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DISTRIBUTIONS OF RANDOM CLOSED SETS VIA CONTAINMENT FUNCTIONALS

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Abstract. It is a celebrated result in the theory of random sets that, in a locally compact second countable Hausdorff space, distributions of random closed sets are in one-to-one correspondence with certain capacities. In this paper we generalize the correspondence to locally compact σ-compact Hausdorff spaces, showing that second countability is not necessary.

1. Introduction

A key result in the theory of random sets, the Choquet or Choquet–Kendall–Matheron theorem, states that the distribution of a random closed set X is characterized by its *hitting functional*

$$T_X : K \mapsto \mathbb{P}(X \cap K \neq \emptyset), \quad K \text{ compact},$$

and hitting functionals are characterized by the property of being completely alternating, outer continuous capacities. In the usual presentation (e.g. [10, Theorem 1.13, p.10] or [12, Section 5.3]), the carrier space is locally compact, second countable and Hausdorff (LCSH). Norberg has shown that Hausdorffness can be replaced by the weaker property of sobriety [13, Theorem 6.1]. Another approach is to try a similar characterization for the *containment functional*

$$C_X : F \mapsto \mathbb{P}(X \subset F), \quad F \text{ closed},$$

the natural analog of the cumulative distribution function. Although equivalent in an LCSH space, the two approaches depart in more general spaces: refining the topology leads to having more closed sets and fewer compact sets, so the natural σ-algebra associated to the containment functional grows while that associated to the hitting functional shrinks.

This paper presents such a characterization of distributions by identifying their containment functionals as the completely monotone, outer continuous capacities. The result is valid in any Hausdorff space which is both locally compact and σ-compact, thus dropping the assumption of second countability. Although σ-compactness and second countability are logically independent in general, under local compactness the former is strictly weaker than the latter. Note that a LCSH space is metrizable and separable, so it has at most the cardinality of the continuum; in contrast, the cardinality of a locally compact σ-compact Hausdorff space (even a compact Hausdorff space) can be arbitrarily large.

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Since

$$C_X(F) = 1 - T_X(F^c),$$

it is also clear that the description of distributions via the containment functional is tantamount to a Choquet-Kendall-Matheron theorem with the hitting functional being defined on a class of open instead of compact sets.

The method of proof differs from those in the literature. To establish the result for a compact Hausdorff space, an integral representation theorem due to Rébillé [14] is used. Then the Alexandrov compactification is used to derive the extension to locally compact spaces.

The structure of the paper is as follows. Section 2 collects the basic notation and definitions. The main result is presented in Section 3 and proved in Section 4 and 5. Section 6 shows examples of random closed sets in spaces without the second countability axiom. Section 7 closes the paper with some final remarks.

2. Preliminaries

Let $(\mathbb{E}, \tau)$ be a locally compact Hausdorff space. If $\mathbb{E}$ is the union of a countable family of compact spaces, then $\mathbb{E}$ is called σ-compact. The closure and the complement of a subset A are denoted by $\operatorname{cl} A, A^c$. The space of all real continuous functions on $\mathbb{E}$ will be denoted by $\mathcal{C}(\mathbb{E})$, while we will use $\mathcal{C}_0(\mathbb{E})$ if the functions additionally vanish at infinity (i.e. the sets $\{|f| \geq n^{-1}\}$ are compact for all $n \in \mathbb{N}$). Clearly, if $\mathbb{E}$ is compact then $\mathcal{C}(\mathbb{E}) = \mathcal{C}_0(\mathbb{E})$.

Some classes of sets we will be using are the following. If necessary, we will specify the carrier space in parentheses.

- $\mathcal{F}$ Closed sets
- $\mathcal{F}^*$ Non-empty closed sets
- $\mathcal{F}_f$ Functionally closed sets, i.e. those having the form $\{f = 0\}$ (equivalently, $\{f \geq 0\}$) for some $f \in \mathcal{C}(\mathbb{E})$
- $\mathcal{G}_f$ Functionally open sets, i.e. those having the form $\{f > 0\}$ for some $f \in \mathcal{C}(\mathbb{E})$
- $\mathcal{F}_0$ Sets having the form $\{f = 0\}$ (equivalently, $\{f \geq 0\}$) for some $f \in \mathcal{C}_0(\mathbb{E})$
- $\mathcal{K}$ Compact sets
- $\mathcal{K}^*$ Non-empty compact sets
- $\mathcal{B}$ The Borel σ-algebra $\sigma(\tau)$
- $\mathcal{B}_0$ The Baire σ-algebra $\sigma(\mathcal{G}_f)$, the coarsest σ-algebra making all real continuous functions measurable
- $\mathcal{K}_0$ Baire compact sets ($\mathcal{K}_0 = \mathcal{K} \cap \mathcal{B}_0$)
- $\mathcal{G}_{f\delta}$ Countable intersections of functionally open sets

The topology τ coincides with the weak topology generated by $\mathcal{C}_0(\mathbb{E})$ [1, Corollary 2.2], therefore every closed subset of $\mathbb{E}$ is the intersection of sets in $\mathcal{F}_0$. Moreover, it is easy to check that $\mathcal{F}_f$ is closed under countable intersections and $\mathcal{F}_f \subset \mathcal{G}_{f\delta}$.

Let $\mathcal{H}$ be a lattice of subsets of $\mathbb{E}$ such that $\emptyset, \mathbb{E} \in \mathcal{H}$. A mapping $\nu : \mathcal{H} \to [0,1]$ is a *capacity* if it is monotone ($\nu(A) \leq \nu(B)$ whenever $A \subset B$), $\nu(\mathbb{E}) = 1$ and $\nu(\emptyset) = 0$. Further properties which a capacity may have are the following.

Outer continuity: $A_n \searrow A$ implies $\nu(A_n) \to \nu(A)$.

Inner continuity: $A_n \nearrow A$ implies $\nu(A_n) \to \nu(A)$.

Complete monotony: $\nu(\bigcup_{i=1}^n A_i) \geq \sum_{\emptyset \neq I \subset \{1,\dots,n\}} (-1)^{|I|+1} \nu(\bigcap_{i \in I} A_i)$ for any $n \in \mathbb{N}$.

Complete alternation: $\nu(\bigcap_{i=1}^n A_i) \leq \sum_{\emptyset \neq I \subset \{1,\dots,n\}} (-1)^{|I|+1} \nu(\bigcup_{i \in I} A_i)$ for any $n \in \mathbb{N}$.

Minitivity: $\nu(A \cap B) = \min\{\nu(A), \nu(B)\}$.

Let ν be a capacity defined on $\mathcal{H}$. If a bounded function $f : \mathbb{E} \to \mathbb{R}$ is $\mathcal{H}$-*measurable*, in the sense that the level sets $\{f \geq t\}$ are in $\mathcal{H}$ for each $t \in \mathbb{R}$, its *Choquet integral* against ν is defined to be

$$\int f \mathrm{d}\nu = \int_0^\infty \nu(\{f \geq t\}) \mathrm{d}t - \int_{-\infty}^0 [1 - \nu(\{f \geq t\})] \mathrm{d}t,$$

where both Riemann integrals exist due to the monotonicity of the function $t \mapsto \nu(\{f \geq t\})$.

3. Main result

If $\mathbb{E}$ is an LCSH space, a function from a measurable space to $\mathcal{F}^*$ is a random closed set if and only if it is *Effros measurable*, namely the events $\{X \subset F\}$ are measurable for all closed F. In this paper, X will be called a *random closed set* if

$$\{X \subset F\} \text{ is measurable for each } F \in \mathcal{F}_0.$$

This notion is formally less stringent than the usual one, but they coincide in LCSH spaces.

Proposition 3.1. *Let $\mathbb{E}$ be a locally compact Hausdorff space, and let X be a function from a measurable space to $\mathcal{F}^*$. Then,*

(a) *X is a random closed set if and only if $\inf f(X)$ is a random variable for every $f \in \mathcal{C}_0(\mathbb{E})$.*
(b) *If $\mathbb{E}$ is second countable, then X is a random closed set if and only if X is Effros measurable.*

Proof. Part (a) is routine. Only the necessity in part (b) is not trivial. It will suffice to show that $\mathcal{F}_0 = \mathcal{F}$ in a LCSH space.

Let $F \in \mathcal{F}$. From [2, Lemma 1.1], $F \in \mathcal{F}_0$ if and only if F^c is locally compact and σ-compact. The former is clear, since every open subset of a locally compact Hausdorff space is locally compact. As to σ-compactness, note that an LCSH space is both metrizable (by Urysohn's metrizability theorem, e.g. [8, Theorem 16, p.125]) and σ-compact (e.g. [9, Lemma 6.8, p.169]). Denote by G_n the open set of all points at a distance less than n^{-1} from F. Then $F^c = \bigcup_n G_n^c$ and each of the G_n^c is σ-compact since it is closed in $\mathbb{E}$. Thus F^c is indeed σ-compact, and the proof is complete. □

Our main theorem is as follows.

Theorem 3.2. *Let $\mathbb{E}$ be a locally compact σ-compact Hausdorff space. Then,*

(a) *For every random closed set X in $\mathbb{E}$, its containment functional C_X on $\mathcal{F}_0$ is an outer continuous, completely monotone capacity.*

(b) *For every outer continuous, completely monotone capacity μ on $\mathcal{F}_0$, there exists a random closed set X in $\mathbb{E}$ such that $C_X = \mu$.*

Note that the existence result in part (b) easily implies uniqueness. That is so because two distributions having the same containment functional coincide on the π-system generating the σ-algebra and so are identical.

The proof of part (a) is straightforward and relies on basic properties of probabilities, see e.g. [10, Sections 1.2 and 1.6]. The following two sections are devoted to proving part (b).

4. The compact Hausdorff case

In this section, we assume $\mathbb{E}$ is a compact Hausdorff space. Therefore, $\mathcal{F}_0 = \mathcal{F}_f = \mathcal{K}_0$ (for the last identity, see e.g. [5]) and we will use $\mathcal{K}_0$ throughout.

Proof of Theorem 3.2.(b) in the compact case. Fix an outer continuous, completely monotone capacity μ on $\mathcal{K}_0$ (while a generic capacity is denoted ν in the proof). The task is to construct a random closed set X, on some probability space, such that $C_X = \mu$ on $\mathcal{K}_0$.

Let $\mathfrak{C}[\mathcal{B}_0; \mathcal{K}_0]$ be the family of all outer continuous, completely monotone capacities ν on $\mathcal{B}_0$ which are $\mathcal{K}_0$-inner regular, namely

$$\nu(A) = \sup_{A \supset K \in \mathcal{K}_0} \nu(K)$$

for all $A \in \mathcal{B}_0$. The following lemma is [14, Proposition 2]; in view of equation (5.2) in [15], it essentially follows from [15, Theorem 5.1.(2)].

Lemma 4.1. *Let ν be an outer continuous, completely monotone capacity on $\mathcal{K}_0$. Then ν has a unique extension $\nu_* \in \mathfrak{C}[\mathcal{B}_0; \mathcal{K}_0]$.*

Each function in $\mathcal{C}(\mathbb{E})$ is bounded and $\mathcal{K}_0$-measurable. Let $\mathfrak{C}[\mathcal{B}_0; \mathcal{K}_0]$ be endowed with the topology τ_C of pointwise convergence of the functionals $f \in \mathcal{C}(\mathbb{E}) \mapsto \int f \mathrm{d}\nu \in \mathbb{R}$ (analogous to the weak topology of probability measures).

In the language of cooperative game theory, a *filter game* is a 0-1 valued minitive capacity. Let $\mathfrak{F}[\mathcal{B}_0; \mathcal{K}_0]$ be the set of all outer continuous, $\mathcal{K}_0$-inner regular filter games on $\mathcal{B}_0$. Let $\sigma_{\mathfrak{F}}$ be the σ-algebra on $\mathfrak{F}[\mathcal{B}_0; \mathcal{K}_0]$ generated by the restrictions to $\mathfrak{F}[\mathcal{B}_0; \mathcal{K}_0]$ of all τ_C-continuous functionals $\varphi : \mathfrak{C}[\mathcal{B}_0; \mathcal{K}_0] \to \mathbb{R}$ which are linear in the sense that

$$\varphi(\lambda\nu_1 + (1-\lambda)\nu_2) = \lambda\varphi(\nu_1) + (1-\lambda)\varphi(\nu_2)$$

for $\lambda \in [0,1], \nu_1, \nu_2 \in \mathfrak{C}[\mathcal{B}_0; \mathcal{K}_0]$.

The σ-algebra $\sigma_{\mathfrak{F}}$ provides a measurable structure for certain 0-1 valued capacities. Our main tool is the following integral representation of Choquet integrals by Rébillé [14, Theorem 1 plus Proposition 4].

Lemma 4.2. *Let $\nu \in \mathfrak{C}[\mathcal{B}_0; \mathcal{K}_0]$. Then, there exists a probability measure m_ν on $\sigma_{\mathfrak{F}}$ such that, for every $f \in \mathcal{C}(\mathbb{E})$,*

$$\int f \mathrm{d}\nu = \int \int f \mathrm{d}\eta \, \mathrm{d}m_\nu(\eta).$$

The *unanimity game* of $B \in \mathcal{B}_0$ is the 0-1 valued capacity on $\mathcal{B}_0$ given by

$$u_B(A) = \begin{cases} 1, & B \subset A \\ 0, & B \not\subset A. \end{cases}$$

Let us introduce a closely related notion which we will call *qualified unanimity games.* For any fixed $B \in \mathcal{B}_0$, the $\mathcal{K}_0$-*qualified unanimity game* of B is the 0-1 valued capacity on $\mathcal{B}_0$ such that

$$q_B(A) = 1 \text{ if and only if } B \subset K \subset A \text{ for some } K \in \mathcal{K}_0.$$

The name 'qualified unanimity' corresponds to the fact that the unanimity of all players in B is necessary but more players may need to be enrolled as only coalitions in $\mathcal{K}_0$ are qualified to reach the goal of the game.

The set of $\mathcal{K}_0$-qualified unanimity games of all compact sets will be denoted by $\mathfrak{Q}[\mathcal{B}_0; \mathcal{K}_0; \mathcal{K}]$. In general, every unanimity game is minitive and so is a filter game, but there exist filter games which are not unanimity games. On the other hand, the unanimity game of a set in $\mathcal{K}_0$ is always a $\mathcal{K}_0$-qualified unanimity game.

The key to our result is proving that the filter games we are considering are exactly the $\mathcal{K}_0$-qualified unanimity games.

Theorem 4.3. $\mathfrak{F}[\mathcal{B}_0; \mathcal{K}_0] = \mathfrak{Q}[\mathcal{B}_0; \mathcal{K}_0; \mathcal{K}]$.

Proof. We begin with the inclusion $\mathfrak{Q}[\mathcal{B}_0; \mathcal{K}_0; \mathcal{K}] \subset \mathfrak{F}[\mathcal{B}_0; \mathcal{K}_0]$. Let $K \in \mathcal{K}$; to prove the minitivity of q_K, one checks easily with the definition that $q_K(A \cap B) = 1$ if and only if $q_K(A) = 1$ and $q_K(B) = 1$. For outer continuity, let $\{A_n\}_n$ be Baire sets with $A_n \searrow A$. If $q_K(A_n) = 1$ for all n, then there exist $K_n \in \mathcal{K}_0$ with $K \subset K_n \subset A_n$. Then $K \subset \bigcap_n K_n \subset A$ so $q_K(A_n) \to q_K(A)$. Hence, q_K is outer continuous. Finally, as regards inner regularity, if $q_K(A) = 0$ then it is trivial; otherwise, there is $K' \in \mathcal{K}_0$ such that $K \subset K' \subset A$. Thence, $q_K(K') = 1$ and indeed $q_K(A) = \max_{A \supset L \in \mathcal{K}_0} q_K(L)$.

We proceed now to proving the inclusion $\mathfrak{F}[\mathcal{B}_0; \mathcal{K}_0] \subset \mathfrak{Q}[\mathcal{B}_0; \mathcal{K}_0; \mathcal{K}]$.

Step 1. The proof makes an essential use of singletons and other sets which may fail to be Baire sets, so we must start by extending each $\eta \in \mathfrak{F}[\mathcal{B}_0; \mathcal{K}_0]$ to deal with such sets. For any Borel set A, we define

$$\eta_{\mathcal{B}}(A) = \sup_{A \supset K \in \mathcal{K}_0} \eta(C).$$

Since η is $\mathcal{K}_0$-inner regular, $\eta_{\mathcal{B}}$ is indeed an extension of η. We do not need $\eta_{\mathcal{B}}$ to have more properties than monotony (which is clear from the definition) and minitivity. Indeed, using the monotony and minitivity of η,

$$\begin{aligned} \min\{\eta_{\mathcal{B}}(A), \eta_{\mathcal{B}}(B)\} \geq \eta_{\mathcal{B}}(A \cap B) &\geq \sup_{A \subset K_A \in \mathcal{K}_0, B \subset K_B \in \mathcal{K}_0} \min\{\eta_{\mathcal{B}}(K_A), \eta_{\mathcal{B}}(K_B)\} \\ &= \min\{\eta_{\mathcal{B}}(A), \eta_{\mathcal{B}}(B)\}. \end{aligned}$$

Fix $\eta \in \mathfrak{F}[\mathcal{B}_0; \mathcal{K}_0]$ and set

$$Z = \{x \in \mathbb{E} \mid \eta_{\mathcal{B}}(\{x\}^c) = 0\}.$$

Our task is to show that $\eta = q_Z$ and Z is non-empty and closed.

Step 2. Let us prove the identity

$$\eta(G) = \min_{x \notin G} \eta_{\mathcal{B}}(\{x\}^c)$$

for $G \in \mathcal{G}_f$. If $\eta(G) = 1$, it follows from the monotony of η; therefore we assume $\eta(G) = 0$.

Let $i : \mathcal{P}(\mathbb{E}) \backslash \{\emptyset\} \to \mathbb{E}$ select a point from each non-empty subset of $\mathbb{E}$. Define the set-valued mapping

$$W : (a, b) \in \{(a, b) \in \mathbb{E} \times \mathbb{E} \mid a \neq b\} \mapsto \{G \in \mathcal{G}_f \mid a \in G, b \notin \operatorname{cl} G\}.$$

The images $W(a, b)$ are non-empty. Indeed, a and b are separated by disjoint open neighbourhoods G_a, G_b so $b \notin \operatorname{cl} G_a$. By the complete regularity of $\mathbb{E}$, there is a continuous function f with $f(b) = 0$ and $f(\operatorname{cl} G_a) = \{1\}$, therefore $\{f > 1/2\} \in W(a, b)$. Then the axiom of choice ensures the existence of a selection w of W (i.e. $w(a, b)$ is a functionally open neighbourhood of a whose closure misses b).

Our main tool now is a mapping $\Phi : \mathcal{K}^* \to \mathcal{K}^*$ defined as follows. For any $K \in \mathcal{K}^*$,

(1) If $K \subset Z$ or K is a singleton, then let $\Phi(K) = K$.
(2) Else, if $\eta_{\mathcal{B}}(\{i(K)\}^c) = 0$ then let $\Phi(K) = \{i(K)\}$.
(3) Else, let $K' = K \cap \{i(K)\}^c$. If $\eta_{\mathcal{B}}(\{i(K')\}^c) = 0$ then let $\Phi(K) = \{i(K')\}$.

If $\Phi(K)$ still remains undefined, set

$$K_1 = \operatorname{cl}(K \cap w(i(K), i(K'))),$$
$$K_2 = \operatorname{cl}(K \cap w(i(K), i(K'))^c).$$

Since $K = K_1 \cup K_2$, we have $K^c = K_1^c \cap K_2^c$ and so $\eta_{\mathcal{B}}(K^c) = \min\{\eta_{\mathcal{B}}(K_1^c), \eta_{\mathcal{B}}(K_2^c)\}$.

(4) If $\eta_{\mathcal{B}}(K^c) = \eta_{\mathcal{B}}(K_1^c)$, let $\Phi(K) = K_1$.
(5) Else, let $\Phi(K) = K_2$.

The relevant fact following from the construction of Φ is that, for every $K \in \mathcal{K}^*$,

$$\Phi(K) \in \mathcal{K}^*, \quad \Phi(K) \subset K, \quad \eta_{\mathcal{B}}(\Phi(K)^c) = \eta_{\mathcal{B}}(K^c).$$

We use transfinite recursion now to define a family $\{K_\alpha\}_\alpha$ as follows:

$K_0 = G^c$.
If α is an ordinal, then $K_{\alpha+1} = \Phi(K_\alpha)$.
If α is a limit ordinal, then $K_\alpha = \bigcap_{\beta<\alpha} K_\beta$.

Note that K_α is non-empty for every ordinal α. And, since they are decreasing in α, there exists $K_\infty \in \mathcal{K}^*$ such that, for some β, $K_\alpha = K_\infty$ for any $\alpha \geq \beta$.

We claim $K_\infty \subset Z$. Reasoning by contradiction, assume some $x \in K_\infty$ has $\eta_{\mathcal{B}}(\{x\}^c) = 1$. That rules out the possibility that K_∞ is a singleton, since

$$\eta_{\mathcal{B}}(K_\infty^c) = \eta_{\mathcal{B}}(K_\beta^c) = \eta_{\mathcal{B}}(G) = \eta(G) = 0.$$

Consequently, rule (1) does not apply to K_∞. Since $\Phi(K_\infty) = K_{\beta+1} = K_\infty$ is not a singleton, (2) and (3) do not apply either. But the same happens to (4) and (5), since both $(K_\infty)_1$ and $(K_\infty)_2$ are proper subsets of K_∞ (indeed, $i(K_\infty) \notin (K_\infty)_2$, $i(K_\infty \cap \{i(K_\infty)\}^c) \notin (K_\infty)_1$). Cases (1) through (5) exhaust all possibilities but none of them is the case, a contradiction.

Since

$$\emptyset \neq K_\infty \subset G^c \cap Z,$$

there exists $x \notin G$ with $\eta_{\mathcal{B}}(\{x\}^c) = 0$ and so indeed

$$\eta(G) = \min_{x \notin G} \eta_{\mathcal{B}}(\{x\}^c).$$

Step 3. We will show now that $\eta = q_Z$ and $Z \in \mathcal{K}$. Since it contains K_∞, Z is non-empty. From the formula above, $\eta(G) = 1$ if and only if $Z \subset G$, so $\eta = u_Z$ on $\mathcal{G}_f$. By the outer continuity of η and u_Z, the identity extends to $\mathcal{G}_{f\delta}$ (in particular, it holds in $\mathcal{K}_0$.)

At this point, let us check that Z is compact. Take $x \in Z^c$, then $\eta_{\mathcal{B}}(\{x\}^c) = 1$ so there is $K \in \mathcal{K}_0$ such that $\eta(K) = 1$ for some $K \in \mathcal{K}_0$ with $x \notin K$. By the regularity of $\mathbb{E}$, there is an open neighbourhood G_x separating x from K. For each $y \in G_x$ we have $\eta_{\mathcal{B}}(\{y\}^c) \geq \eta(K) = 1$, and so $G \subset Z^c$. Accordingly, Z^c is open.

To conclude the proof, we recap three facts: (i) $q_Z = u_Z = \eta$ on $\mathcal{K}_0$; (ii) η is outer continuous and $\mathcal{K}_0$-inner regular; (iii) q_Z is so as well since $\mathfrak{Q}[\mathcal{B}_0; \mathcal{K}_0; \mathcal{K}] \subset \mathfrak{F}[\mathcal{B}_0; \mathcal{K}_0]$. By Lemma 4.1 (uniqueness of the outer continuous, inner regular extension to $\mathcal{B}_0$), we have $\eta = q_Z$. ☐

Finally, denote by $\mathbb{P}$ the probability measure m_{μ_*} obtained by applying Lemma 4.2 to the extended capacity μ_*, and consider the function

$$X : (\mathfrak{F}[\mathcal{B}_0; \mathcal{K}_0], \sigma_{\mathfrak{F}}, \mathbb{P}) \to \mathcal{K}^*$$

which maps each filter game η to the unique $Z \in \mathcal{K}^*$ such that $\eta = q_Z$. By Theorem 4.3, X is well defined and bijective. The following result concludes the proof; note that its general part has some independent significance, cf. [10, Theorem 5.1, p.70].

Proposition 4.4. *The function X is a random closed set and, for every upper semicontinuous Baire function $f : \mathbb{E} \to \mathbb{R}$,*

$$\int f \mathrm{d}\mu = \int \inf f(X) \mathrm{d}\mathbb{P}.$$

In particular, $C_X = \mu$.

Proof. Denote the Lebesgue measure in $\mathbb{R}$ by Leb. Note that any $g \in \mathcal{C}(\mathbb{E})$ is bounded and $\mathcal{B}_0$-measurable. For any $\eta \in \mathfrak{F}[\mathcal{B}_0; \mathcal{K}_0]$, if g is non-negative we have

$$\begin{aligned} \int g \mathrm{d}\eta &= \int_0^\infty q_{X(\eta)}(\{g \geq t\}) \mathrm{d}t \\ &= \mathrm{Leb}(\{t \in [0, \infty) \mid \exists H \in \mathcal{K}_0 \mid X(\eta) \subset H \subset \{g \geq t\}\}). \end{aligned}$$

Choosing

$$H = \{g \geq \inf g(X(\eta))\} \in \mathcal{F}_f = \mathcal{K}_0,$$

we readily show that the right-hand side above equals $\inf g(X(\eta))$. And the identity

$$\int g \mathrm{d}\eta = \inf g(X(\eta))$$

extends to the general case by applying it to $g - \inf g(\mathbb{E})$.

Taking into account Lemma 4.2 now, we have

$$\int g \mathrm{d}\mu = \int g \mathrm{d}\mu_* = \int \inf g(X) \mathrm{d}\mathbb{P}$$

for $g \in \mathcal{C}(\mathbb{E})$. An upper semicontinuous Baire function f on $\mathbb{E}$ is the pointwise limit of a decreasing sequence $\{f_n\}_n$ of continuous functions [11, Theorem 3]. Since the Choquet integral against an outer continuous capacity is continuous on decreasing sequences [17, Theorem 1],

$$\int f \mathrm{d}\mu = \inf_n \int f_n \mathrm{d}\mu = \inf_n \int \inf f_n(X) \mathrm{d}\mathbb{P}.$$

The sequence $\{\inf f_n(X)\}_n$ being non-decreasing with limit $\inf f(X)$, the monotone convergence theorem yields the sought identity.

To show that X is a random closed set, fix an arbitrary $F \in \mathcal{F}_0 = \mathcal{F}_f$. We can write $F = \{f = 1\}$ for some continuous function $f : \mathbb{E} \to [0,1]$. Define $\varphi : \nu \in \mathfrak{C}[\mathcal{B}_0; \mathcal{K}_0] \mapsto \int f \mathrm{d}\nu \in \mathbb{R}$. Clearly, φ is linear and τ_C-continuous. By the definition of $\sigma_{\mathfrak{F}}$, the restriction $\varphi|_{\mathfrak{F}[\mathcal{B}_0;\mathcal{K}_0]}$ is $\sigma_{\mathfrak{F}}$-measurable. But then the event

$$\begin{aligned}\{X \subset F\} &= \{f(X) = \{1\}\} = \{\eta \in \mathfrak{F}[\mathcal{B}_0; \mathcal{K}_0] \mid \inf f(X(\eta)) = 1\} \\ &= \{\eta \in \mathfrak{F}[\mathcal{B}_0; \mathcal{K}_0] \mid \int f \mathrm{d}\eta = 1\} = (\varphi|_{\mathfrak{F}[\mathcal{B}_0;\mathcal{K}_0]})^{-1}(1)\end{aligned}$$

is measurable.

Finally, for any $K \in \mathcal{K}_0$, its indicator function I_K is an upper semicontinuous Baire function and so

$$\mu(K) = \int I_K \mathrm{d}\mu = \int \inf I_K(X) \mathrm{d}\mathbb{P} = \mathbb{P}(X \subset K) = C_X(K).$$

□

5. Extension to the noncompact case

Let $\mathbb{E}$ be a locally compact σ-compact Hausdorff space. We regard $\mathbb{E}$ as a subset of its Alexandrov compactification $\alpha\mathbb{E} = \mathbb{E} \cup \{\infty\}$. Every $f \in \mathcal{C}_0(\mathbb{E})$ has a continuous extension $\hat{f} \in \mathcal{C}(\alpha\mathbb{E})$ with $\hat{f}(\infty) = 0$. Since we need to consider classes of subsets of both $\mathbb{E}$ and $\alpha\mathbb{E}$, we will indicate the carrier space in parentheses.

From [2, Lemma 1.1], there exists a function $h \in \mathcal{C}_0(\mathbb{E})$ with $\mathbb{E} = \{h > 0\}$. Therefore $\emptyset = \{h = 0\} \in \mathcal{F}_0(\mathbb{E})$ and $\{\infty\} = \{\hat{h} = 0\} \in \mathcal{K}_0(\alpha\mathbb{E})$.

Proof of Theorem 3.2.(b) in the general case. Fix an outer continuous, completely monotone capacity $\mu : \mathcal{F}_0(\mathbb{E}) \to [0,1]$. Let us find a random closed set X such that $C_X = \mu$. Set $\hat{\mu} : \mathcal{K}_0(\alpha\mathbb{E}) \to [0,1]$ given by $\hat{\mu}(B) = \mu(B \cap \mathbb{E})$. The mapping $\hat{\mu}$ is well defined: if $B \in \mathcal{K}_0(\alpha\mathbb{E})$ then $B = \{f = 0\}$ for some $f \in \mathcal{C}(\alpha\mathbb{E})$, whence $B \cap \mathbb{E} = \{f|_{\mathbb{E}} = 0\} \in \mathcal{F}_0(\mathbb{E})$.

It is immediate that $\hat{\mu}(\emptyset) = 0, \hat{\mu}(\alpha\mathbb{E}) = 1$. The outer continuity of $\hat{\mu}$ is clear as well. And $\hat{\mu}$ is completely monotone because the mapping $B \mapsto B \cap \mathbb{E}$ is a $\cap$-homomorphism [15, Section 2].

The proof for the compact case yields the probability space $(\mathfrak{F}[\mathcal{B}_0(\alpha\mathbb{E}); \mathcal{K}_0(\alpha\mathbb{E})], \sigma_{\mathfrak{F}}, \mathbb{P})$ and a random closed set $Y : \mathfrak{F}[\mathcal{B}_0(\alpha\mathbb{E}); \mathcal{K}_0(\alpha\mathbb{E})] \to \mathcal{F}(\alpha\mathbb{E})$ such that

$$\mathbb{P}(Y \subset B) = \hat{\mu}(B) \text{ for all } B \in \mathcal{K}_0(\alpha\mathbb{E}).$$

Set

$$X = Y \cap \mathbb{E} : \mathfrak{F}[\mathcal{B}_0(\alpha\mathbb{E}); \mathcal{K}_0(\alpha\mathbb{E})] \to \mathcal{F}^*(\mathbb{E}).$$

The values of X are indeed non-empty almost surely, since

$$\mathbb{P}(X = \emptyset) = \mathbb{P}(Y \subset \{\infty\}) = \hat{\mu}(\{\infty\}) = \mu(\emptyset) = 0.$$

We must show that X is a random closed set and $C_X = \mu$. For any $F = \{f = 0\} \in \mathcal{F}_0(\mathbb{E})$,

$$\{X \subset F\} = \{Y \subset F \cup \{\infty\}\} = \{Y \subset \{\hat{f} = 0\}\},$$

which is measurable. Therefore X is a random closed set in $\mathbb{E}$. Finally,

$$C_X(F) = \mathbb{P}(Y \subset F \cup \{\infty\}) = \hat{\mu}(F \cup \{\infty\}) = \mu(F)$$

for each $F \in \mathcal{F}_0(\mathbb{E})$, and the proof is complete. □

Observe that Y may differ from $X \cup \{\infty\}$. Indeed, for any $K \in \mathcal{K}_0(\mathbb{E})$, since $\infty \notin K$ we have

$$\mathbb{P}(X \subset K) = \mathbb{P}(Y \subset K),$$

whence $\{\infty \notin Y\} \supset \{X \subset K\}$ modulo a null set.

6. Some examples

In this section we present examples of random closed sets in locally compact Hausdorff spaces which may fail to be second countable.

Example 6.1. Powers of probability measures. Let P be the distribution of a random element of $\mathbb{E}$. Then the simplest example of a non-trivial containment functional is P^k since, for i.i.d. random elements ξ_i of $\mathbb{E}$ with distribution P,

$$P^k(F) = \prod_{i=1}^{k} \mathbb{P}(\xi_i \in F) = \mathbb{P}(\bigcap_{i=1}^{k}\{\xi_i \in F\}) = \mathbb{P}(\{\xi_i\}_{i=1}^k \subset F) = C_{\{\xi_i\}_{i=1}^k}(F).$$

In general, $C_X^k = C_{X_1 \cup \ldots \cup X_k}$, with X_k i.i.d. as X.

Example 6.2. Poisson point processes. Point processes can be formulated in an abstract measurable space [7], in particular a locally compact Hausdorff space with its Baire σ-algebra. A *Poisson point process* with intensity measure Λ is a locally finite process whose random counting measure N satisfies the assumptions

(i) For every measurable A, the count $N(A) = \mathrm{card}(X \cap A)$ is Poisson distributed with mean $\Lambda(A)$,
(ii) Disjoint measurable sets $A_1, \ldots, A_k$ have independent counts $N(A_1), \ldots, N(A_k)$.

Then X is a random closed set in our sense since

$$\{X \subset F\} = \{N(F^c) = 0\},$$

and its containment functional is given by

$$C_X(F) = \exp\{-\Lambda(F^c)\}.$$

Example 6.3. Random walk in a locally compact group. Let $\{\xi_n\}_n$ be a random walk in a locally compact group $\mathbb{E}$ (see e.g. [16]), and consider its closed trajectory $X = \mathrm{cl}\{\xi_n\}_n$. For any $f \in \mathcal{C}_0(\mathbb{E})$,

$$\inf f(X) = \inf f(\{\xi_n\}_n) = \inf_n f(\xi_n)$$

is a random variable (e.g. [3, Corollary 6.3.7]) and so, by Proposition 3.1, X is a random closed set.

Example 6.4. Random subtrees. A *real tree* is a metric space $(\mathbb{E}, d)$ such that, for all $x, y \in \mathbb{E}$, the following uniqueness properties hold:

(i) There exists a unique isometry $f_{x,y} : [0, d(x, y)] \to \mathbb{E}$ such that $f_{x,y}(0) = x$ and $f_{x,y}(d(x, y)) = y$.

(ii) If $q : [0, 1] \to \mathbb{E}$ is continuous and injective, with $q(0) = x$ and $q(1) = y$, then $q([0, 1]) = f_{x,y}([0, d(x, y)])$.

Let $\mathbb{E}$ be a locally compact real tree (see e.g. [6]). Then a random closed set in $\mathbb{E}$ which is a tree is a random subtree of $\mathbb{E}$.

Example 6.5. Discrete spaces. Any set $\mathbb{E}$ with its discrete topology is locally compact and Hausdorff. In that case, Theorem 3.2 states that the probabilities $\mathbb{P}(X \subset A)$ for all $A \in \mathcal{P}(\mathbb{E})$ determine uniquely the probabilities $\mathbb{P}(X \in \mathcal{A})$ for all $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathbb{E}))$.

7. Concluding remarks

Since the method of proof is new, it is interesting to remark what it looks like in the LCSH case. Apart from minor language changes (e.g. every upper semicontinuous function is Baire, discussion of neighbourhoods can be done with balls, and so on), the only significant difference is that the transfinite recursion argument in Step 2 of the proof of Theorem 4.3 can be replaced by a much simpler, standard argument of compactness and subsequencing in a metric space.

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