(\eta(s)),y(\eta(s)))\right| ds \\
& + \int_{q(t)}^{q(t')} |h(t,s,x(\eta(s)),y(\eta(s)))|\, ds \\
\leq\ & \frac{k}{2}((\omega^L(x,\omega^L(\xi,\varepsilon))) + (\omega^L(y,\omega^L(\xi,\varepsilon))) \\
& + \omega_r^L(f,\varepsilon) + \int_0^{q_L} \omega_r^{q_L}(h,\varepsilon) ds + \omega_r^{q_L}(q,\varepsilon) H_r^L, \qquad (3.13)
\end{aligned}
$$

where

$$\omega^L(\xi,\varepsilon) = \sup\{\left|\xi(t) - \xi(t')\right| :\ t,\ t' \in [0,L]\,,\ \ \left|t - t'\right| \leq \varepsilon\}$$

$$\omega^L(x,\omega^L(\xi,\varepsilon)) = \sup\{\left|x(t) - x(t')\ \right| :\ t,\ t' \in [0,L]\,,\ \left|t - t'\right| \leq \omega^L(\xi,\varepsilon)\},$$

$$\omega_r^L(f,\varepsilon) = \sup\left\{\left|f(t,x,y) - f(t',x,y)\right| :\ t,\ t' \in [0,L]\,,\ \left|t - t'\right| \leq \varepsilon,\ \ |x| + |y| \leq r\right\},$$

$$
\begin{aligned}
\omega_r^{q_L}(h,\varepsilon) = \sup\Big\{&\left|h(t,s,x,y) - h(t',s,x,y)\right| :\ t,\ t' \in [0,L]\,, \\
& \left|t - t'\right| \leq \varepsilon,\ \ s \in [0,q_L]\,,\ \ |x| + |y| \leq r\Big\}
\end{aligned}
$$

and

$$H_r^L = \sup\{|h(t,s,x,y)| : \ t \in [0,L]\,, s \in [0,q_L] \text{ and } |x|+|y| \le r\}.$$

Since (x,y) was an arbitrary element of $X_1 \times X_2$, the inequality (3.13) implies that

$$\begin{aligned}\omega^L(G(X_1\times X_2),\varepsilon) \quad &\le \quad \frac{k}{2}(\omega^L(X_1,\omega^L(\xi,\varepsilon))+\omega^L(X_2,\omega^L(\xi,\varepsilon)))+\omega_r^L(f,\varepsilon)\\ &+\int_0^{q_L}\omega_r^{q_L}(h,\varepsilon)ds+\omega_r^L(q,\varepsilon)H_r^L. \qquad (3.14)\end{aligned}$$

On the other hand, since f and h are uniformly continuous on $[0,L]\times[0,L]\times[-r,r]$ and $[0,L]\times[0,q_L]\times[-r,r]\times[-r,r]$, respectively, also because of the uniform continuity of q and ξ on $[0,L]$ we have $\omega_r^L(f,\varepsilon)\to 0$, $\omega_r^{q_L}(h,\varepsilon)\to 0$, $\omega^L(\xi,\varepsilon)\to 0$ and $\omega_r^L(q,\varepsilon)\to 0$ as $\varepsilon\to 0$. Now, since H_r^L is finite, (3.14) implies that

$$\omega_0^L(G(X_1\times X_2)) \le \frac{k}{2}(\omega_0^L(X_1)+\omega_0^L(X_2)). \qquad (3.15)$$

So, by taking $L\to\infty$ from (3.15), we have

$$\omega_0(G(X_1\times X_2)) \le \frac{k}{2}(\omega_0(X_1)+\omega_0(X_2)). \qquad (3.16)$$

Moreover for $(x,y),(u,\ v)\in X_1\times X_2$ and $t\in\mathbb{R}_+$ we obtain

$$\begin{aligned}&|(G(x,y)(t)-(G(u,v))(t)|\\ \le\ & \frac{k}{2}(|x(\xi(t))-u(\xi(t))|+|y(\xi(t))-v(\xi(t))|)\\ &+\int_0^{q(t)}|h(t,s,x(\eta(s)),y(\eta(s)))-h(t,s,u(\eta(s)),v(\eta(s)))|\,ds\\ \le\ & \frac{k}{2}(diam X_1(\xi(t))+diam X_2(\xi(t)))\\ &+\int_0^{q(t)}|h(t,s,x(\eta(s)),y(\eta(s)))-h(t,s,u(\eta(s)),v(\eta(s)))|\,ds.\end{aligned}$$

This yields

$$\begin{aligned}diam(G(X_1\times X_2))(t) &\le \frac{k}{2}(diam X_1(\xi(t))+diam X_2(\xi(t)))\\ &+\int_0^{q(t)}|h(t,s,x(\eta(s)),y(\eta(s)))-h(t,s,u(\eta(s)),v(\eta(s)))|\,ds. \qquad (3.17)\end{aligned}$$

Letting $t\to\infty$ in (3.17) and using (3.4) we get

(3.18)

$$\limsup_{t\to\infty} diam(G(X_1\times X_2)(t)) \le \frac{k}{2}(\limsup_{t\to\infty} diam(X_1(\xi(t)))+\limsup_{t\to\infty} diam(X_2(\xi(t)))).$$

Adding (3.16) and (3.18) we obtain

$$\begin{aligned}&\omega_0(G(X_1\times X_2))+\limsup_{t\to\infty} diam(G(X_1\times X_2)(t))\\ \le\ & \frac{k}{2}(\omega_0(X_1)+\omega_0(X_2))\end{aligned}$$

$$
\begin{aligned}
& +\frac{k}{2}(\limsup_{t\to\infty} diam(X_1(\xi(t))) + \limsup_{t\to\infty} diam(X_2(\xi(t)))) \\
= & \ \frac{k}{2}(\omega_0(X_1) + \limsup_{t\to\infty} diam(X_1(\xi(t)) \\
& +\omega_0(X_2) + \limsup_{t\to\infty} diam(X_2(\xi(t)))).
\end{aligned} \tag{3.19}
$$

Therefore, since $\xi(t) \to \infty$ as $t \to \infty$, from (1.3) and (3.19) we earn

$$\mu(G(X_1 \times X_2) \leq \frac{k}{2}(\mu(X_1) + \mu(X_2)).$$

Finally, applying Corollary 2.5, we obtain the desired result. □

Now we give an example to illustrate how the above theorem can be used in practice.

Example 3.2. Consider the following system of integral equations

$$
\begin{cases}
x(t) = e^{-3t} + \frac{arctg(x(\frac{t}{3}))+\sin(x(\frac{t}{3}))}{4\pi+t^4} + \frac{\ln(1+\left|y(\frac{t}{3})\right|)}{2\pi+t^2} \\
+\int_0^{t^2} \frac{\ln(1+s\left|\sin(x^2(\sqrt{s}))\right|)y^2(\sqrt{s})+s^2(1+\left|\sin(x^2(\sqrt{s}))\right|)(1+y^4(\sqrt{s})}{(1+\left|\sin(x^2(\sqrt{s}))\right|)(1+t^6)(1+y^4(\sqrt{s})} ds, \\
y(t) = e^{-3t} + \frac{arctg(y(\frac{t}{3}))+\sin(y(\frac{t}{3}))}{4\pi+t^4} + \frac{\ln(1+\left|x(\frac{t}{3})\right|)}{2\pi+t^2} \\
+\int_0^{t^2} \frac{\ln(1+s\left|\sin(y^2(\sqrt{s}))\right|)x^2(\sqrt{s})+s^2(1+\left|\sin(y^2(\sqrt{s}))\right|)(1+x^4(\sqrt{s})}{(1+\left|\sin(y^2(\sqrt{s}))\right|)(1+t^6)(1+x^4(\sqrt{s})} ds.
\end{cases} \tag{3.20}
$$

Comparing (3.20) with (3.1), we get

$$
\begin{aligned}
f(t,x,y) &= e^{-3t} + \frac{arctg(x) + \sin(x)}{4\pi + t^4} + \frac{\ln(1+|y|)}{2\pi + t^2}, \\
h(t,s.x,y) &= \frac{\ln(1+s\left|\sin(x^2)\right|)y^2 + s^2(1+\left|\sin(x^2)\right|)(1+y^4)}{(1+|\sin(x^2)|)(1+t^6)(1+y^4)} \\
\xi(t) &= \frac{t}{3},\ \eta(s) = \sqrt{s}, q(t) = t^2.
\end{aligned}
$$

Now, we show that all the conditions of Theorem 3.1 are satisfied. It is obvious that $f(t,0,0) = e^{-3t}$ satisfies the condition (i). On other hand, let $(x,y),(u,v) \in \mathbb{R}\times \mathbb{R}$ with $|v| \geq |y|$, then we get

$$
\begin{aligned}
|f(t,x,y) - f(t,u,v)| &\leq \frac{1}{2\pi}\Big(|x-u| + \frac{1}{2\pi+t^2}\left|\ln\Big(\frac{1+|y|}{1+|v|}\Big)\right|\Big) \\
&\leq \frac{1}{2\pi}\Big(|x-u| + \frac{1}{2\pi}\ln\Big(1 + \frac{|v|-|y|}{1+|v|}\Big) \\
&\leq \frac{1}{2\pi}(|x-u| + \ln(1+|u-v|)) \\
&= \frac{1}{2\pi}(|x-y| + |u-v|))
\end{aligned}
$$

for all $t > 0$. Thus f satisfies condition (ii). Also it is clear that ξ, η, q are continuous and $\xi(t) \to \infty$ as $t \to \infty$. In addition, g is continuous on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ and

$$\lim_{t\to\infty} \int_0^{t^2} |h(t,s,x,y)|\, ds$$

$$= \lim_{t\to\infty} \int_0^{t^2} \left| \frac{\ln(1+s\left|\sin(x^2)\right|)y^2 + s^2(1+\left|\sin(x^2)\right|)(1+y^4)}{(1+|\sin(x^2)|)(1+t^6)(1+y^4)} \right| ds$$

$$= \frac{1}{3}$$

and

$$\begin{aligned} \lim_{t\to\infty} \int_0^{t^2} |h(t,s,x,y) - h(t,s,u,v)|\, ds &\le \lim_{t\to\infty} \int_0^{t^2} 2\frac{s}{1+t^6} ds \\ &= 0, \end{aligned}$$

which imply that assumptions (iii), (iv) are satisfied. Therefore, as a result of Theorem 3.1, the system of integral equations (28) has a solution.

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References

[1] M. Abbas, M. A. Khan and S. Radenović, *Common coupled fixed point theorems in cone metric spaces for w−compatible mappings*, Applied Mathematics and Computation **217** (2010), 195–202.

[2] R. Agarwal, M. Meehan and D. O'regan, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2004.

[3] A. Aghajani, J. Bans and Y. Jalilian, *Existence of solution for a class nonlinearVoltrra sigular integral*, Computer and Mathematics with Applications **62** (2011), 1215–1227.

[4] A. Aghajani and Y. Jalilian, *Existence and global attractivity of solutions of a nonlinear functional integral equation*, Commun. Nonlinear Sci. numer. Simulat. **15** (2010), 3306–3312.

[5] A. Aghajani and Y. Jalilian, *Existence of Nondecreasing Positive Solution for a system of sigular integral equations*, Mediterr. J. Math. **8** (2010), 563–576.

[6] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina and B. N Sadovskii, *Measures of Noncompactness and Condensing Operators*, Birkhäuser Verlag Basel, Boston, Berlin 1992.

[7] H. Aydi, E. Karapinar and W. Shatanawi, *Coupled fixed point results for $(\psi, \phi)-$weakly contractive condition in ordered partial metric spaces*, Computers and Mathematics with Applications, (2011), doi:10.1016/ j.camwa. 2011.10.021.

[8] J. Banaś, *On measures of noncompactness in Banach spaces*, Comment. Math. Univ. Carolinae **21** (1980), 131–143.

[9] J. Banaś, *Measures of noncompactness in the spaces of continuous tempered functions*, Demonstration Math. **14** (1981), 127–133.

[10] J. Banaś and B. C. Dhage, *Global asymptotic stability of solutions of a functional integral equation*, Nonlinear Analysis **69** (2008), 1945–1952.

[11] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Space*, Lecture Notes in Pure and Applied Mathematics, vol. 60, Dekker, New York, 1980.

[12] J. Banaś and R. Rzepka, *An application of a measure of noncompactness in the study of asymptotic stability*, Appl. Math. Lett. **16** (2003), 1–6.

[13] T. G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered partial metric spaces and applications*, Nonlinear Analysis **65** (2006), 1379–1393.

[14] S. S. Chang, Y. J. Cho and N. J. Huang, *Coupled fixed point theorems with applications*, J. Korean Math. Soc. **33** (1996), 575–585.

[15] S. S. Chang and Y. H. Ma, *Coupled fixed point theorems for mixed monotone condensing operators and an existence theorem of a solutions for for a class of functional equations*, J. Math, Anal. Appl. **160** (1991), 468–479.

[16] Y. J. Cho, M. H. Shah and N. Hussain, *Coupled fixed points of weakly F-contractive mappings in topological spaces*, Appl. Math. Lett. **24** (2011), 1185–1190.

[17] B. C. Dhage and S. S. Bellale, *Local asymptotic stability for nonlinear quadratic functional integral equations*, Electronic Journal of Qualitative Theory of Differential Equatctions **10** (2008), 1–13.

[18] D. Guo and V. Lakshmikantham, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal. TMA. **11** (1987), 623–632.

[19] K. Kuratowski, *Sur les espaces complets*, Fund. Math. **15** (1930), 301–309.

[20] V. Lakshmikantham and L. Ćiriċ, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Analysis **70** (2009), 4341–4349.

[21] W. Shatanawi, B. Samet and M. Abbas, *Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces*, Mathematical and Computer Modelling (2011), doi:10.1016/j.mcm.2011.08.042.

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