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THE PROPERTY WORTH* AND THE WEAK FIXED POINT PROPERTY

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Abstract. A Banach space, X, has the weak fixed point property (w-FPP) if every nonexpansive mapping, T, on every weak compact convex nonempty subset, C, has a fixed point. A Banach space, X^*, has WORTH* if for every weak* null sequence (x_n^*) and every $x^* \in X^*$,

$$\limsup_n \|x_n^* - x^*\| = \limsup_n \|x_n^* + x^*\|.$$

A new proof is given of the recent result that WORTH* implies the weak fixed point property.

1. Introduction

A Banach space, X, has the weak fixed point property (w-FPP) if every nonexpansive mapping, T, on every weak compact convex nonempty subset, C, has a fixed point. If the subsets are closed and bounded instead of weak compact then the property is called the fixed point property (FPP). The past 40 or so years has seen a number of Banach space properties shown to imply the w-FPP. Some such properties are weak normal structure, Opial's condition and Property(M). A conjecture of Sims [23] from 1988 that a property called WORTH may lead to the w-FPP has recently been shown to be partially correct. In [8] Fetter and Gamboa de Buen show that if X is a separable Banach space and its dual, X^*, has the property WORTH* then X has the w-FPP. Some previous results of this nature are X^* has WORTH* and is uniformly nonsquare then X normal structure and so has the FPP in [6] and X^* has WORTH* and is ϵ-inquadrate in every direction for some $\epsilon \in (0, 2)$ then X has the w-FPP in [6].

Fetter and Gamboa de Buen [8] use a theorem, from the recent paper from Cowell and Kalton [3], that states that if X is a separable Banach space where X^* has WORTH* then, for every $\delta > 0$, X $(1 + \delta)$-embeds into a space with shrinking 1-unconditional basis. To achieve their main result, Fetter and Gamboa de Buen show that if X is a Banach space which $(1 + \delta)$-embeds into a space with 1-unconditional basis where $\delta < \frac{\sqrt{13}-3}{2}$ then X has the AMC property and so the w-FPP. Here the AMC property is a Banach space property close to the w-FPP introduced by García Falset and Lloréns-Fuster in [10]. This present paper also relies on the result of Cowell and Kalton but then travels down a different path that avoids the AMC property. We use the fact that a space with a 1-unconditional basis is a Banach lattice in the natural ordering and so highlighting the fundamental order properties

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of WORTH*, WORTH and the w-FPP. This pathway allows the incorporation of the Banach-Mazur distance in the result.

2. Preliminaries

The most common way to prove that a Banach space has the w-FPP is to assume the opposite is true and try for a contradiction. If the Banach space, X, does not have the w-FPP then there is a nonempty weak compact convex set, C, and a nonexpansive mapping $T : C \to C$ which is fixed point free. This means that $\operatorname{diam}(C) > 0$ and leads to the existence of a sequence (x_n) in C where

$$\lim_n \|Tx_n - x_n\| = 0.$$

Such a sequence is called an approximate fixed point sequence.

The next step is to use weak compactness and Zorn's lemma to show the existence of minimal invariant subsets of C. If K is a minimal invariant subset of C for T then K is nonempty weak compact convex and $T(K) \subseteq K$. K also has the property that it has no nonempty weak compact convex proper subsets that are T-invariant.

The Goebel-Karlovitz Lemma states that if K is a minimal invariant subset and (x_n) is an approximate fixed point sequence in K then

$$\lim_n \|x - x_n\| = \operatorname{diam}(K) \text{ for all } x \in K.$$

More properties of (x_n) can be developed using the fact that any subsequence of an approximate fixed point sequence is another approximate fixed point sequence. We may assume that (x_n) is weak convergent and by a dilation and translation we can further assume that $x_n \rightharpoonup 0$ and $\operatorname{diam}(K) = 1$. This means that $0 \in K$ and $\lim_n \|x_n\| = 1$. By taking subsequences if necessary, we also assume that $\lim_n \|x_n - x_{n+1}\| = 1$.

Now consider $\tilde{X} := l_\infty(X)/c_0(X)$. The elements of $\tilde{X}$ are $\tilde{x} := [x_n]$ where $[x_n]$ represents the equivalence class (x_n) in $l_\infty(X)$.

The mapping $J : Z \to \tilde{X}, J(x) = [x_n]$ where $x_n = x$ for all n is the natural isometric embedding of X into $\tilde{X}$.

Define $\tilde{C}$ in $\tilde{X}$ by

$$\tilde{C} := \{[x_n] : x_n \in C \text{ for all } n\}$$

then $\tilde{C}$ is a closed bounded subset with $\operatorname{diam}(\tilde{C}) = \operatorname{diam}(C)$ and $JC \subset \tilde{C}$.

Define $\tilde{T} : \tilde{C} \to \tilde{C}$ by

$$\tilde{T}[x_n] := [Tx_n]$$

then $\tilde{T}$ is a well defined nonexpansive mapping where $\tilde{T}(JC) \subset JC$. The fact that $\tilde{C}$ is closed bounded and convex means that there are approximate fixed point sequences for $\tilde{T}$ in $\tilde{C}$.

If (x_n) is an approximate fixed point sequence for T then $[x_n]$ is a fixed point of $\tilde{T}$. Conversely if $[x_n]$ is a fixed point for $\tilde{T}$ then (x_n) is an approximate fixed point sequence for T. This leads to a result of Lin [18] where the Goebel-Karlovitz lemma is translated into $\tilde{X}$. The results states:

> *If C is a weak compact minimal invariant set for T and $(\tilde{x}_n)$ is an approximate fixed point sequence for $\tilde{T}$ in $\tilde{C}$ then*
>
> $$\lim_n \|\tilde{x}_n - Jx\| = \operatorname{diam}(C) \quad \textit{for all } x \in C.$$

Combining this with $\operatorname{diam}(C) = 1$ and $0 \in C$ we have $\lim_n \|\tilde{x}_n\| = 1$. Since such an approximate fixed point sequence always exists we may conclude that $\sup\{\|\tilde{x}\| : \tilde{x} \in \tilde{C}\} = 1$.

This property is inherited by any $\tilde{T}$-invariant, closed convex and nonempty subset, W, of $\tilde{C}$. That is

$$\sup\{\|\tilde{w}\| : \tilde{w} \in W\} = 1.$$

In the proof of the main result below, such a W will be constructed and used to force a contradiction.

The w-FPP is separably determined so throughout this paper, X, will be assumed to be a infinite dimensional separable real Banach space.

For more information on any of those above steps see Goebel and Kirk [12] or Kirk and Sims [16].

3. Definitions and notation

A Banach space, X, has WORTH if for every weak null sequence (x_n) and every $x \in X$,

$$\limsup_n \|x_n - x\| = \limsup_n \|x_n + x\|.$$

This concept was introduced independently by Rosenthal [21] and Sims [23]. For separable Banach spaces this is the same as X being asymptotically unconditional (au), a concept used in the Cowell and Kalton paper [3].

If X^* has a similar property involving weak* null sequences then the property is called WORTH*. More specifically, a Banach space, X^*, has WORTH* if for every weak* null sequence (x_n^*) and every $x^* \in X^*$,

$$\limsup_n \|x_n^* - x^*\| = \limsup_n \|x_n^* + x^*\|.$$

Cowell and Kalton [3] refer to this property as X has property(au^*) and Lima [17] calls it property(wM^*). In that paper Lima showed that if X has property(wM^*) then X^* has the RNP and so $\ell_1 \not\hookrightarrow X$.

In [6] Dalby showed that if X^* has WORTH* then X has WORTH. Clearly, WORTH and WORTH* are equivalent if X is reflexive. Cowell and Kalton [3] conjecture WORTH implies WORTH* if $\ell_1 \not\hookrightarrow X$.

The notion of WORTH has its origins in a Banach lattice property, weak orthogonality. In [2] Borwein and Sims said that a Banach lattice, X, was weakly orthogonal if for every sequence (x_n) weakly convergent to x_0,

$$\liminf_n \liminf_m \| \, |x_n - x_0| \wedge |x_m - x_0| \, \| = 0.$$

Using this concept Borwein and Sims were able to generalise Maurey's result that c_0 has the w-FPP [18]. They proved that if X is a weakly orthogonal Banach lattice and its Riesz angle, $\alpha(X)$, is less than 2 then X has the w-FPP [2]. Sims, in [23], was able to drop the Riesz angle condition by slightly strengthening the definition of weak orthogonality. The new definition is, X is a weakly orthogonal Banach lattice if for every weak null sequence, (x_n), in X and every $x \in X$,

$$\lim_n \| \, |x_n| \wedge |x| \, \| = 0.$$

It has become the practice to use the stronger definition when referring to weak orthogonality, see for example Sims [23] and García-Falset [9]. Clearly, a weakly orthogonal Banach lattice has WORTH. Crucially, for this paper, any Banach space with 1-unconditional basis is a weakly orthogonal Banach lattice with the natural ordering and original norm. In addition, the norm is order continuous and the lattice is order complete.

Stability results for weakly orthogonal Banach lattices were established by Sims [22], Khamsi and Turpin [15], and Dalby [4].The Banach-Mazur distance that Sims achieved was $\sqrt{5}-1$, Khamsi and Turpin achieved $\frac{4}{3}$ and Dalby reached $\frac{\sqrt{33}-3}{2}$. The proof used in the latter result will be adapted and used in the next section. That proof, in turn, was based on the ideas of Baillon that were reported in Aksoy and Khamsi [1].

The crucial result needed is theorem 4.2 of Cowell and Kalton [3] which states

Let X be a separable Banach space. Then the following conditions are equivalent:

(i) *X has (au*).*

(ii) *For any $\delta > 0$ there is a Banach space Y with a shrinking 1-unconditional basis and a subspace X_δ of Y such that $d(X, X_\delta) < 1 + \delta$.*

4. Main result

Theorem 4.1. *A Banach space X has the w-FPP if there exists a Banach space, Y, where Y^* has WORTH* and the Banach-Mazur distance $d(X, Y) < \frac{\sqrt{33}-3}{2}$.*

Proof. Assume that X does not satisfy the w-FPP. Then there exists a nonempty weak compact, convex set $C \subset X$, minimal for nonexpansive $T : C \to C$ with $\operatorname{diam}(C) = 1$. Further, let (x_n) be an approximate fixed point sequence in C where $x_n \rightharpoonup 0$, $\lim_n \|x_n\| = 1$ and $\lim_n \|x_{n+1} - x_n\| = 1$.

Let Y be a Banach space with Y^* having WORTH* where $U : X \to Y$ is a linear isomorphism with $\|U\|\|U^{-1}\| = m$ where $m < \frac{\sqrt{33}-3}{2}$.

Using theorem 4.2 of [3], for any $\delta > 0$ there is a Banach space Z with shrinking 1-conditional basis and a subspace Y_δ of Z such that $d(Y, Y_\delta) < 1+\delta$. Let $A : Y \to Y_\delta$ be a linear isomorphism with $\|A\|\|A^{-1}\| = p$ where $p < 1 + \delta$. Note that Z is a weakly orthogonal Banach lattice with order continuous norm.

Then $B := AU : X \to Y_\delta$ is a linear isomorphism with

$$\|B\|\|B^{-1}\| \le mp < \Big(\frac{\sqrt{33}-3}{2}\Big)(1+\delta).$$

So $Bx_n \rightharpoonup 0$ in Y_δ and Z. By taking subsequences if necessary, we may assume, using weak orthogonality, that $\lim_n \| \, |Bx_n| \wedge |Bx_{n+1}| \, \| = 0$. For more details see [22] or [23].

Along with $\tilde{X} := l_\infty(X)/c_0(X)$, let $\tilde{Y_\delta} := l_\infty(Y_\delta)/c_0(Y_\delta)$ and $\tilde{Z} := l_\infty(Z)/c_0(Z)$. Note that $\tilde{Z}$ is a Banach lattice with order continuous norm. Then B leads to a linear isomorphism $\tilde{B} : \tilde{X} \to \tilde{Y_\delta}$ where $\|B\| = \|\tilde{B}\|$.

Let $\tilde{a} := [x_n]$, $\tilde{b} := [x_{n+1}]$ then let $\tilde{u} = \tilde{B}\tilde{a}$ and $\tilde{v} = \tilde{B}\tilde{b}$. This means that $|\tilde{u}| \wedge |\tilde{v}| = 0$.

Since $\tilde{Z}$ is order complete we can use principal band projections $P_{\tilde{u}}, P_{\tilde{v}}$ where

$$P_{\tilde{u}}\tilde{u} = \tilde{u} \text{ and } P_{\tilde{v}}\tilde{v} = \tilde{v},$$

$$\|P_{\tilde{u}}\| = \|P_{\tilde{v}}\| = \|\tilde{I} - P_{\tilde{u}}\| = \|\tilde{I} - P_{\tilde{v}}\| = 1.$$

So

$$P_{\tilde{u}}J_Z = 0 = P_{\tilde{v}}J_Z = P_{\tilde{u}}P_{\tilde{v}}.$$

In addition,

$$\|\tilde{I} - 2P_{\tilde{u}} - 2P_{\tilde{v}}\| \text{ and } \|\tilde{I} - 2P_{\tilde{u}}\| \text{ are } \le 1.$$

Let $Q := \{\tilde{x} \in \tilde{C} : \|\tilde{x} - \tilde{a}\| = \|\tilde{x} - \tilde{b}\| = \frac{1}{2}\}$.

Then Q is a closed convex $\tilde{T}$ invariant set with $\frac{\tilde{a}+\tilde{b}}{2} \in Q$.

Let

$$V := \Big\{\tilde{x} \in \tilde{C} : \exists\, x_0 \in C \text{ such that } \|\tilde{x} - J_X x_0\| \le \frac{m}{2}\Big\}.$$

Then V is a closed convex $\tilde{T}$ invariant set with $\frac{\tilde{a}+\tilde{b}}{2} \in V$.

Therefore $W := Q \cap V$ is a nonempty closed convex subset of $\tilde{C}$ which is invariant under $\tilde{T}$. So $\sup\{\|\tilde{w}\| : \tilde{w} \in W\} = 1$.

For any $\tilde{w} \in W$

$$\begin{aligned} 2\tilde{B}^{-1}(P_{\tilde{u}} + P_{\tilde{v}})\tilde{B}\tilde{w} = {} & \Big[\tilde{B}^{-1}(P_{\tilde{u}} + P_{\tilde{v}})\tilde{B}\tilde{w} + \tilde{w} - \tilde{B}^{-1}(P_{\tilde{u}} + P_{\tilde{v}})\tilde{B}\tilde{w} - J_X x\Big] \\ & + \Big[\tilde{B}^{-1}(P_{\tilde{u}} + P_{\tilde{v}})\tilde{B}\tilde{w} - \tilde{w} + \tilde{B}^{-1}(P_{\tilde{u}} + P_{\tilde{v}})\tilde{B}\tilde{w} + J_X x\Big] \text{ for all } x \in C. \end{aligned}$$

But

$$\tilde{B}^{-1}(P_{\tilde{u}} + P_{\tilde{v}})\tilde{B}\tilde{w} + \tilde{w} - \tilde{B}^{-1}(P_{\tilde{u}} + P_{\tilde{v}})\tilde{B}\tilde{w} - J_X x = \tilde{w} - J_X x$$

and

$$\begin{aligned}
&\|\tilde{B}^{-1}(P_{\tilde{u}}+P_{\tilde{v}})\tilde{B}\tilde{w}-\tilde{w}+\tilde{B}^{-1}(P_{\tilde{u}}+P_{\tilde{v}})\tilde{B}\tilde{w}+J_Xx\| \\
&\qquad = \|\tilde{B}^{-1}(2P_{\tilde{u}}+2P_{\tilde{v}}-\tilde{I})\tilde{B}\tilde{w}+J_Xx\| \\
&\qquad = \|\tilde{B}^{-1}(2P_{\tilde{u}}+2P_{\tilde{v}}-\tilde{I})\tilde{B}\tilde{w}-\tilde{B}^{-1}(2P_{\tilde{u}}+2P_{\tilde{v}}-\tilde{I})\tilde{B}J_Xx\| \\
&\qquad = \|\tilde{B}^{-1}(2P_{\tilde{u}}+2P_{\tilde{v}}-\tilde{I})\tilde{B}(\tilde{w}-J_Xx)\| \\
&\qquad \le mp\|\tilde{w}-J_Xx\|.
\end{aligned}$$

Therefore $\|\tilde{B}^{-1}(P_{\tilde{u}}+P_{\tilde{v}})\tilde{B}\tilde{w}\| \le \frac{(mp+1)m}{4}$.

Now

$$\begin{aligned}
\tilde{w}+\tilde{a}-2\tilde{B}^{-1}P_{\tilde{u}}\tilde{B}\tilde{w} &= \tilde{w}-\tilde{a}+2\tilde{a}-2\tilde{B}^{-1}P_{\tilde{u}}\tilde{B}\tilde{w} \\
&= \tilde{w}-\tilde{a}+2\tilde{B}^{-1}P_{\tilde{u}}\tilde{B}(\tilde{a}-\tilde{w}) \\
&= \tilde{B}^{-1}(\tilde{I}-2P_{\tilde{u}})\tilde{B}(\tilde{w}-\tilde{a}).
\end{aligned}$$

Therefore $\|\tilde{w}+\tilde{a}-2\tilde{B}^{-1}P_{\tilde{u}}\tilde{B}\tilde{w}\| \le mp\|\tilde{w}-\tilde{a}\| = \frac{mp}{2}$.

Similarly $\|\tilde{w}+\tilde{b}-2\tilde{B}^{-1}P_{\tilde{v}}\tilde{B}\tilde{w}\| \le \frac{mp}{2}$.

Therefore

$$\begin{aligned}
&\Big\|\tilde{w}+\frac{\tilde{a}+\tilde{b}}{2}-\tilde{B}^{-1}P_{\tilde{u}}\tilde{B}\tilde{w}-\tilde{B}^{-1}P_{\tilde{v}}\tilde{B}\tilde{w}\Big\| \\
&\qquad \le \frac{1}{2}\|\tilde{w}+\tilde{a}-2\tilde{B}^{-1}P_{\tilde{u}}\tilde{B}\tilde{w}\|+\frac{1}{2}\|\tilde{w}+\tilde{b}-2\tilde{B}^{-1}P_{\tilde{v}}\tilde{B}\tilde{w}\| \\
&\qquad \le \frac{mp}{2}.
\end{aligned}$$

So

$$\begin{aligned}
\Big\|\tilde{w}+\frac{\tilde{a}+\tilde{b}}{2}\Big\| &\le \Big\|\tilde{w}+\frac{\tilde{a}+\tilde{b}}{2}-\tilde{B}^{-1}(P_{\tilde{u}}+P_{\tilde{v}})\tilde{B}\tilde{w}\Big\|+\|\tilde{B}^{-1}(P_{\tilde{u}}+P_{\tilde{v}})\tilde{B}\tilde{w}\| \\
&\le \frac{mp}{2}+\frac{(mp+1)m}{4}.
\end{aligned}$$

But

$$\begin{aligned}
\Big\|\tilde{w}+\frac{\tilde{a}+\tilde{b}}{2}\Big\| &= 2\Big\|\tilde{w}-\frac{1}{2}\Big(\tilde{w}-\frac{\tilde{a}+\tilde{b}}{2}\Big)\Big\| \\
&\ge 2\|\tilde{w}\|-\frac{1}{2} \qquad \text{since } \|\tilde{w}-\frac{\tilde{a}+\tilde{b}}{2}\| \le \frac{1}{2}.
\end{aligned}$$

Therefore

$$2\|\tilde{w}\| \le \frac{mp}{2}+\frac{(mp+1)m}{4}+\frac{1}{2}.$$

So

$$2 \le \frac{mp}{2}+\frac{(mp+1)m}{4}+\frac{1}{2} \text{ where } p<1+\delta.$$

Since δ is arbitrary, we have

$$0 \leq m^2 + 3m - 6$$
$$m \geq \frac{\sqrt{33} - 3}{2}.$$

A contradiction, so X has the w-FPP. □

Corollary 4.2. *Let X be a reflexive Banach space with WORTH then X has the FPP*

This corollary will be followed up in the next section.

5. Discussion

For a number of years it has been conjectured that there is a connection between the FPP and the reflexivity of X. Could one property imply the other? Maurey [20] showed that reflexive subspaces of L_1 have the FPP and then Dowling and Lennard showed that every nonreflexive subspace of L_1 fails the FPP.

Recently Lin [19] showed that there is an equivalent norm on ℓ_1 that has the fixed point property. Since then, Hernández Linares and Japón [13] have found classes of nonreflexive Banach spaces which have, under equivalent norms, the FPP and in doing so have captured Lin's result. So one side of the problem has been answered but there are still intriguing indications of some sort of connection.

One of these involves a property stronger than WORTH, a property introduced in Kalton [14].

X has property(M) if

$$(x_n) \rightharpoonup 0 \text{ and } \|u\| = \|v\| \text{ then } \lim_n \|x_n - u\| = \lim \|x_n - v\|.$$

Clearly, property(M) implies WORTH.

Property(M*) is the corresponding property in X^* but using weak* null sequences. Kalton [14] proved that property(M*) implies property(M) and the reverse implication holds if $\ell_1 \not\hookrightarrow X$. So there are some similarities with WORTH. In [11] García Falset and Sims showed that property(M) implies the w-FPP.

The connection between property(M), property(M*) and X being reflexive is contained in the following from Dalby [5].

Proposition 5.1. *Let X be a Banach space such that X^* has property(M*) then the following are equivalent.*

a) *X is reflexive*
b) $\ell_1 \not\hookrightarrow X$
c) $c_0 \not\hookrightarrow X$
d) *X has the FPP*
e) *X^* has the FPP.*

If X has property(M) and $\ell_1 \not\hookrightarrow X$ then X^* has property(M*) and there is a similar list of equivalent conditions.

If X has property (M) then X is reflexive if and only if $\ell_1 \not\hookrightarrow X$ and $c_0 \not\hookrightarrow X$. So X is acting like a Banach lattice.

If X^* has WORTH* then X is very close to a Banach space with a 1-unconditional basis which can be considered to be a Banach lattice. So maybe X could act like a Banach lattice. To emphasize this point, note that corollary 4.2 above can be rewritten to be something like proposition 5.1.

Proposition 5.2. *Let X be a Banach space with WORTH. Consider the following statements.*

a) *X is reflexive*
b) *$\ell_1 \not\hookrightarrow X$*
c) *X^* has WORTH**
d) *X has the w-FPP*

Then a) $\Rightarrow$ b) $\Leftarrow$ c) $\Rightarrow$ d).

Can this proposition be improved in any way? Clearly, looking at proposition 5.1, the presence or absence of c_0 is crucial. If X^* has WORTH* and Y is the nearby Banach space with the 1-unconditional basis then if $c_0 \not\hookrightarrow Y$ then $c_0 \not\hookrightarrow X$. But since Y is a Banach lattice and $l_1 \not\hookrightarrow Y$, the property that $c_0 \not\hookrightarrow Y$ is equivalent to Y being reflexive. Does this necessarily mean that X is reflexive? A conjecture is that if X^* has WORTH* and $c_0 \not\hookrightarrow X$ then X is reflexive. This conjecture is the same as if X^* has WORTH* then X is reflexive if and only if $c_0 \not\hookrightarrow X$. Or the same as if X has WORTH then X is reflexive if and only if $c_0 \not\hookrightarrow X$ and $l_1 \not\hookrightarrow X$.

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