Journal of Nonlinear and Convex Analysis
Volume 15, Number 5, 2014, 929–939

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL


# DARBOUX TRANSFORMATIONS FOR TWO CLASSES OF INCOMPRESSIBLE FLUID EQUATIONS

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Abstract. This paper is concerned with Darboux transformations for 3D incompressible inviscid Euler equations and 2D quasi-geostrophic equations from fluid dynamics. New and general Darboux transformations are found for these incompressible fluid equations. Moreover, our results show that there is a good similarity between the 3D incompressible inviscid Euler equation and the 2D quasi-geostrophic equation at Darboux transformation for them.

## 1. Introduction

It is well known that Navier-Stokes equations are the governing equations for the incompressible viscous fluid flow, and they reduce to the Euler equations of incompressible inviscid fluid flow, with the Reynolds number set formally to be infinity. Therefore, incompressible inviscid Euler equations as well as quasi-geostrophic equations, which are lower dimensional models of the corresponding incompressible Euler equations in some sense, are also fundamental and important equations in fluid dynamics. On the other hand, Darboux transformations are powerful tools in many studies. As we have seen, Darboux transformations were traditionally employed for generating multisoliton solutions to soliton equations (cf., e.g., [1, 10] and references therein), and they were also very useful in constructing explicit representations for figure eight structures in the phase spaces of Hamiltonian partial differential equations (cf., e.g., [7]). Moreover, Darboux transformations can play key roles in investigating finite time blow-up solutions of 2D quasi-geostrophic equations and 2D or 3D incompressible inviscid Euler equations (cf., e.g., [3, 4, 8] and references therein). Actually, Darboux transformations can also be utilized to study the global well-posedness of 3D Navier-Stokes equations (cf., e.g., [8]). For other kind of investigations of Darboux transformations and applications, the reader can find from, e.g., [6, 12, 13] and references therein.

Consider the following 3D incompressible inviscid Euler equation in vorticity form (cf., e.g., [9])

$$\omega_t + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = 0, \tag{1.1}$$

where $u = (u^{(1)}, u^{(2)}, u^{(3)})$ and $\omega = (\omega^{(1)}, \omega^{(2)}, \omega^{(3)})$ are the vorticities, $\nabla =$

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2010 *Mathematics Subject Classification.* 35Q31, 35Q35.

*Key words and phrases.* Darboux transformation, 3D incompressible inviscid Euler equations, 2D quasi-geostrophic equations.

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The work was supported partly by the NSF of China (11271082, 11371095).

$(\partial_x, \partial_y, \partial_z)$, $\omega = \nabla \times u$, and $\nabla \cdot u = 0$. $u$ can be represented by $\omega$ (for example, through Biot-Savart law), and the following 2D quasi-geostrophic equation comes from the transportation of the potential temperature $\theta$ by an incompressible flow (cf., e.g., [3, 4, 8, 9])

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = 0$$

with initial condition $\theta|_{t=0} = \theta_0$. The relation between the active scalar $\theta$ and the velocity $u$ is given by

$$u = \nabla^{\perp} \psi, \quad \theta = (-\triangle)^{\frac{1}{2}}(-\psi)$$

where

$$\nabla^{\perp} \psi \equiv \Big( -\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x} \Big).$$

Define the bracket $\{,\}$ as

$$\{f, g\} = f_x g_y - f_y g_x.$$

Then, we have

$$\theta_t + \{\psi, \theta\} = 0. \tag{1.2}$$

It is clear that if denoting

$$[u, \omega] = (u \cdot \nabla)\omega - (\omega \cdot \nabla)u,$$

then (1.1) can be written as

$$\omega_t + [u, \omega] = 0. \tag{1.3}$$

As pointed out by Constantin, Majda and Tabak [2], the 2D quasi-geostrophic equation has a striking mathematical and physical analogy to the 3D incompressible Euler equation and they both exhibit similar geometric/analytic structure. For example, there exists a correspondence between $\nabla^{\perp}\theta$ in the 2D quasi-geostrophic equation and the vorticity $\omega$ for the 3D incompressible Euler equation.

Recently it has been realized that there exist Lax pair structures and Darboux transformations for these two systems. In [5, 11], Friedlander and Vishik found a Lax pair for the Euler equations written in the Lagrangian coordinates, while Li [8] gave a Darboux transformation for 2D Euler equations and their Lax pair. In [3], Deng presented a Lax pair for 2D quasi-geostrophic equations, which is similar to that of 3D Euler equations.

In this paper, we will give new and general Darboux transformations for 2D quasi-geostrophic equations as well as for 3D incompressible inviscid Euler equations. As one can see from our results, there is actually a good similarity between the 3D incompressible inviscid Euler equation and the 2D quasi-geostrophic equation at Darboux transformation for them.

## 2. Main results and proofs

First, we recall the definition of Lax pair and two basic interrelated lemmas. For more related information, the reader is referred to the references [3–5, 8].

**Definition 2.1.** Suppose that $L(t)$, $A(t)$ are two time dependent operators. If for any function $\varphi$ in the intersection of $L(t)$ and $A(t)$, we have

$$\frac{dL}{dt}\varphi = (LA - AL)\varphi,$$

then we call $L(t)$, $A(t)$ a Lax pair.

Let

$$L(t)\xi(t) := \lambda(t) \cdot \xi(t),$$

where $\lambda(t)$ is the eigenvalue of $L(t)$, and $\xi(t)$ the eigenfunction with respect to $\lambda(t)$. Then $\lambda(t)$ is independent of time; in other words,

$$\frac{d\lambda(t)}{dt} = 0.$$

As for the evolution manner of $\xi(t)$, we know that

$$\frac{d\xi(t)}{dt} = -A(t)\xi(t).$$

Note that (1.2) has exactly the same form as the equation governing the evolution of vorticity in the 2D Euler equation. So, from the idea of the Euler equation, we can get the Lax pairs of 2D quasi-geostrophic equations.

**Lemma 2.2.** *The Lax pair of the 2D quasi-geostrophic equation (1.2) is given by*

$$\begin{cases} \{\theta, p\} = \lambda p, \\ p_t + \{\psi, p\} = 0, \end{cases} \tag{2.1}$$

*where $\lambda$ is a complex constant, and $p$ a complex-valued function.*

The proof of the lemma is similar to that of [8, Theorem 1]. We omit it here.

From [8], we also know the following lemma holds.

**Lemma 2.3.** *A Lax pair of the 3D Euler equation (1.3) is given by*

$$L\varphi = \lambda\varphi, \tag{2.2}$$

$$\varphi_t + A\varphi = 0. \tag{2.3}$$

*Here $L\varphi = (\omega \cdot \nabla)\varphi - (\varphi \cdot \nabla)\omega = [\omega, \varphi]$, $A\varphi = (u \cdot \nabla)\varphi - (\varphi \cdot \nabla)u = [u, \varphi]$, $\lambda$ is a complex constant, and $\varphi = (\varphi^{(1)}, \varphi^{(2)}, \varphi^{(3)})$ is a complex 3-vector valued function.*

Consider the Lax pair (2.1) at $\lambda = 0$, i.e.,

$$\{\theta, p\} = 0, \tag{2.4}$$

$$p_t + \{\psi, p\} = 0. \tag{2.5}$$

Clearly, $p = \theta$ is a solution to (2.4) and (2.5).

**Theorem 2.4.** *Let $f = f(t, x, y)$ be any solution to (2.4) and (2.5). Define*

$$\tilde{p} = p\frac{f_x}{\theta_x} \tag{2.6}$$

*and*

$$\tilde{\psi} = \psi + F, \qquad \tilde{\theta} = \theta + (-\Delta)^{\frac{1}{2}}(-F), \tag{2.7}$$

*for the potentials $\theta$ and $\psi$, where $F$ is subject to the following constraints*

$$\{\theta, (-\Delta)^{\frac{1}{2}}(-F)\} = 0, \qquad \{\theta + (-\Delta)^{\frac{1}{2}}(-F), F\} = 0. \tag{2.8}$$

*Then*

(i) *$\tilde{p}$ solves the system (2.4) and (2.5) at $(\tilde{\theta}, \tilde{\psi})$.*
(ii) *(2.6) and (2.7) form the Darboux transformation for the 2D quasi-geostrophic equation (1.2) and its Lax pair (2.4) and (2.5).*

*Proof.* By (2.4), (2.7) and (2.8) , we have

$$\{\theta, \tilde{\theta}\} = 0,$$

and

$$\{\tilde{\theta}, F\} = 0.$$

Therefore, $\theta$ and $\tilde{\theta}$ have the same level set, and $\tilde{\theta}$ and $F$ have the same level set. So, $\theta$ and $F$ have the same level set, that is,

$$\{\theta, F\} = 0.$$

Similarly, we can see that $p$ and $F$ have the same level set, that is,

$$\{p, F\} = 0.$$

Clearly, to prove the theorem, we only need to prove that

$$\{\tilde{\theta}, \tilde{p}\} = 0, \tag{2.9}$$

and

$$\tilde{p}_t + \{\tilde{\psi}, \tilde{p}\} = 0. \tag{2.10}$$

We will prove (2.9) and (2.10) by two steps.

**Step 1**. First, we verify the following two identities

$$\{\theta, \tilde{p}\} = 0, \tag{2.11}$$

and

$$\tilde{p}_t + \{\psi, \tilde{p}\} = 0. \tag{2.12}$$

From

$$\{\theta, p\} = 0, \quad \{\theta, f\} = 0,$$

it follows that

$$\begin{aligned}
\{\theta, \tilde{p}\} &= \Big\{\theta, \frac{pf_x}{\theta_x}\Big\} \\
&= \theta_x\Big(\frac{(p_y f_x + pf_{xy})\theta_x - pf_x\theta_{xy}}{(\theta_x)^2}\Big) - \theta_y\Big(\frac{(p_x f_x + pf_{xx})\theta_x - pf_x\theta_{xx}}{(\theta_x)^2}\Big) \\
&= \frac{1}{(\theta_x)^2}\left[\theta_x(p_y f_x + pf_{xy})\theta_x - \theta_x pf_x\theta_{xy} - \theta_y(p_x f_x + pf_{xx})\theta_x + \theta_y pf_x\theta_{xx}\right] \\
&= \frac{f_x}{\theta_x}(\theta_x p_y - \theta_y p_x) + \frac{1}{(\theta_x)^2}p(\theta_x\theta_x f_{xy} - \theta_x f_x\theta_{xy} - \theta_x\theta_y f_{xx} + \theta_y f_x\theta_{xx}) \\
&= \frac{f_x}{\theta_x}\{\theta, p\} + \frac{p}{\theta_x}p(\theta_x f_y - \theta_y f_x)_x \\
&= \frac{p}{\theta_x}\{\theta, f\}_x \\
&= 0.
\end{aligned}$$

This means that (2.11) is true.

On the other hand, since

$$\begin{aligned}
f_t + \{\psi, f\} &= 0; \\
p_t + \{\psi, p\} &= 0; \\
\theta_t + \{\psi, \theta\} &= 0,
\end{aligned}$$

we obtain

$$\begin{aligned}
& \tilde{p}_t + \{\psi, \tilde{p}\} \\
= \; & \tilde{p}_t + \psi_x\tilde{p}_y - \psi_y\tilde{p}_x \\
= \; & \Big(\frac{pf_x}{\theta_x}\Big)_t + \psi_x\Big(\frac{pf_x}{\theta_x}\Big)_y - \psi_y\Big(\frac{pf_x}{\theta_x}\Big)_x \\
= \; & \frac{f_x}{\theta_x}p_t + \frac{f_x}{\theta_x}\psi_x p_y - \psi_y\frac{f_x}{\theta_x}p_x + p\Big(\frac{f_x}{\theta_x}\Big)_t + p\psi_x\Big(\frac{f_x}{\theta_x}\Big)_y - p\psi_y\Big(\frac{f_x}{\theta_x}\Big)_x \\
= \; & \frac{f_x}{\theta_x}(p_t + \{\psi, p\}) + p\frac{1}{\theta_x}((f_t)_x + \psi_x f_{xy} - \psi_y f_{xx}) \\
& \qquad -\frac{pf_x}{(\theta_x)^2}((\theta_t)_x + \psi_x\theta_{xy} - \psi_y\theta_{xx}) \\
= \; & \frac{p}{\theta_x}((f_t)_x + \psi_x f_{xy} - \psi_y f_{xx} + \psi_{xx}f_y - \psi_{xy}f_x) - \frac{p}{\theta_x}\psi_{xx}f_y + \frac{p}{\theta_x}\psi_{xy}f_x \\
& \qquad -\frac{pf_x}{(\theta_x)^2}((\theta_t)_x + \psi_x\theta_{xy} - \psi_y\theta_{xx} + \psi_{xx}\theta_y - \psi_{xy}\theta_x) + \frac{pf_x}{(\theta_x)^2}\varphi_{xx}\theta_y \\
& \qquad\quad -\frac{pf_x}{(\theta_x)^2}\psi_{xy}\theta_x \\
= \; & \frac{p}{\theta_x}(f_t + \{\psi, f\})_x - \frac{pf_x}{(\theta_x)^2}(\theta_t + \{\psi, \theta\})_x - \frac{p}{\theta_x}\psi_{xx}f_y + \frac{pf_y}{\theta_x\theta_y}\psi_{xx}\theta_y
\end{aligned}$$

$$= \quad 0.$$

This means that (2.12) is true.

**Step 2**. We prove (2.9) and (2.10).

From (2.11), the hypothesis $\{\theta, (-\Delta)^{\frac{1}{2}}(-F)\} = 0$, and the argument as in the beginning of the proof, it follows that

$$\{(-\Delta)^{\frac{1}{2}}(-F), \tilde{p}\} = 0.$$

This, together with (2.11), yields that

$$\{\theta + (-\Delta)^{\frac{1}{2}}(-F), \tilde{p}\} = 0.$$

So we have (2.9).

Similarly, we know, by (2.7) and (2.12), that

$$\{F, \tilde{p}\} = 0.$$

This implies (2.10).

Thus, the proof of Theorem 2.4 is completed. □

**Remark 2.5.** By the similar arguments as those in the proof above, we can prove the same conclusion as in Theorem 2.4 for the following $\tilde{p}$

$$\tilde{p} = p\frac{f_y}{\theta_y}.$$

Moreover, we have

**Theorem 2.6.** *Let $f = f(t,x,y)$ be any solution to (2.4) and (2.5), and let $G(\cdot,\cdot)$ and $H(\cdot,\cdot)$ be continuously differentiable functions. Define*

$$\tilde{p} = \frac{1}{\theta_x}G(f,p)H(f,p)_x \tag{2.13}$$

*and*

$$\tilde{\psi} = \psi + F, \qquad \tilde{\theta} = \theta + (-\Delta)^{\frac{1}{2}}(-F), \tag{2.14}$$

*for the potentials $\theta$ and $\psi$, where $F$ is subject to the constraints*

$$\{\theta, (-\Delta)^{\frac{1}{2}}(-F)\} = 0, \qquad \{\theta + (-\Delta)^{\frac{1}{2}}(-F), F\} = 0.$$

*Then*

(i) *$\tilde{p}$ solves the system (2.4) and (2.5) at $(\tilde{\theta}, \tilde{\psi})$.*
(ii) *(2.13) and (2.14) form the Darboux transformation for the 2D quasi-geostrophic equation (1.2) and its Lax pair (2.4) and (2.5).*

*Proof.* By Theorem 2.4, we know that if $f$ and $p$ are solutions to the system (2.4) and (2.5), then

(i) $\tilde{p}$ solves the system (2.4) and (2.5) at $(\tilde{\theta}, \tilde{\psi})$.
(ii) (2.6) and (2.7) form the Darboux transformation for the 2D quasi-geostrophic equation (1.2) and its Lax pair (2.4) and (2.5).

Therefore, in order to prove Theorem 2.6, we only need to verify that $G(f,p)$ and $H(f,p)$ satisfy (2.4) and (2.5).

It is not hard to see that

$$\{\theta, G(f,p)\} = G_f\{\theta, f\} + G_p\{\theta, p\} = 0,$$
$$(G(f,p))_t + \{\psi, G(f,p)\} = G_f(f_t + \{\psi + f\}) + G_p(p_t + \{\psi, p\}) = 0.$$

So $G(f,p)$ satisfies (2.4) and (2.5). Likewise, we infer that $H(f,p)$ satisfies (2.4) and (2.5) too. Thus, the proof of Theorem 2.6 is complete.

□

**Remark 2.7.** By virtue of Theorem 2.6, we get the following Darboux transformation for the Euler equation given in [8],

$$\tilde{p} = \frac{1}{\Omega_x}[p_x - (\partial_x lnf)p],$$

when taking $G(f,p)$ and $H(f,p)$ as two special functions.

Consider the Lax pair (2.2), (2.3) at $\lambda = 0$, i.e.,

$$[\omega, \varphi] = (\omega \cdot \nabla)\varphi - (\varphi \cdot \nabla)\omega = 0, \tag{2.15}$$

$$\varphi_t + [u, \varphi] = (u \cdot \nabla)\varphi - (\varphi \cdot \nabla)u = 0. \tag{2.16}$$

**Theorem 2.8.** *Let* $f = f(t,x,y,z) = (f^{(1)}, f^{(2)}, f^{(3)}) = \frac{f^{(1)}}{\omega^{(1)}}\omega$, *and* $f^{(1)}$ *be any solution to*

$$(\omega \cdot \nabla)f^{(1)} - (f \cdot \nabla)\omega^{(1)} = 0,$$

*and*

$$f_t^{(1)} + (u \cdot \nabla)f^{(1)} - (f \cdot \nabla)u^{(1)} = 0.$$

*Define*

$$\tilde{\varphi} = \frac{f^{(1)}}{\omega^{(1)}}\varphi, \tag{2.17}$$

*and*

$$\tilde{u} = u + H, \qquad \tilde{\omega} = \omega + \nabla \times H, \tag{2.18}$$

*for the potentials $u$ and $\omega$, where $H$ is a vector valued function and subject to the constraints*

$$[\tilde{\varphi}, \nabla \times H] = 0, \qquad [\tilde{\varphi}, H] = 0. \tag{2.19}$$

*Then*

(i) *$\tilde{\varphi}$ solves the system (2.15) and (2.16) at $(\tilde{u}, \tilde{\omega})$.*
(ii) *(2.17) and (2.18) form the Darboux transformation for the 3D Euler equation (1.3) and its Lax pair (2.15) and (2.16).*

*Proof.* Clearly, to prove the theorem, we only need to prove the following two equations

$$[\tilde{\omega}, \tilde{\varphi}] = 0,$$

and

$$\tilde{\varphi}_t + [\tilde{u}, \tilde{\varphi}] = 0. \tag{2.20}$$

(2.20) implies that it suffices to show that

$$[\omega, \tilde{\varphi}] = 0, \tag{2.21}$$

and

$$\tilde{\varphi}_t + [u, \tilde{\varphi}] = 0. \tag{2.22}$$

Now we begin to prove (2.21).
Write

$$M = \omega \cdot \nabla, \quad N = \varphi \cdot \nabla, \quad \tilde{N} = \tilde{\varphi} \cdot \nabla.$$

Then,

$$M(N\phi) = ((\omega \cdot \nabla)\varphi) \cdot (\nabla\phi) + \varphi \cdot (\omega \cdot \nabla)(\nabla\phi),$$
$$N(M\phi) = ((\varphi \cdot \nabla)\omega) \cdot (\nabla\phi) + \omega \cdot (\varphi \cdot \nabla)(\nabla\phi).$$

Noting that

$$(\omega \cdot \nabla)\varphi = (\varphi \cdot \nabla)\omega,$$

we get

$$M(N\phi) = N(M\phi),$$

i.e., $M$ is commutative with $N$. So

$$[\omega, \varphi] = 0.$$

Therefore it suffices to prove

$$\begin{aligned} M\tilde{N} &= \tilde{N}M, \\ M(\tilde{N}\phi) &= \omega \cdot \nabla\Big(\frac{f^{(1)}}{\omega^{(1)}}\Big)(N\phi) + \frac{f^{(1)}}{\omega^{(1)}} M(N\phi), \\ \tilde{N}(M\phi) &= \frac{f^{(1)}}{\omega^{(1)}} N(M\phi) = \frac{f^{(1)}}{\omega^{(1)}} M(N\phi), \end{aligned}$$

i.e, we just need to show that

$$\omega \cdot \nabla\Big(\frac{f^{(1)}}{\omega^{(1)}}\Big) = 0. \tag{2.23}$$

Next we prove (2.23). Since,

$$\begin{aligned} \omega \cdot \nabla f^{(1)} &= f \cdot \nabla\omega^{(1)}, \\ f &= \frac{f^{(1)}}{\omega^{(1)}}\omega, \end{aligned}$$

we have

$$\begin{aligned}\omega\cdot\nabla\Big(\frac{f^{(1)}}{\omega^{(1)}}\Big)&=\frac{1}{\omega^{(1)}}\omega\cdot\nabla f^{(1)}-\frac{f^{(1)}}{(\omega^{(1)})^2}\omega\cdot\nabla\omega^{(1)}\\&=\frac{1}{\omega^{(1)}}f\cdot\nabla\omega^{(1)}-\frac{1}{\omega^{(1)}}f\cdot\nabla\omega^{(1)}\\&=0,\end{aligned}$$

that is, (2.23) holds. Thus we see that (2.21) holds.

Next we check (2.22). Since

$$\begin{aligned}&\varphi_t+[u,\varphi]=0,\\&w_t^{(1)}+u\cdot\nabla w^{(1)}-\omega\cdot\nabla u^{(1)}=0,\\&f_t^{(1)}+u\cdot\nabla f^{(1)}-f\cdot\nabla u^{(1)}=0,\\&f=\frac{f^{(1)}}{\omega^{(1)}}\omega,\end{aligned}$$

we obtain

$$\begin{aligned}\tilde\varphi_t+[u,\tilde\varphi]&=\Big(\frac{f^{(1)}}{\omega^{(1)}}\Big)_t\varphi+\frac{f^{(1)}}{\omega^{(1)}}\varphi_t+(u\cdot\nabla)\Big(\frac{f^{(1)}}{\omega^{(1)}}\varphi\Big)-\frac{f^{(1)}}{\omega^{(1)}}\varphi\cdot u\\&=\frac{f^{(1)}}{\omega^{(1)}}(\varphi_t+(u\cdot\nabla)\varphi-(\varphi\cdot\nabla)u)+\Big(\frac{f^{(1)}}{\omega^{(1)}})_t\varphi+(u\cdot\nabla\frac{f^{(1)}}{\omega^{(1)}}\Big)\varphi\\&=\frac{f^{(1)}}{\omega^{(1)}}(\varphi_t+[u,\varphi])+\frac{1}{\omega^{(1)}}\varphi(f_t^{(1)}+u\cdot\nabla f^{(1)})\\&-\frac{f^{(1)}}{(w^{(1)})^2}\varphi(w_t^{(1)}+u\cdot\nabla w^{(1)})\\&=\frac{1}{\omega^{(1)}}\varphi(f_t^{(1)}+u\cdot\nabla f^{(1)}-f\cdot\nabla u^{(1)})+\frac{1}{\omega^{(1)}}\varphi f\cdot\nabla u^{(1)}\\&-\frac{f^{(1)}}{(w^{(1)})^2}\varphi(w_t^{(1)}+u\cdot\nabla w^{(1)}-\omega\cdot\nabla u^{(1)})-\frac{f^{(1)}}{(w^{(1)})^2}\varphi(\omega\cdot\nabla u^{(1)})\\&=0.\end{aligned}$$

Hence, (2.22) holds. By (2.21), (2.22) and the assumption, we get

$$[\tilde\omega,\tilde\varphi]=0,$$

and

$$\tilde\varphi_t+[\tilde u,\tilde\varphi]=0.$$

Thus the proof is complete. □

**Remark 2.9.** From Theorem 2.4 and Theorem 2.8, we see that 3-D Euler equations and 2D quasi-geostrophic equations also have a good similarity at Darboux transformations.

**Theorem 2.10.** *Let* $i = 1, 2, 3,$

$$\begin{aligned} &f = \frac{f^{(i)}}{\omega^{(i)}}\omega, \\ &[\omega, f] = 0, \\ &f_t + [u, f] = 0. \end{aligned}$$

*Define*

$$\tilde{\varphi} = \frac{f^{(i)}}{\omega^{(i)}}\varphi, \tag{2.24}$$

*and*

$$\tilde{u} = u + H, \qquad \tilde{\omega} = \omega + \nabla \times H, \tag{2.25}$$

*for the potentials* $u$ *and* $\omega$*, where* $H$ *is a vector valued function and subject to the constraints*

$$[\tilde{\varphi}, \nabla \times H] = 0, \qquad [\tilde{\varphi}, H] = 0.$$

*Then*

(i) $\tilde{\varphi}$ *solves the system (2.15) and (2.16) at* $(\tilde{u}, \tilde{\omega})$.

(ii) *(2.24) and (2.25) form the Darboux transformation for the 3D Euler equation (1.3) and its Lax pair (2.15) and (2.16).*

*Proof.* Define

$$\widehat{f} = fA, \quad \widehat{\varphi} = \varphi A, \quad \widehat{\omega} = \omega A, \quad \widehat{u} = uA,$$

where $A = P_{1i}$ is a elementary matrix, and it is get by exchanging the first row with the $i$ row of identity matrix. Then $\widehat{f}$, $\widehat{\varphi}$, $\widehat{\omega}$, $\widehat{u}$ satisfy all conditions of Theorem 2.8, as well as (2.15) and (2.16). Therefore, Theorem 2.10 is true in view of Theorem 2.8.

□

## References


[1] V. Chrastinová, *On the Darboux transformation*, I, Georgian Math. J. **2** (1995), 237–240.

[2] P. Constantin, A. Majda and E. Tabak, *Formation of strong fronts in the 2D quasi-geostrophic thermal active scalar*, Nonlinearity **7** (1994), 1495–1533.

[3] J. Deng and M. Ji, *Geometric structure and non-blowup of 2D quasi-geostrophic equation*, J. Fudan Uni. **47** (2008), 224–231.

[4] J. Deng, T. Y. Hou, R. Li and X. Yu, *Level set dynamics and the non-blowup of 2D quasi-geostrophic equation*, Methods Appl. Anal. **13** (2006), 157–180.

[5] S. Friedlander and M. Vishik, *Lax pair formulation for the Euler equation*, Phys. Lett. A **148** (1990), 313–319.

[6] J. S. He, Y. Cheng and Y. S. Li, *The Darboux transformation for NLS-MB equations, Commun*, Theor. Phys. (Beijing) **38** (2002), 493–496.

[7] Y. Li, *Bäcklund-Darboux transformations and Melnikov analysis for Davey-Stewartson II equations*, J. Nonlinear Sci. **10** (2000), 103–131.

[8] Y. Li and A. V. Yurov, *Lax pairs and Darboux transformations for Euler equations*, Stud. Appl. Math. **111** (2003), 101–113.

[9] A. Majda and A. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2002.

[10] V. B. Matveev and M. A. Salli, Darboux *Transformations and Solitons*, Vol. 4, Springer Series in Nonlinear Dynamics, Springer-Verlag, Berlin, 1991.

[11] M. Vishik and S. Friedlander, *An inverse scattering treatment for the flow of an ideal fluid in two dimensions*, Nonlinearity **6** (1993), 231–249.

[12] S. W. Xu, J. S. He and L. H. Wang, *The Darboux transformation of the derivative nonlinear Schrödinger equation*, J. Phys. **A 44** (2011), 305203, 22 pp.

[13] H. Q. Zhao and Z. N. Zhu, *Darboux transformation and exact solutions for a three-field lattice equation*, in Nonlinear and Modern Mathematical Physics, AIP Conf. Proc., 1212, Amer. Inst. Phys., Melville, NY, 2010, pp. 162–169.

*Manuscript received November 18, 2012*
*revised February 26, 2013*

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