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ITERATIVE METHODS FOR A SYSTEM OF NONLINEAR GENERALIZED MIXED IMPLICIT EQUILIBRIUM PROBLEMS IN BANACH SPACES

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Abstract. A new system of nonlinear generalized mixed implicit equilibrium problems involving non-monotone set-valued mappings with non-compact values in q-uniformly smooth Banach spaces is introduced. By using the Yosida approximation, a system of generalized Wiener-Hopf equations is considered. The equivalence between the system of nonlinear generalized mixed implicit equilibrium problems and a system of generalized Wiener-Hopf equations is established. By using this equivalence, a fixed point formulation is derived. Two new iterative algorithms with mixed errors are proposed and the existence theorems for solutions of the aforesaid systems are established. Under some suitable conditions, the convergence analysis of the sequences generated by the proposed iterative algorithms is discussed. Some fatal errors in the results in [20] are pointed out and the correct versions of these results are presented. Some comments related to the work in [20] are given at the end. The results presented in this paper extend and improve some known results in the literature.

1. Introduction

The equilibrium problem (EP) is an unified model of several problems, namely, variational inequalities, optimization problems, problems of Nash equilibria, saddle point problems, fixed point problems and complementarity problems; See, for example, [1, 2, 7, 8, 14] and the references therein. Several extensions of EP have been studied in the literature; See, for example, [3, 10–12, 19, 20, 22, 23, 30] and the references therein. The system of equilibrium problems is one of the strongest tools to study Nash equilibrium problem [25, 26]. In the last two decades, the Nash equilibrium problem is studied by using system of variational inequalities or system of equilibrium problems; See, for example, [4–6, 13, 21] and the references therein. The existence of a solution of system of variational inequalities or systems of equilibrium problems is studied in [4–6, 21]. While in [13, 30], the solution methods for these systems are studied. In the early ninety's, Robinson [27] and Shi [28] initially used the Wiener-Hopf equation to study the variational inequalities. Recently, Kazmi and Khan [20] considered a generalized mixed equilibrium problem (in short, GMEP) involving non-monotone set-valued mappings with non-compact values in real Hilbert space. They extended the notions of the Yosida approximation and its corresponding regularized operator given in [22, 23] and discussed some of their properties. Related to GMEP, they considered a generalized Wiener-Hopf

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equation problem (in short, GWHEP) and proved the equivalence between GMEP and GWHEP. They gave a fixed point formulation of GWHEP and constructed an iterative algorithm for solving GWHEP. They extended the notion of stability given by Harder and Hicks [15], and discussed the existence of solution of GWHEP, and convergence and stability analysis for the their proposed iterative algorithm. One of the main motivations of this paper is to show that the paper [20] has some fatal errors.

In this paper, we introduce a system of nonlinear generalized mixed implicit equilibrium problems (in short, SNGMIEP) involving non-monotone set-valued mappings with non-compact values in q-uniformly smooth Banach spaces, which includes GMEP, the problems of finding a zero of a maximal monotone operator, and Nash equilibria problems as special cases. By using the Yosida approximation, we consider a system of generalized Wiener-Hopf equations (in short, SGWHE) associated with SNGMIEP. We prove that SNGMIEP and SGWHE have the same solution set. We get fixed point formulations of SNGMIEP and SGWHE and construct two new iterative algorithms with mixed errors for solving SNGMIEP and SGWHE. We establish the existence theorems for solutions of the aforesaid systems and discuss the convergence analysis of the sequences generated by the our proposed iterative algorithms. Finally, we show that the verified theorem in related to stability of the iterative algorithm introduced by Kazmi and Khan [20] is incorrect. The results presented in this paper improve and extend the corresponding results in the literature.

2. Formulations and preliminaries

Let X be a real Banach space with its dual space X^* and $\langle .,. \rangle$ be the dual pairing between X and X^*. Let K be a nonempty, closed and convex subset of X and $CB(X)$ be the family of all nonempty, closed and bounded subsets of X. The *Hausdorff metric* $\mathcal{H}(.,.)$ on $CB(X)$ is defined by

$$\mathcal{H}(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}, \quad \forall A,B\in CB(X).$$

The generalized duality mapping $J_q : X \multimap X^*$ is defined by

$$J_q(x) = \left\{f^*\in X^* : \langle x, f^*\rangle = \|x\|^q,\ \|f^*\| = \|x\|^{q-1}\right\}, \quad \forall x\in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \neq 0$ and J_q is single-valued if X^* is strictly convex. In the sequel, we always assume that X is a real Banach space such that J_q is single-valued. If X is a Hilbert space, then J_2 becomes the identity mapping on X.

The *modulus of smoothness* of X is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}\left(\|x+y\|+\|x-y\|\right)-1 : \|x\|\leq 1,\ \|y\|\leq t\right\}.$$

A Banach space X is called *uniformly smooth* if

$$\lim_{t\to 0}\frac{\rho_X(t)}{t} = 0.$$

X is called *q-uniformly smooth* if there exists a constant $c > 0$ such that

$$\rho_X(t) \leq ct^q, \quad q > 1.$$

Note that J_q is single-valued if X is uniformly smooth. For further detail on the geometry of Banach spaces, we refer to [2] and the references therein.

For each $j \in \{1, 2, \ldots, l\}$, let X_j be a real reflexive q_j-uniformly smooth Banach space with norm $\|.\|_j$ and its dual X_j^*, $\langle ., . \rangle_j$ be the dual pairing of X_j^* and X_j, and K_j be a nonempty closed convex subset of X_j. For each $i \in \{1, 2, \ldots, p\}$, $p \leq l$, let $F_i : K_i \times K_i \to \mathbb{R}$ be a bifunction such that $F_i(x, x) = 0$, $\forall x \in K_i$. Further, for each $i \in \{1, 2, \ldots, p\}$, assume that $g_i : K_i \to K_i$, $N_i : \prod_{j=1}^{l} X_j \to X_i$, $\eta_i : X_i \times X_i \to X_i^*$ are nonlinear single-valued mappings, and let $T_{i,1}, T_{i,2}, \ldots, T_{i,l} : X_i \to CB(X_i)$ be set-valued mappings. We consider the problem of finding $(\bar{x}_1, \ldots, \bar{x}_p) \in \prod_{i=1}^{p} K_i$ and $(\bar{u}_{1,1}, \ldots, \bar{u}_{1,l}, \bar{u}_{2,1}, \ldots, \bar{u}_{2,l}, \ldots, \bar{u}_{p,1}, \bar{u}_{p,2}, \ldots, \bar{u}_{p,l}) \in \prod_{i=1}^{p} \prod_{j=1}^{l} T_{i,j}(\bar{x}_i)$ such that for each $i = 1, 2, \ldots, p$,

$$F_i(g_i(\bar{x}_i), y_i) + \langle N_i(\bar{u}_{i,1}, \bar{u}_{i,2}, \ldots, \bar{u}_{i,l}), \eta_i(y_i, g_i(\bar{x}_i)) \rangle \geq 0, \quad \forall y_i \in K_i. \tag{2.1}$$

Problem (2.1) is called a *system of nonlinear generalized mixed implicit equilibrium problems* (in short, SNGMIEP) in uniformly smooth Banach spaces.

If $p = 1$, $l = 2$, $X_1 = \mathcal{H}$ is a real Hilbert space, $F_1 = F$, $T_{1,1} = T$, $T_{1,2} = B$, $N_1 = N$, $\eta_1 = \eta$, $K_1 = K$, then SNGMIEP (2.1) collapses to the following *generalized mixed implicit equilibrium problem* considered and studied in [20].

$$\begin{array}{l} \text{Find } \bar{x}_1 = \bar{x} \in K,\ \bar{u}_{1,1} = \bar{u} \in T(\bar{x}),\ \bar{u}_{1,2} = \bar{v} \in B(\bar{x}) \text{ such that} \\ F(g(\bar{x}), y) + \langle N(\bar{u}, \bar{v}), \eta(y, g(\bar{x})) \rangle \geq 0, \quad \forall y \in K. \end{array} \tag{2.2}$$

For different choices of the mappings, we obtain different problems considered and studied in [10, 16, 22] and the references therein.

We present some definitions and results which will be used in the sequel.

Definition 2.1 ([24])**.** A set-valued mapping $T : X \to CB(X)$ is called *$\mathcal{H}$-Lipschitz continuous* if there exists a constant $\xi > 0$ such that

$$\mathcal{H}(T(x), T(y)) \leq \xi \|x - y\|, \quad \forall x, y \in X,$$

where $\mathcal{H}$ is the Hausdorff metric on X.

Lemma 2.2 ([24])**.** *Let $T : X \to CB(X)$ be a set-valued mapping. Then for any given $\epsilon > 0$, $x, y \in X$ and $u \in T(x)$, there exists $v \in T(y)$ such that*

$$\|u - v\| \leq (1 + \epsilon)\mathcal{H}(T(x), T(y)). \tag{2.3}$$

If $T : X \to C(X)$, then the above inequality holds for $\epsilon = 0$, where $C(X)$ denotes the family of all nonempty compact subsets of X.

Xu [29] proved the following result concerning the characteristic inequalities in q-uniformly smooth Banach spaces.

Lemma 2.3 ([29])**.** *The real Banach space X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x) \rangle + c_q \|y\|^q, \quad \forall x, y \in X.$$

Definition 2.4. Let X be a q-uniformly smooth Banach space. A nonlinear mapping $g : X \to X$ is said to be

(a) *accretive* if
$$\langle g(x) - g(y), J_q(x-y)\rangle \geq 0, \quad \forall x, y \in X;$$
(b) *δ-strongly accretive* if there exists a constant $\delta > 0$ such that
$$\langle g(x) - g(y), J_q(x-y)\rangle \geq \delta\|x-y\|^q, \quad \forall x, y \in X;$$
(c) *σ-Lipschitz continuous* if there exists a constant $\sigma > 0$ such that
$$\|g(x) - g(y)\| \leq \sigma\|x-y\|, \quad \forall x, y \in X.$$

Definition 2.5. For each $j = 1, 2, \dots, l$, let X_j be a q_j-uniformly smooth Banach space with norm $\|.\|_j$, and for each $i \in \{1, 2, \dots, p\}$, $p \leq l$, let $N_i : \prod_{j=1}^{l} X_j \to X_i$ be a single-valued mapping. The mapping N_i is said to be *$\gamma_{i,j}$-Lipschitz continuous in the jth argument*, for each $i \in \{1, 2, \dots, p\}$ and $j \in \{1, 2, \dots, l\}$, if there exists a constant $\gamma_{i,j} > 0$ such that
$$\|N_i(x_1, x_2, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_l) - N_i(x_1, x_2, \dots, x_{j-1}, \hat{x}_j, x_{j+1}, \dots, x_l)\|_i \\ \leq \gamma_{i,j}\|x_j - \hat{x}_j\|_j, \quad \forall x_j, \hat{x}_j \in X_j.$$

Definition 2.6. Let K be a nonempty closed convex subset of a Hausdorff topological vector space E. A real-valued bifunction $F : K \times K \to \mathbb{R}$ is said to be

(a) *monotone* if
$$F(x, y) + F(y, x) \leq 0, \quad \forall x, y \in K;$$
(b) *strictly monotone* if
$$F(x, y) + F(y, x) < 0, \quad \forall x, y \in K \text{ with } x \neq y;$$
(c) *α-strongly monotone* if $E = X$ is a Banach space and there exists a constant $\alpha > 0$ such that
$$F(x, y) + F(y, x) \leq -\alpha\|x-y\|^2, \quad \forall x, y \in K;$$
(d) *upper hemicontinuous in the first argument* if
$$\limsup_{t \to 0} F(tz + (1-t)x, y) \leq F(x, y), \quad \forall x, y, z \in K.$$

Obviously, the strong monotonicity of F implies the monotonicity of F.

Definition 2.7. A mapping $\eta : X \times X \to X^*$ is said to be

(a) *monotone* if
$$\langle x - y, \eta(x, y)\rangle \geq 0, \quad \forall x, y \in X;$$
(b) *κ-strongly monotone* if there exists a constant $\kappa > 0$ such that
$$\langle x - y, \eta(x, y)\rangle \geq \kappa\|x-y\|^2, \quad \forall x, y \in X;$$
(c) *affine* in the first argument if
$$\eta(\beta x + (1-\beta)z, y) = \beta\eta(x, y) + (1-\beta)\eta(z, y), \quad \forall \beta \in [0, 1], x, y \in X;$$
(d) *γ-Lipschitz continuous* if there exists a constant $\gamma > 0$ such that
$$\|\eta(x, y)\| \leq \gamma\|x-y\|, \quad \forall x, y \in X.$$

In similar to part (c), we can define the affineness of the mapping η in the second argument.

3. Regularization, generalized Wiener-Hopf equations and iterative algorithms

In this section, we recall some auxiliary results which will be used in the sequel. Associated with SNGMIEP (2.1), we introduce a system of generalized Wiener-Hopf equations and propose two algorithms to solve this system of generalized Wiener-Hopf equations and the system of nonlinear generalized mixed implicit equilibrium problems (2.1).

The following result is a special case of [9, Theorem 3.9.3].

Lemma 3.1. *Let K be a closed convex subset of a Hausdorff topological vector space E and $F : K \times K \to \mathbb{R}$ be a bifunction such that the following conditions hold:*

(a) $F(x,x) \geq 0$, $\forall x \in K$;
(b) *F is monotone and for each $y \in K$, $x \mapsto F(x,y)$ is upper hemicontinuous;*
(c) *For each $x \in K$, $y \mapsto F(x,y)$ is convex and lower-semicontinuous;*
(d) *There exists a compact subset C of E and there exists $y_0 \in C \cap K$ such that $F(x,y_0) < 0$ for each $x \in K \setminus C$.*

Then the set of solutions to the following equilibrium problem (EP)*: Find $\hat{x} \in K$ such that*

$$F(\hat{x},y) \geq 0, \quad \forall y \in K,$$

is nonempty, convex and compact.

By using Lemma 3.1, Ding [12] deduced the following result under some conditions.

Lemma 3.2 ([12, Lemma 2.2])**.** *Let K be a nonempty closed convex bounded subset of a reflexive Banach space X. Let $F : K \times K \to (-\infty,+\infty)$ be a function, $\eta : X \times X \to X^*$ be a mapping and $\rho > 0$ be positive number. Suppose the following conditions are satisfied:*

(a) *F satisfies conditions (a)–(c) of Lemma 3.1;*
(b) *η is monotone with $\eta(x,y) + \eta(y,x) = 0$, $\forall x,y \in X$;*
(c) *η is affine in the second argument and continuous from weak topology in X to weak* topology in X^* in the first argument.*

Then for each $x \in X$, there exists a point $z \in K$ such that

$$\rho F(z,y) + \langle z - x, \eta(y,z)\rangle \geq 0, \quad \forall y \in K.$$

Remark 3.3. By a careful reading of the proof of Lemma 3.2, we found that the affineness in the first argument of η is needed. Moreover, we can also deduce the conclusion of Lemma 3.2 with the assumption that the mapping η is sequentially continuous in the first argument from the weak topology in X to the weak* topology in X^* instead of the assumption of continuity of the mapping η in the first argument from the weak topology in X to the weak* topology in X^*.

We present the correct version of above lemma as follows:

Lemma 3.4. *Let K be a nonempty closed convex bounded subset of a reflexive Banach space X and ρ be a positive number. Let $F : K \times K \to \mathbb{R}$ and $\eta : X \times X \to X^*$ satisfy conditions (a) and (b) of Lemma 3.2. Further, assume that η is affine in* *the both arguments and sequentially continuous in the first argument from the weak topology in X to the weak* topology in X^*. Then for each $x \in X$, there exists a point $z \in K$ such that*

$$\rho F(z,y) + \langle z - x, \eta(y,z)\rangle \geq 0, \quad \forall y \in K. \tag{3.1}$$

Proof. For each fixed $x \in X$, define $\varphi : K \times K \to \mathbb{R}$ by

$$\varphi(z,y) = \rho F(z,y) + \langle z - x, \eta(y,z)\rangle, \quad \forall y, z \in K. \tag{3.2}$$

From the proof of [12, Lemma 2.2], we deduce that $\varphi(z,z) \geq 0$ for all $z \in K$, the mapping φ is monotone and for each $y \in K$, the mapping $\varphi(.,y)$ is upper hemicontinuous. Now, we show that for each $z \in K$, the mapping $\varphi(z,.)$ is convex and lower semicontinuous. Since η is affine in the first argument and for each $x \in K$, the mapping $F(x,.)$ is convex, it follows that for each $u, y, z \in K$,

$$\begin{aligned}
&\varphi(z, tu + (1-t)y) \\
&= \rho F(z, tu + (1-t)y) + \langle z - x, \eta(tu + (1-t)y, z)\rangle \\
&\leq \rho(tF(z,u) + (1-t)F(z,y)) + \langle z - x, t\eta(u,z) + (1-t)\eta(y,z)\rangle \\
&= t\rho F(z,u) + (1-t)\rho F(z,y) + t\langle z - x, \eta(u,z)\rangle \\
&\quad + (1-t)\langle z - x, \eta(y,z)\rangle \\
&= t(\rho F(z,u) + \langle z - x, \eta(u,z)\rangle) + (1-t)(\rho F(z,y) + \langle z - x, \eta(y,z)\rangle) \\
&= t\varphi(z,u) + (1-t)\varphi(z,y).
\end{aligned} \tag{3.3}$$

Therefore, for each $z \in K$, the mapping $\varphi(z,.)$ is convex. Consider an arbitrary point $y_0 \in K$ and an arbitrary sequence $\{y_n\}$ in K such that $y_n \to y_0$ as $n \to \infty$. Since for each $x \in K$, $F(x,.)$ is lower semicontinuous and η is sequentially continuous in the first argument from the weak topology in X to the weak* topology in X^*, for each $z \in K$, we have

$$\begin{aligned}
\lim_{n\to\infty} \varphi(z,y_n) &= \lim_{n\to\infty} (\rho F(z,y_n) + \langle z - x, \eta(y_n,z)\rangle \\
&= \rho \lim_{n\to\infty} F(z,y_n) + \lim_{n\to\infty} \langle z - x, \eta(y_n,z)\rangle \\
&\geq \rho F(z,y_0) + \langle z - x, \eta(y_0,z)\rangle \\
&= \varphi(z,y_0).
\end{aligned} \tag{3.4}$$

Accordingly, for each $z \in K$, $\varphi(z,.)$ is lower semicontinuous. Since K is a nonempty closed convex bounded subset of reflexive Banach space X, K is compact in the weak topology, and thus, the condition (d) of Lemma 3.1 is satisfied. By Lemma 3.1, there exists a point $z \in K$ such that

$$\varphi(z,y) \geq 0, \quad \forall y \in K.$$

This completes the proof. □

If we write $z = J_\rho^F(x)$, then it follows that for each $x \in X$, there exists a point $J_\rho^F(x) \in K$ such that

$$\rho F(J_\rho^F(x), y) + \langle J_\rho^F(x) - x, \eta(y, J_\rho^F(x))\rangle \geq 0, \quad \forall y \in K. \tag{3.5}$$

Remark 3.5. If F is strictly accretive, then the solution of EP (3.1) is unique. Thus, for each $x \in X$, there exists a unique point $J_\rho^F(x) \in K$ such that the inequality (3.5) holds.

Definition 3.6 ([12])**.** Let ρ be a positive number. For a given bifunction F, the associated Yosida approximation, F_ρ, over K and the corresponding regularized operator, A_ρ^F, are defined as follows:

$$F_\rho(x,y) = \Big\langle \frac{1}{\rho}(x - J_\rho^F(x)), \eta(y,x) \Big\rangle \quad \text{and} \quad A_\rho^F(x) = \frac{1}{\rho}\Big(x - J_\rho^F(x)\Big),$$

in which $J_\rho^F(x) \in K$ is the unique solution of (3.1), that is,

$$\rho F(J_\rho^F(x), y) + \Big\langle J_\rho^F(x) - x, \eta(y, J_\rho^F(x)) \Big\rangle \geq 0, \quad \forall y \in K.$$

Remark 3.7. If $K = X = H$ is a Hilbert space and

$$F(x,y) = \sup_{\xi \in M(x)} \langle \xi, \eta(y,x) \rangle, \quad \forall x, y \in K,$$

where M is a maximal η-monotone operator (see [18]), then it directly yields $J_\rho^F(x) = (I + \rho M)^{-1}(x)$ and $A_\rho^F(x) = M_\rho(x)$, where $M_\rho := \frac{1}{\rho}\left(I - (I + \rho M)^{-1}\right)$ is the Yosida approximation of M.

By using Lemma 3.2 and the Yosida approximation, Ding [12] derived Lipschitz continuity of the mapping J_ρ^F. In view of Remark 3.3 and Lemma 3.4, we state [12, Theorem 2.1] as follows.

Theorem 3.8. *Let K be a nonempty closed convex bounded subset of a reflexive Banach space X and ρ be a positive number. Let $F : K \times K \to (-\infty, +\infty)$ and $\eta : X \times X \to X^*$ satisfy the following conditions.*

(a) *F is α-strongly monotone and satisfies conditions (a)–(c) of Lemma 3.1;*
(b) *η is δ-strongly monotone and τ-Lipschitz continuous with $\eta(x,y) + \eta(y,x) = 0$, $\forall x, y \in X$;*
(c) *η is affine in both the arguments and sequentially continuous in first argument from weak topology in X to weak* topology in X^*.*

Then the mapping J_ρ^F is $\frac{\tau}{\delta + \rho\alpha}$-Lipschitz continuous.

Associated with the system of nonlinear generalized mixed implicit equilibrium problems (2.1), we consider the following *system of generalized Wiener-Hopf equations* (in short, SGWHE):

(3.6)
Find $(z_1, z_2, \ldots, z_p) \in \prod_{i=1}^p X_i$ and
$(\bar{x}_1, \ldots, \bar{x}_p, \bar{u}_{1,1}, \ldots, \bar{u}_{1,l}, \ldots, \bar{u}_{p,1}, \ldots, \bar{u}_{p,l}) \in \prod_{i=1}^p K_i \times \prod_{i=1}^p \prod_{j=1}^l T_{i,j}(\bar{x}_i)$
such that
$N_i(\bar{u}_{i,1}, \bar{u}_{i,2}, \ldots, \bar{u}_{i,l}) + A_{\rho_i}^{F_i}(z_i) = 0, \; g_i(\bar{x}_i) = J_{\rho_i}^{F_i}(z_i), \; i = 1, 2, \ldots, p.$

For a suitable and appropriate choice of the mappings $T_{i,j}$, F_i, N_i, g_i, η_i, the spaces X_i, the subsets K_i of X_i, and the constants ρ_i $(i = 1, 2, \ldots, p)$ $(j = 1, 2, \ldots l)$, we obtain the generalized Wiener-Hopf equations (1.23) in [10], the generalized Wiener-Hopf equation (10) in [22] and the generalized Wiener-Hopf equation problem (3.3) in [20] as special cases of the generalized Wiener-Hopf equations (3.6).

Directly from the definition of $A_{\rho_i}^{F_i}$, we obtain the following result.

Lemma 3.9. *$(z_1,\dots,z_p,\bar{x}_1,\dots,\bar{x}_p,\bar{u}_{1,1},\dots,\bar{u}_{1,l},\dots,\bar{u}_{p,1},\dots,\bar{u}_{p,l}) \in \prod_{i=1}^p X_i \times \prod_{i=1}^p K_i \times \prod_{i=1}^p \prod_{j=1}^l T_{i,j}(\bar{x}_i)$ is a solution of* SGWHE (3.6) *if and only if*

$$g_i(\bar{x}_i) = J_{\rho_i}^{F_i}(z_i),\ z_i = g_i(\bar{x}_i) - \rho_i N_i(\bar{u}_{i,1},\bar{u}_{i,2},\dots,\bar{u}_{i,l}),\ i=1,2,\dots,p. \tag{3.7}$$

The following equivalence between SGMIEP (2.1) and SGWHE (3.6) is derived by using the definition of $J_{\rho_i}^{F_i}$.

Lemma 3.10. *$(\bar{x}_1,\dots,\bar{x}_p,\bar{u}_{1,1},\dots,\bar{u}_{1,l},\dots,\bar{u}_{p,1},\dots,\bar{u}_{p,l}) \in \prod_{i=1}^p K_i \times \prod_{i=1}^p \prod_{j=1}^l T_{i,j}(\bar{x}_i)$ is a solution of* SNGMIEP (2.1) *if and only if*

$$\begin{aligned}&(z_1,\dots,z_p,\bar{x}_1,\dots,\bar{x}_p,\bar{u}_{1,1},\dots,\bar{u}_{1,l},\dots,\bar{u}_{p,1},\dots,\bar{u}_{p,l})\\ &\in \prod_{i=1}^p X_i \times \prod_{i=1}^p K_i \times \prod_{i=1}^p \prod_{j=1}^l T_{i,j}(\bar{x}_i)\end{aligned}$$

is a solution of SGWHE (3.6) *satisfying* (3.7).

From Lemmas 3.9 and 3.10, we obtain the following result.

Lemma 3.11. *$(\bar{x}_1,\dots,\bar{x}_p,\bar{u}_{1,1},\dots,\bar{u}_{1,l},\dots,\bar{u}_{p,1},\dots,\bar{u}_{p,l}) \in \prod_{i=1}^p K_i \times \prod_{i=1}^p \prod_{j=1}^l T_{i,j}(\bar{x}_i)$ is a solution of* SNGMIEP (2.1) *if and only if*

$$g_i(\bar{x}_i) = J_{\rho_i}^{F_i}(g_i(\bar{x}_i) - \rho_i N_i(\bar{u}_{i,1},\bar{u}_{i,2},\dots,\bar{u}_{i,l})),\ i=1,2,\dots,p. \tag{3.8}$$

Fixed point formulation (3.7) allows us to construct the following iterative algorithm with mixed errors for computing a solution of SGWHE (3.6).

Algorithm 3.12. Suppose that X_j, K_i, $T_{i,j}$, N_i, F_i, g_i and η_i $(i=1,2,\dots,p)$ $(j=1,2,\dots,l)$, are the same as in system (2.1) and let for each $i \in \{1,2,\dots,p\}$, the mapping g_i is onto. For arbitrary chosen initial point $(z_{1,0},z_{2,0},\dots,z_{p,0}) \in \prod_{i=1}^p X_i$, we take $(x_{1,0},x_{2,0},\dots,x_{p,0}) \in \prod_{i=1}^p K_i$ such that $g_i(x_{i,0}) = J_{\rho_i}^{F_i}(z_{i,0})$. For each $i \in \{1,2,\dots,p\}$ and $j \in \{1,2,\dots,l\}$, we take $u_{i,j,0} \in T_{i,j}(x_{i,0})$ and set

$$z_{i,1} = (1-\lambda_i)z_{i,0} + \lambda_i[g_i(x_{i,0}) - \rho_i N_i(u_{i,1,0},u_{i,2,0},\dots,u_{i,l,0})] + \lambda_i e_{i,0} + r_{i,0},$$

where for each $i \in \{1,2,\dots,p\}$, $\rho_i > 0$ is a constant and $\lambda_i \in (0,1)$ is a relaxation parameter.

For $(z_{1,1},z_{2,1},\dots,z_{p,1})$, we take $(x_{1,1},x_{2,1},\dots,x_{p,1}) \in \prod_{i=1}^p K_i$ such that we have $g_i(x_{i,1}) = J_{\rho_i}^{F_i}(z_{i,1})$. By Lemma 2.2, for each $i \in \{1,2,\dots,p\}$ and $j \in \{1,2,\dots,l\}$, there exists $u_{i,j,1} \in T_{i,j}(x_{i,1})$ such that

$$\|u_{i,j,1} - u_{i,j,0}\|_i \le (1+(1+0)^{-1})\mathcal{H}_i(T_{i,j}(x_{i,1}),T_{i,j}(x_{i,0})).$$

For each $i \in \{1,2,\dots,p\}$, let

$$\begin{aligned} z_{i,2} \;=\; & (1-\lambda_i)z_{i,1} + \lambda_i\big[g_i(x_{i,1}) \\ & -\rho_i N_i\left(u_{i,1,1},u_{i,2,1},\dots,u_{i,l,1}\right)\big] + \lambda_i e_{i,1} + r_{i,1}.\end{aligned}$$

By induction, we can define the iterative sequences $\{z_{i,n}\}$, $\{x_{i,n}\}$, $\{u_{i,j,n}\}$ $(i = 1, 2, \ldots, p)$ $(j = 1, 2, \ldots, l)$ satisfying

$$\text{(3.9)} \quad \begin{cases} g_i(x_{i,n}) &= J^{F_i}_{\rho_i}(z_{i,n}), \\ z_{i,n+1} &= (1-\lambda_i)z_{i,n} + \lambda_i\big[g_i(x_{i,n}) \\ & \quad -\rho_i N_i(u_{i,1,n}, u_{i,2,n}, \ldots, u_{i,l,n})\big] + \lambda_i e_{i,n} + r_{i,n}, \\ u_{i,j,n} &\in T_{i,j}(x_{i,n}) \ : \ \|u_{i,j,n+1} - u_{i,j,n}\|_i \\ & \qquad \leq (1 + (1+n)^{-1})\mathcal{H}_i(T_{i,j}(x_{i,n+1}), T_{i,j}(x_{i,n})), \end{cases}$$

where $n = 0, 1, 2, \ldots$, and for each $i \in \{1, 2, \ldots, p\}$, $\rho_i > 0$ is a constant, $\lambda_i \in (0, 1)$ is a relaxation parameter, and the sequences $\{e_{i,n}\}$ and $\{r_{i,n}\}$ are errors to take into account a possible inexact computation of the resolvent mapping point satisfying the following conditions:

$$\text{(3.10)} \quad \begin{cases} \lim\limits_{n\to\infty} \|e_{i,n}\|_i = \lim\limits_{n\to\infty} \|r_{i,n}\|_i = 0; \\ \sum\limits_{n=0}^{\infty} \|e_{i,n} - e_{i,n-1}\|_i < \infty; \\ \sum\limits_{n=0}^{\infty} \|r_{i,n} - r_{i,n-1}\|_i < \infty. \end{cases}$$

Fixed point formulation (3.8) enables us to suggest the following iterative algorithm with mixed errors for finding a solution of SNGMIEP (2.1).

Algorithm 3.13. Let X_j, K_i, $T_{i,j}$, N_i, F_i, g_i and η_i $(i = 1, 2, \ldots, p)$ $(j = 1, 2, \ldots, l)$, be the same as in system (2.1). For arbitrary chosen initial point $(x_{1,0}, x_{2,0}, \ldots, x_{p,0}) \in \prod_{i=1}^{p} K_i$, for each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, l\}$, we take $u_{i,j,0} \in T_{i,j}(x_{i,0})$ and set

$$\begin{aligned} x_{i,1} &= (1-\lambda_i)x_{i,0} + \lambda_i\big[x_{i,0} - g_i(x_{i,0}) \\ &\quad + J^{F_i}_{\rho_i}(g_i(x_{i,0}) - \rho_i N_i(u_{i,1,0}, u_{i,2,0}, \ldots, u_{i,l,0}))\big] \\ &\quad + \lambda_i e_{i,0} + r_{i,0}, \end{aligned}$$

where for each $i \in \{1, 2, \ldots, p\}$, $\rho_i > 0$ is a constant and $\lambda_i \in (0, 1)$ is a relaxation parameter. In view of Lemma 2.2, for each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, l\}$, there exists $u_{i,j,1} \in T_{i,j}(x_{i,1})$ such that for each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, l\}$,

$$\|u_{i,j,1} - u_{i,j,0}\|_i \leq (1 + (1+0)^{-1})\mathcal{H}_i(T_{i,j}(x_{i,1}), T_{i,j}(x_{i,0})).$$

For each $i \in \{1, 2, \ldots, p\}$, let

$$\begin{aligned} x_{i,2} &= (1-\lambda_i)x_{i,1} + \lambda_i\big[x_{i,1} - g_i(x_{i,1}) \\ &\quad + J^{F_i}_{\rho_i}(g_i(x_{i,1}) - \rho_i N_i(u_{i,1,1}, u_{i,2,1}, \ldots, u_{i,l,1}))\big] \\ &\quad + \lambda_i e_{i,1} + r_{i,1}. \end{aligned}$$

By induction, we can define the iterative sequences $\{x_{i,n}\}$, $\{u_{i,j,n}\}$ $(i = 1, 2, \ldots, p)$ $(j = 1, 2, \ldots, l)$ satisfying

$$\text{(3.11)} \quad \begin{cases} x_{i,n+1} &= (1-\lambda_i)x_{i,n} + \lambda_i\big[x_{i,n} - g_i(x_{i,n}) \\ & \quad + J^{F_i}_{\rho_i}(g_i(x_{i,n}) - \rho_i N_i(u_{i,1,n}, u_{i,2,n}, \ldots, u_{i,l,n}))\big] \\ & \quad + \lambda_i e_{i,n} + r_{i,n}, \\ u_{i,j,n} &\in T_{i,j}(x_{i,n}) \ : \ \|u_{i,j,n+1} - u_{i,j,n}\|_i \\ & \quad \leq (1 + (1+n)^{-1})\mathcal{H}_i(T_{i,j}(x_{i,n+1}), T_{i,j}(x_{i,n})), \end{cases}$$

where $n = 0, 1, 2, \ldots$, and for each $i \in \{1, 2, \ldots, p\}$, $\rho_i > 0$ is a constant, $\lambda_i \in (0, 1)$ is a relaxation parameter and the sequences $\{e_{i,n}\}$ and $\{r_{i,n}\}$ are the same as Algorithm 3.12 and satisfy (3.10).

Remark 3.14. If $e_{i,n} = r_{i,n} = 0$ for each $n \in \mathbb{N} \cup \{0\}$ and $i \in \{1, 2, \ldots, p\}$, then Algorithms 3.12 and 3.13 reduce to the iterative methods without error. We note that the Algorithm 3.1 in [20] is a special case of Algorithm 3.12.

4. Existence of solution and convergence analysis

In this section, we establish the existence theorems for solutions of SNGMIEP (2.1) and SGWHE (3.6) and discuss the convergence analysis of the sequences generated by iterative Algorithms 3.12 and 3.13 under some suitable conditions.

Theorem 4.1. *Let X_j, K_i, $T_{i,j}$, N_i, F_i, g_i and η_i $(i = 1, 2, \ldots, p)$, $(j = 1, 2, \ldots, l)$, be the same as in system* (2.1) *such that for each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, l\}$,*

(a) *F_i is α_i-strongly monotone and satisfy conditions (a)–(c) of Lemma 3.1;*
(b) *η_i is δ_i-strongly monotone and τ_i-Lipschitz continuous with $\eta_i(x, y)+\eta_i(y, x) = 0$, $\forall x, y \in X_i$;*
(c) *η_i is affine in the both arguments and sequentially continuous from the weak topology in X_i to the weak* topology in X_i^*;*
(d) *N_i is $\nu_{i,j}$-Lipschitz continuous in the jth argument;*
(e) *g_i is κ_i-strongly accretive and σ_i-Lipschitz continuous;*
(f) *$T_{i,j}$ is $\mu_{i,j}$-$\mathcal{H}_i$-Lipschitz continuous;*
(g) *there exists the constant $\rho_i > 0$ such that*

$$\begin{cases} \rho_i < \dfrac{\theta_i(\delta_i-\tau_i)}{\tau_i \sum\limits_{j=1}^{l} \nu_{i,j}\mu_{i,j}-\alpha_i\theta_i}, \\ \theta_i = 1 - \sqrt[q_i]{1 - q_i\kappa_i + c_{q_i}\sigma_i^{q_i}} > 0, \\ \tau_i \sum\limits_{j=1}^{l} \nu_{i,j}\mu_{i,j} > \alpha_i\theta_i, \end{cases} \tag{4.1}$$

where c_{q_i} is the same as in Lemma 2.3.

Then, the sequences $\{z_{i,n}\}$, $\{x_{i,n}\}$, $\{u_{i,j,n}\}$ $(i = 1, 2, \ldots, p)$ $(j = 1, 2, \ldots, l)$, generated by Algorithm 3.12 converge strongly to $z_i \in X_i$, $\bar{x}_i \in K_i$, $\bar{u}_{i,j} \in T_{i,j}(\bar{x}_i)$, respectively, and $(z_1, \ldots, z_p, \bar{x}_1, \ldots, \bar{x}_p, \bar{u}_{1,1}, \ldots, \bar{u}_{1,l}, \ldots, \bar{u}_{p,1}, \ldots, \bar{u}_{p,l}) \in \prod_{i=1}^{p} X_i \times \prod_{i=1}^{p} K_i \times \prod_{i=1}^{p}\prod_{j=1}^{l} T_{i,j}(\bar{x}_i)$ is a solution of SGWHE (3.6).

Proof. For each $i \in \{1, 2, \ldots, p\}$, it follows from (3.9) that

$$\begin{aligned} &\|z_{i,n+1} - z_{i,n}\|_i \\ &\le (1 - \lambda_i)\|z_{i,n} - z_{i,n-1}\|_i + \lambda_i\|g_i(x_{i,n}) - g_i(x_{i,n-1})\|_i \\ &\quad +\lambda_i\rho_i\|N_i(u_{i,1,n}, u_{i,2,n}, \ldots, u_{i,l,n}) - N_i(u_{i,1,n-1}, u_{i,2,n-1}, \ldots, u_{i,l,n-1})\|_i \\ &\quad +\lambda_i\|e_{i,n} - e_{i,n-1}\|_i + \|r_{i,n} - r_{i,n-1}\|_i \\ &\le (1 - \lambda_i)\|z_{i,n} - z_{i,n-1}\|_i + \lambda_i\|g_i(x_{i,n}) - g_i(x_{i,n-1})\|_i \\ &\quad +\lambda_i\rho_i \sum_{j=1}^{l} \|N_i(u_{i,1,n-1}, u_{i,2,n-1}, \ldots, u_{i,j-1,n-1}, u_{i,j,n}, u_{i,j+1,n}, \ldots, u_{i,l,n}) \\ &\quad -N_i(u_{i,1,n-1}, u_{i,2,n-1}, \ldots, u_{i,j-1,n-1}, u_{i,j,n-1}, u_{i,j+1,n}, \ldots, u_{i,l,n})\|_i \\ &\quad +\lambda_i\|e_{i,n} - e_{i,n-1}\|_i + \|r_{i,n} - r_{i,n-1}\|_i. \end{aligned} \tag{4.2}$$

By using (3.9) and Theorem 3.8, for each $i \in \{1, 2, \ldots, p\}$, we have

$$\begin{aligned} \|g_i(x_{i,n}) - g_i(x_{i,n-1})\|_i &= \left\| J^{F_i}_{\rho_i}(z_{i,n}) - J^{F_i}_{\rho_i}(z_{i,n-1}) \right\|_i \\ &\le \frac{\tau_i}{\delta_i + \rho_i\alpha_i} \|z_{i,n} - z_{i,n-1}\|_i. \end{aligned} \tag{4.3}$$

Since N_i is $\nu_{i,j}$-Lipschitz continuous in the jth argument and $T_{i,j}$ is $\mu_{i,j}$-$\mathcal{H}_i$-Lipschitz continuous, for each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, l\}$, we conclude that

$$\begin{aligned} &\|N_i(u_{i,1,n-1}, u_{i,2,n-1}, \ldots, u_{i,j-1,n-1}, u_{i,j,n}, u_{i,j+1,n}, \ldots, u_{i,l,n}) \\ &- N_i(u_{i,1,n-1}, u_{i,2,n-1}, \ldots, u_{i,j-1,n-1}, u_{i,j,n-1}, u_{i,j+1,n}, \ldots, u_{i,l,n})\|_i \\ &\le \nu_{i,j}\|u_{i,j,n} - u_{i,j,n-1}\|_i \le (1 + (1+n)^{-1})\nu_{i,j}\mathcal{H}_i(T_{i,j}(x_{i,n}), T_{i,j}(x_{i,n-1})) \\ &\le (1 + n^{-1})\nu_{i,j}\mu_{i,j}\|x_{i,n} - x_{i,n-1}\|_i. \end{aligned} \tag{4.4}$$

For each $i \in \{1, 2, \ldots, p\}$, we make an estimation for $\|x_{i,n} - x_{i,n-1}\|_i$.

Applying (3.9) and (4.3), for each $i \in \{1, 2, \ldots, p\}$, we get

$$\begin{aligned} \|x_{i,n} - x_{i,n-1}\|_i &\le \|x_{i,n} - x_{i,n-1} - (g_i(x_{i,n}) - g_i(x_{i,n-1}))\|_i + \|g_i(x_{i,n}) - g_i(x_{i,n-1})\|_i \\ &\le \|x_{i,n} - x_{i,n-1} - (g_i(x_{i,n}) - g_i(x_{i,n-1}))\|_i + \frac{\tau_i}{\delta_i + \rho_i\alpha_i}\|z_{i,n} - z_{i,n-1}\|_i. \end{aligned} \tag{4.5}$$

Lemma 2.3 guarantees that for each $i \in \{1, 2, \ldots, p\}$, there exists $c_{q_i} > 0$ such that

$$\begin{aligned} &\|x_{i,n} - x_{i,n-1} - (g_i(x_{i,n}) - g_i(x_{i,n-1}))\|_i^{q_i} \\ &\le \|x_{i,n} - x_{i,n-1}\|_i^{q_i} - q_i\langle g_i(x_{i,n}) - g_i(x_{i,n-1}), J_{q_i}(x_{i,n} - x_{i,n-1})\rangle \\ &\quad + c_{q_i}\|g_i(x_{i,n}) - g_i(x_{i,n-1})\|_i^{q_i}. \end{aligned} \tag{4.6}$$

Since g_i is κ_i-strongly accretive and σ_i-Lipschitz continuous, by using (4.6), for $i \in \{1, 2, \ldots, p\}$, we have

$$\begin{aligned} &\|x_{i,n} - x_{i,n-1} - (g_i(x_{i,n}) - g_i(x_{i,n-1}))\|_i^{q_i} \\ &\le (1 - q_i\kappa_i)\|x_{i,n} - x_{i,n-1}\|_i^{q_i} + c_{q_i}\|g_i(x_{i,n}) - g_i(x_{i,n-1})\|_i^{q_i} \\ &\le (1 - q_i\kappa_i + c_{q_i}\sigma_i^{q_i})\|x_{i,n} - x_{i,n-1}\|_i^{q_i}. \end{aligned} \tag{4.7}$$

From (4.5) and (4.7), it follows that for each $i \in \{1, 2, \ldots, p\}$,

$$\begin{aligned} \|x_{i,n} - x_{i,n-1}\|_i \le\ & \sqrt[q_i]{1 - q_i\kappa_i + c_{q_i}\sigma_i^{q_i}}\|x_{i,n} - x_{i,n-1}\|_i \\ & + \frac{\tau_i}{\delta_i + \rho_i\alpha_i}\|z_{i,n} - z_{i,n-1}\|_i, \end{aligned}$$

which leads to

$$\|x_{i,n} - x_{i,n-1}\|_i \le \frac{\tau_i}{(1 - \sqrt[q_i]{1 - q_i\kappa_i + c_{q_i}\sigma_i^{q_i}})(\delta_i + \rho_i\alpha_i)}\|z_{i,n} - z_{i,n-1}\|_i. \tag{4.8}$$

Substituting (4.3), (4.4) and (4.8) in (4.2), for $i \in \{1, 2, \ldots, p\}$, we obtain

$$\begin{aligned} \|z_{i,n+1} - z_{i,n}\|_i \le\ & (1 - \lambda_i)\|z_{i,n} - z_{i,n-1}\|_i + \frac{\lambda_i\tau_i}{\delta_i + \rho_i\alpha_i}\|z_{i,n} - z_{i,n-1}\|_i \\ & + \frac{\lambda_i\rho_i\tau_i(1 + n^{-1})}{(1 - \sqrt[q_i]{1 - q_i\kappa_i + c_{q_i}\sigma_i^{q_i}})(\delta_i + \rho_i\alpha_i)}\sum_{j=1}^{l}\nu_{i,j}\mu_{i,j}\|z_{i,n} - z_{i,n-1}\|_i \\ & + \lambda_i\|e_{i,n} - e_{i,n-1}\|_i + \|r_{i,n} - r_{i,n-1}\|_i \end{aligned} \tag{4.9}$$

$$= (1-\lambda_i+\lambda_i\omega_i(n))\|z_{i,n}-z_{i,n-1}\|_i+\lambda_i\|e_{i,n}-e_{i,n-1}\|_i$$
$$+\|r_{i,n}-r_{i,n-1}\|_i,$$

where for each $i \in \{1,2,\dots,p\}$,

$$\omega_i(n) = \frac{\tau_i}{\delta_i+\rho_i\alpha_i}\left(1+\frac{\rho_i(1+n^{-1})\sum\limits_{j=1}^{l}\nu_{i,j}\mu_{i,j}}{1-\sqrt[q_i]{1-q_i\kappa_i+c_{q_i}\sigma_i^{q_i}}}\right).$$

Letting $\vartheta_i(n) = 1-\lambda_i+\lambda_i\omega_i(n)$, for each $i \in \{1,2,\dots,p\}$, we know that $\vartheta_i(n) \to \vartheta_i = 1-\lambda_i+\lambda_i\omega_i$, as $n\to\infty$, where for each $i \in \{1,2,\dots,p\}$,

$$\omega_i = \frac{\tau_i}{\delta_i+\rho_i\alpha_i}\left(1+\frac{\rho_i\sum\limits_{j=1}^{l}\nu_{i,j}\mu_{i,j}}{1-\sqrt[q_i]{1-q_i\kappa_i+c_{q_i}\sigma_i^{q_i}}}\right).$$

In view of condition (4.1), $\omega_i \in (0,1)$ for each $i \in \{1,2,\dots,p\}$, and so $\vartheta_i \in (0,1)$ for each $i \in \{1,2,\dots,p\}$. Hence, there exists $n_0 \in \mathbb{N}$ such that for each $i \in \{1,2,\dots,p\}$, $\vartheta_i(n) \le \hat{\vartheta}_i$ for all $n \ge n_0$. Accordingly, for each $i \in \{1,2,\dots,p\}$ and for all $n > n_0$, by (4.9), we have

$$\begin{aligned}
&\|z_{i,n+1}-z_{i,n}\|_i\\
&\le \hat{\vartheta}_i\|z_{i,n}-z_{i,n-1}\|_i+\lambda_i\|e_{i,n}-e_{i,n-1}\|_i+\|r_{i,n}-r_{i,n-1}\|_i\\
&\le \hat{\vartheta}_i[\hat{\vartheta}_i\|z_{i,n-1}-z_{i,n-2}\|_i+\lambda_i\|e_{i,n-1}-e_{i,n-2}\|_i+\|r_{i,n-1}-r_{i,n-2}\|_i]\\
&\quad+\lambda_i\|e_{i,n}-e_{i,n-1}\|_i+\|r_{i,n}-r_{i,n-1}\|_i\\
&= \hat{\vartheta}_i^2\|z_{i,n-1}-z_{i,n-2}\|_i+\lambda_i[\hat{\vartheta}_i\|e_{i,n-1}-e_{i,n-2}\|_i\\
&\quad+\|e_{i,n}-e_{i,n-1}\|_i]+\hat{\vartheta}_i\|r_{i,n-1}-r_{i,n-2}\|_i+\|r_{i,n}-r_{i,n-1}\|_i\\
&\le\\
&\vdots\\
&\le \hat{\vartheta}_i^{n-n_0}\|z_{i,n_0+1}-z_{i,n_0}\|_i+\sum_{s=1}^{n-n_0}\lambda_i\hat{\vartheta}_i^{s-1}\|e_{i,n-(s-1)}-e_{i,n-s}\|_i\\
&\quad+\sum_{s=1}^{n-n_0}\hat{\vartheta}_i^{s-1}\|r_{i,n-(s-1)}-r_{i,n-s}\|_i.
\end{aligned} \tag{4.10}$$

By using inequality (4.10), it follows that for each $i \in \{1, 2, \ldots, p\}$, and for any $m \geq n > n_0$,

$$
\begin{aligned}
\|z_{i,m} - z_{i,n}\|_i &\leq \sum_{t=n}^{m-1} \|z_{i,t+1} - z_{i,t}\|_i \leq \sum_{t=n}^{m-1} \hat{\vartheta}_i^{t-n_0} \|z_{i,n_0+1} - z_{i,n_0}\|_i \\
&\quad + \sum_{t=n}^{m-1} \sum_{s=1}^{t-n_0} \lambda_i \hat{\vartheta}_i^{s-1} \|e_{i,n-(s-1)} - e_{i,n-s}\|_i \\
&\quad + \sum_{t=n}^{m-1} \sum_{s=1}^{t-n_0} \hat{\vartheta}_i^{s-1} \|r_{i,n-(s-1)} - r_{i,n-s}\|_i.
\end{aligned} \tag{4.11}
$$

Since $\hat{\vartheta}_i < 1$ for each $i \in \{1, 2, \ldots, p\}$, it follows from (3.10) and (4.11) that for each $i \in \{1, 2, \ldots, p\}$, $\|z_{i,m} - z_{i,n}\|_i \to 0$, as $n \to \infty$ and so for each $i \in \{1, 2, \ldots, p\}$, $\{z_{i,n}\}$ is a Cauchy sequence in X_i. In view of completeness of X_i for each $i \in \{1, 2, \ldots, p\}$, there exists $z_i \in X_i$ such that $z_{i,n} \to z_i$, as $n \to \infty$. By using (3.9) and $\mu_{i,j}$-$\mathcal{H}_i$-Lipschitz continuity of $T_{i,j}$ for each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, l\}$, we have

$$
\begin{aligned}
\|u_{i,j,n+1} - u_{i,j,n}\|_i &\leq (1 + (1+n)^{-1}) \mathcal{H}_i(T_{i,j}(x_{i,n+1}), T_{i,j}(x_{i,n})) \\
&\leq (1 + (1+n)^{-1}) \mu_{i,j} \|x_{i,n+1} - x_{i,n}\|_i \\
&\leq \frac{(1 + (1+n)^{-1}) \mu_{i,j} \tau_i}{(1 - \sqrt[q_i]{1 - q_i \kappa_i + c_{q_i} \sigma_i^{q_i}})(\delta_i + \rho_i \alpha_i)} \|z_{i,n+1} - z_{i,n}\|_i.
\end{aligned} \tag{4.12}
$$

Similarly, since for each $i \in \{1, 2, \ldots, p\}$, $\{z_{i,n}\}$ is a Cauchy sequence in X_i, (4.8) and (4.12) imply that $\{x_{i,n}\}$ and $\{u_{i,j,n}\}$ are also Cauchy sequences in $K_i \subseteq X_i$ and X_i, respectively, for each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, l\}$. Since K_i is closed for each $i \in \{1, 2, \ldots, p\}$, it follows that for each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, l\}$, there exists $\bar{x}_i \in K_i$ and $\bar{u}_{i,j} \in T_{i,j}(\bar{x}_i)$ such that $x_{i,n} \to \bar{x}_i$ and $u_{i,j,n} \to \bar{u}_{i,j}$, as $n \to \infty$. Further, by using $\mu_{i,j}$-$\mathcal{H}_i$-Lipschitz continuity of $T_{i,j}$ for each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, l\}$, we have

$$
\begin{aligned}
d(\bar{u}_{i,j}, T_{i,j}(\bar{x}_i)) &= \inf\{\|\bar{u}_{i,j} - u\|_i : u \in T_{i,j}(\bar{x}_i)\} \\
&\leq \|\bar{u}_{i,j} - u_{i,j,n}\|_i + d(u_{i,j,n}, T_{i,j}(\bar{x}_i)) \\
&\leq \|\bar{u}_{i,j} - u_{i,j,n}\|_i + \mathcal{H}_i(T_{i,j}(x_{i,n}), T_{i,j}(\bar{x}_i)) \\
&\leq \|\bar{u}_{i,j} - u_{i,j,n}\|_i + \mu_{i,j} \|x_{i,n} - \bar{x}_i\|_i.
\end{aligned} \tag{4.13}
$$

The right-hand side of the above inequality tends to zero as $n \to \infty$. Since $T_{i,j}(\bar{x}_i) \in CB(X_i)$, for each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, l\}$, deduce that $\bar{u}_{i,j} \in T_{i,j}(\bar{x}_i)$. Since the mappings $J_{\rho_i}^{F_i}$, N_i and g_i $(i = 1, 2, \ldots, p)$ are continuous, it follows from (3.9) and (3.10) that for $i \in \{1, 2, \ldots, p\}$,

$$
g_i(\bar{x}_i) = J_{\rho_i}^{F_i}(z_i), \qquad z_i = g_i(\bar{x}_i) - \rho_i N_i(\bar{u}_{i,1}, \bar{u}_{i,2}, \ldots, \bar{u}_{i,l}).
$$

Lemma 3.9 guarantees that $(z_1, \ldots, z_p, \bar{x}_1, \ldots, \bar{x}_p, \bar{u}_{1,1}, \ldots, \bar{u}_{1,l}, \ldots, \bar{u}_{p,1}, \ldots, \bar{u}_{p,l}) \in \prod_{i=1}^p X_i \times \prod_{i=1}^p K_i \times \prod_{i=1}^p \prod_{j=1}^l T_{i,j}(\bar{x}_i)$ is a solution of SGWHE (3.6). This completes the proof. □

Theorem 4.2. *Let X_j, K_i, $T_{i,j}$, N_i, F_i, g_i and η_i $(i = 1, 2, \dots, p)$, $(j = 1, 2, \dots, l)$, be the same as in Theorem 4.1 and let for each $i \in \{1, 2, \dots, p\}$ and $j \in \{1, 2, \dots, l\}$, conditions (a)–(f) in Theorem 4.1 hold. If for each $i \in \{1, 2, \dots, p\}$, there exists the constant $\rho_i > 0$ such that*

$$\tag{4.14} \begin{cases} \rho_i < \frac{\theta_i\delta_i - \tau_i\sigma_i}{\tau_i \sum\limits_{j=1}^{l} \nu_{i,j}\mu_{i,j} - \alpha_i\theta_i}, \\ \theta_i = 1 - \sqrt[q_i]{1 - q_i\kappa_i + c_{q_i}\sigma_i^{q_i}} > 0, \\ \tau_i \sum\limits_{j=1}^{l} \nu_{i,j}\mu_{i,j} > \alpha_i\theta_i, \end{cases}$$

where c_{q_i} is the same as in Lemma 2.3, then the sequences $\{x_{i,n}\}$ and $\{u_{i,j,n}\}$ $(i = 1, 2, \dots, p)$ $(j = 1, 2, \dots, l)$, generated by Algorithm 3.13 converge strongly to $\bar{x}_i \in K_i$ and $\bar{u}_{i,j} \in T_{i,j}(\bar{x}_i)$, respectively, and

$$(\bar{x}_1, \dots, \bar{x}_p, \bar{u}_{1,1}, \dots, \bar{u}_{1,l}, \dots, \bar{u}_{p,1}, \dots, \bar{u}_{p,l}) \in \prod_{i=1}^{p} K_i \times \prod_{i=1}^{p}\prod_{j=1}^{l} T_{i,j}(\bar{x}_i)$$

is a solution of SNGMIEP (2.1).

Proof. For each $i \in \{1, 2, \dots, p\}$, by using (3.11) and Theorem 3.8, we get

$$\tag{4.15} \begin{aligned} &\|x_{i,n+1} - x_{i,n}\|_i \\ &\leq (1 - \lambda_i)\|x_{i,n} - x_{i,n-1}\|_i + \lambda_i\|x_{i,n} - x_{i,n-1} - (g_i(x_{i,n}) - g_i(x_{i,n-1}))\|_i \\ &\quad + \lambda_i\|J_{\rho_i}^{F_i}(g_i(x_{i,n}) - \rho_i N_i(u_{i,1,n}, u_{i,2,n}, \dots, u_{i,l,n})) \\ &\quad - J_{\rho_i}^{F_i}(g_i(x_{i,n-1}) - \rho_i N_i(u_{i,1,n-1}, u_{i,2,n-1}, \dots, u_{i,l,n-1}))\|_i \\ &\quad + \lambda_i\|e_{i,n} - e_{i,n-1}\|_i + \|r_{i,n} - r_{i,n-1}\|_i \\ &\leq (1 - \lambda_i)\|x_{i,n} - x_{i,n-1}\|_i + \lambda_i\|x_{i,n} - x_{i,n-1} - (g_i(x_{i,n}) - g_i(x_{i,n-1}))\|_i \\ &\quad + \frac{\lambda_i\tau_i}{\delta_i + \rho_i\alpha_i}\Big(\|g_i(x_{i,n}) - g_i(x_{i,n-1})\|_i + \rho_i\|N_i(u_{i,1,n}, u_{i,2,n}, \dots, u_{i,l,n}) \\ &\quad - N_i(u_{i,1,n-1}, u_{i,2,n-1}, \dots, u_{i,l,n-1})\|_i\Big) \\ &\quad + \lambda_i\|e_{i,n} - e_{i,n-1}\|_i + \|r_{i,n} - r_{i,n-1}\|_i \\ &\leq (1 - \lambda_i)\|x_{i,n} - x_{i,n-1}\|_i + \lambda_i\|x_{i,n} - x_{i,n-1} - (g_i(x_{i,n}) - g_i(x_{i,n-1}))\|_i \\ &\quad + \frac{\lambda_i\tau_i}{\delta_i + \rho_i\alpha_i}\Big(\|g_i(x_{i,n}) - g_i(x_{i,n-1})\|_i \\ &\quad + \rho_i\sum_{j=1}^{l}\|N_i(u_{i,1,n-1}, u_{i,2,n-1}, \dots, u_{i,j-1,n-1}, u_{i,j,n}, u_{i,j+1,n}, \dots, u_{i,l,n}) \\ &\quad - N_i(u_{i,1,n-1}, u_{i,2,n-1}, \dots, u_{i,j-1,n-1}, u_{i,j,n-1}, u_{i,j+1,n}, \dots, u_{i,l,n})\|_i\Big) \\ &\quad + \lambda_i\|e_{i,n} - e_{i,n-1}\|_i + \|r_{i,n} - r_{i,n-1}\|_i. \end{aligned}$$

Since for each $i \in \{1, 2, \dots, p\}$, g_i is κ_i-strongly accretive and σ_i-Lipschitz continuous, in a similar way to the proofs of (4.6) and (4.7), for $i \in \{1, 2, \dots, p\}$, we get

$$\|x_{i,n}-x_{i,n-1}-(g_i(x_{i,n})-g_i(x_{i,n-1}))\|_i \leq \sqrt[q_i]{1-q_i\kappa_i+c_{q_i}\sigma_i^{q_i}}\|x_{i,n}-x_{i,n-1}\|_i. \tag{4.16}$$

From σ_i-Lipschitz continuity of g_i for each $i \in \{1,2,\ldots,p\}$, it follows that

$$\|g_i(x_{i,n})-g_i(x_{i,n-1})\|_i \leq \sigma_i\|x_{i,n}-x_{i,n-1}\|_i. \tag{4.17}$$

Since for each $i \in \{1,2,\ldots,p\}$, N_i is $\nu_{i,j}$-Lipschitz continuous in the jth argument and $T_{i,j}$ is $\mu_{i,j}$-$\mathcal{H}_i$-Lipschitz continuous for each $j \in \{1,2,\ldots,l\}$, in a similar way to the proof of (4.4), for each $i \in \{1,2,\ldots,p\}$ and $j \in \{1,2,\ldots,l\}$, we obtain

$$\begin{aligned}
&\|N_i(u_{i,1,n-1},u_{i,2,n-1},\ldots,u_{i,j-1,n-1},u_{i,j,n},u_{i,j+1,n},\ldots,u_{i,l,n})\\
&-N_i(u_{i,1,n-1},u_{i,2,n-1},\ldots,u_{i,j-1,n-1},u_{i,j,n-1},u_{i,j+1,n},\ldots,u_{i,l,n})\|_i\\
&\leq (1+n^{-1})\nu_{i,j}\mu_{i,j}\|x_{i,n}-x_{i,n-1}\|_i.
\end{aligned} \tag{4.18}$$

Combining (4.15)–(4.18), we deduce that

$$\begin{aligned}
&\|x_{i,n+1}-x_{i,n}\|_i\\
&\leq (1-\lambda_i)\|x_{i,n}-x_{i,n-1}\|_i+\lambda_i\Big(\sqrt[q_i]{1-q_i\kappa_i+c_{q_i}\sigma_i^{q_i}}\\
&\quad+\frac{\tau_i}{\delta_i+\rho_i\alpha_i}(\sigma_i+\rho_i(1+n^{-1})\sum_{j=1}^{l}\nu_{i,j}\mu_{i,j}\Big)\|x_{i,n}-x_{i,n-1}\|_i\\
&\quad+\lambda_i\|e_{i,n}-e_{i,n-1}\|_i+\|r_{i,n}-r_{i,n-1}\|_i\\
&=(1-\lambda_i+\lambda_i\psi_i(n))\|x_{i,n}-x_{i,n-1}\|_i+\lambda_i\|e_{i,n}-e_{i,n-1}\|_i+\|r_{i,n}-r_{i,n-1}\|_i,
\end{aligned} \tag{4.19}$$

where for each $i \in \{1,2,\ldots,p\}$,

$$\psi_i(n)=\sqrt[q_i]{1-q_i\kappa_i+c_{q_i}\sigma_i^{q_i}}+\frac{\tau_i}{\delta_i+\rho_i\alpha_i}\left(\sigma_i+\rho_i(1+n^{-1})\sum_{j=1}^{l}\nu_{i,j}\mu_{i,j}\right). \tag{4.20}$$

Letting $\gamma_i(n)=1-\lambda_i+\lambda_i\psi_i(n)$, for each $i \in \{1,2,\ldots,p\}$, we know that $\gamma_i(n) \to \gamma_i = 1-\lambda_i+\lambda_i\psi_i$, as $n \to \infty$, where for each $i \in \{1,2,\ldots,p\}$,

$$\psi_i=\sqrt[q_i]{1-q_i\kappa_i+c_{q_i}\sigma_i^{q_i}}+\frac{\tau_i}{\delta_i+\rho_i\alpha_i}(\sigma_i+\rho_i\sum_{j=1}^{l}\nu_{i,j}\mu_{i,j}).$$

Condition (4.14) implies that $\psi_i \in (0,1)$ for each $i \in \{1,2,\ldots,p\}$, and so, $\gamma_i \in (0,1)$ for each $i \in \{1,2,\ldots,p\}$. Accordingly, there exists $n_0 \in \mathbb{N}$ such that for each $i \in \{1,2,\ldots,p\}$, $\gamma_i(n) \leq \hat{\gamma}_i$ for all $n \geq n_0$. Hence, for each $i \in \{1,2,\ldots,p\}$ and for all $n > n_0$, by (4.19), we conclude that

$$\begin{aligned}
\|x_{i,n+1}-x_{i,n}\|_i &\leq \hat{\gamma}_i\|x_{i,n}-x_{i,n-1}\|_i+\lambda_i\|e_{i,n}-e_{i,n-1}\|_i+\|r_{i,n}-r_{i,n-1}\|_i\\
&\leq \hat{\gamma}_i[\hat{\gamma}_i\|x_{i,n-1}-x_{i,n-2}\|_i+\lambda_i\|e_{i,n-1}-e_{i,n-2}\|_i+\|r_{i,n-1}-r_{i,n-2}\|_i]\\
&\quad+\lambda_i\|e_{i,n}-e_{i,n-1}\|_i+\|r_{i,n}-r_{i,n-1}\|_i\\
&=\hat{\gamma}_i^2\|x_{i,n-1}-x_{i,n-2}\|_i+\lambda_i[\hat{\gamma}_i\|e_{i,n-1}-e_{i,n-2}\|_i\\
&\quad+\|e_{i,n}-e_{i,n-1}\|_i]+\hat{\gamma}_i\|r_{i,n-1}-r_{i,n-2}\|_i+\|r_{i,n}-r_{i,n-1}\|_i\\
&\leq
\end{aligned} \tag{4.21}$$

$$\vdots$$

$$\leq \hat{\gamma}_i^{n-n_0}\|x_{i,n_0+1} - x_{i,n_0}\|_i + \sum_{s=1}^{n-n_0} \lambda_i \hat{\gamma}_i^{s-1}\|e_{i,n-(s-1)} - e_{i,n-s}\|_i + \sum_{s=1}^{n-n_0} \hat{\gamma}_i^{s-1}\|r_{i,n-(s-1)} - r_{i,n-s}\|_i.$$

From inequality (4.21), it follows that for each $i \in \{1, 2, \ldots, p\}$, and for any $m \geq n > n_0$,

$$\begin{aligned} \|x_{i,m} - x_{i,n}\|_i \leq \sum_{t=n}^{m-1} \|x_{i,t+1} - x_{i,t}\|_i &\leq \sum_{t=n}^{m-1} \hat{\gamma}_i^{t-n_0}\|x_{i,n_0+1} - x_{i,n_0}\|_i \\ &+ \sum_{t=n}^{m-1}\sum_{s=1}^{t-n_0} \lambda_i \hat{\gamma}_i^{s-1}\|e_{i,n-(s-1)} - e_{i,n-s}\|_i \\ &+ \sum_{t=n}^{m-1}\sum_{s=1}^{t-n_0} \hat{\gamma}_i^{s-1}\|r_{i,n-(s-1)} - r_{i,n-s}\|_i. \end{aligned} \tag{4.22}$$

Since $\hat{\gamma}_i < 1$ for each $i \in \{1, 2, \ldots, p\}$, from (3.10) and (4.22), it follows that for each $i \in \{1, 2, \ldots, p\}$, $\|x_{i,m} - x_{i,n}\|_i \to 0$, as $n \to \infty$ and so for each $i \in \{1, 2, \ldots, p\}$, $\{x_{i,n}\}$ is a Cauchy sequence in $K_i \subseteq X_i$. Since K_i is closed and X_i is complete for each $i \in \{1, 2, \ldots, p\}$, it follows that for each $i \in \{1, 2, \ldots, p\}$, there exists $\bar{x}_i \in K_i$ such that $x_{i,n} \to \bar{x}_i$, as $n \to \infty$. Applying (3.11) and $\mu_{i,j}$-$\mathcal{H}_i$-Lipschitz continuity of $T_{i,j}$ for each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, l\}$, we obtain

$$\begin{aligned} \|u_{i,j,n+1} - u_{i,j,n}\|_i &\leq (1 + (1+n)^{-1})\mathcal{H}_i(T_{i,j}(x_{i,n+1}), T_{i,j}(x_{i,n})) \\ &\leq (1 + (1+n)^{-1})\mu_{i,j}\|x_{i,n+1} - x_{i,n}\|_i. \end{aligned} \tag{4.23}$$

Similarly, since for each $i \in \{1, 2, \ldots, p\}$, $\{x_{i,n}\}$ is a Cauchy sequence in $K_i \subseteq X_i$, inequality (4.23) implies that for each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, l\}$, $\{u_{i,j,n}\}$ is also a Cauchy sequence in X_i and so for each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, l\}$, there exists $\bar{u}_{i,j} \in T_{i,j}(\bar{x}_i)$ such that $u_{i,j,n} \to \bar{u}_{i,j}$, as $n \to \infty$. In a similar way to the proof of (4.13), conclude that $\bar{u}_{i,j} \in T_{i,j}(\bar{x}_i)$, for each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, l\}$. From continuity of the mappings $J_{\rho_i}^{F_i}$, N_i and g_i $(i = 1, 2, \ldots, p)$, (3.10) and (3.11), it follows that for $i \in \{1, 2, \ldots, p\}$,

$$g_i(\bar{x}_i) = J_{\rho_i}^{F_i}(g_i(\bar{x}_i) - \rho_i N_i(\bar{u}_{i,1}, \bar{u}_{i,2}, \ldots, \bar{u}_{i,l})).$$

Lemma 3.11 implies that $(\bar{x}_1, \ldots, \bar{x}_p, \bar{u}_{1,1}, \ldots, \bar{u}_{1,l}, \ldots, \bar{u}_{p,1}, \ldots, \bar{u}_{p,l}) \in \prod_{i=1}^p K_i \times \prod_{i=1}^p \prod_{j=1}^l T_{i,j}(\bar{x}_i)$ is a solution of SNGMIEP (2.1). This completes the proof. □

Remark 4.3. (a) If for each $i \in \{1, 2, \ldots, p\}$, X_i be a 2-uniformly smooth Banach space and there exists the constant $\rho_i > 0$ such that

$$\begin{cases} \rho_i < \frac{\theta_i(\delta_i - \tau_i)}{\tau_i \sum\limits_{j=1}^{l} \nu_{i,j}\mu_{i,j} - \alpha_i\theta_i}, & \theta_i = 1 - \sqrt{1 - 2\kappa_i + c_2\sigma_i^2} > 0, \\ \tau_i \sum\limits_{j=1}^{l} \nu_{i,j}\mu_{i,j} > \alpha_i\theta_i, & 2\kappa_i \le 1 + c_2\sigma_i^2, \end{cases}$$

then (4.1) holds.

(b) If for each $i \in \{1, 2, \ldots, p\}$, X_i be a 2-uniformly smooth Banach space and there exists the constant $\rho_i > 0$ such that

$$\begin{cases} \rho_i < \frac{\theta_i\delta_i - \tau_i\sigma_i}{\tau_i \sum\limits_{j=1}^{l} \nu_{i,j}\mu_{i,j} - \alpha_i\theta_i}, & \theta_i = 1 - \sqrt{1 - 2\kappa_i + c_2\sigma_i^2} > 0, \\ \tau_i \sum\limits_{j=1}^{l} \nu_{i,j}\mu_{i,j} > \alpha_i\theta_i, & 2\kappa_i \le 1 + c_2\sigma_i^2, \end{cases}$$

then (4.14) holds.

5. Some comments

Kazmi and Khan [20] studied the convergence analysis of the sequences generated by Algorithm 3.1 in [20]. We show that the conditions in the statement of Theorem 4.1 of [20] should be corrected. For this end, we rewrite the statement of Theorem 4.1 of [20].

Theorem 5.1 ([20, Theorem 4.1])**.** *Let K be a nonempty, closed and convex subset of H; let the mapping $\eta : H \times H \to H$ be δ-strongly monotone and τ-Lipschitz continuous with $\eta(x, y) + \eta(y, x) = 0$, $\forall x, y \in H$; let the bifunction $F : K \times K \to \mathbb{R}$ be α-strongly monotone and satisfy the assumptions of Theorem 2.2 in [20]; let the mapping $N : H \times H \to H$ be (σ_1, σ_2)-Lipschitz continuous; let the mappings $T, B : H \to CB(H)$ be μ_1-$\mathcal{H}$-Lipschitz continuous and μ_2-$\mathcal{H}$-Lipschitz continuous, respectively, and the mapping $g : K \to K$ be γ-strongly monotone and ξ-Lipschitz continuous. If the following conditions hold for $\rho > 0$:*

$$\rho < \frac{b(\delta - \tau)}{e\tau - \alpha b}; \tag{5.1}$$

$$e\tau < \alpha b; \quad b > 0, \tag{5.2}$$

where $e := \sigma_1\mu_1 + \sigma_2\mu_2$ and $b = 1 - \sqrt{1 - 2\gamma + \xi^2}$, then the sequences $\{z_n\}$, $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ generated by Algorithm 3.1 strongly converge to $z \in H$, $x \in H$, $u \in T(x)$, $v \in B(x)$, respectively, and (z, x, u, v) is a solution of GWHE *(3.3).*

By reviewing the proof of Theorem 5.1, we note that the authors concluded

$$\theta := \frac{\tau}{\delta + \rho\alpha}\left[1 + \frac{\rho(\sigma_1\mu_1 + \sigma_2\mu_2)}{1 - \sqrt{1 - 2\gamma + \xi^2}}\right] < 1,$$

by using conditions (5.1) and (5.2) (conditions (4.1) and (4.2) in [20]). However, it is easy to check that the conditions (5.1) and (5.2) do not guarantee that $\theta < 1$.

For this aim, condition (5.1) should be replaced by the condition $e\tau > \alpha b$, $b > 0$. Further, condition $2\gamma \leq 1 + \xi^2$ should be added to the conditions (5.1) and (5.2).

In view of the above facts, the correct version of above Theorem 4.1 in [20] is the following.

Theorem 5.2. *Let K be a nonempty, closed and convex subset of H and let the mapping $\eta : H \times H \to H$ be δ-strongly monotone and τ-Lipschitz continuous with $\eta(x, y) + \eta(y, x) = 0$, $\forall x, y \in H$. Let the bifunction $F : K \times K \to \mathbb{R}$ be α-strongly monotone and satisfy the assumptions of Theorem 2.2 in [20] and suppose that the mapping $N : H \times H \to H$ is (σ_1, σ_2)-Lipschitz continuous. Further, let the mappings $T, B : H \to CB(H)$ be μ_1-$\mathcal{H}$-Lipschitz continuous and μ_2-$\mathcal{H}$-Lipschitz continuous, respectively, and the mapping $g : K \to K$ be γ-strongly monotone and ξ-Lipschitz continuous. If the constant $\rho > 0$ satisfies following condition:*

$$\begin{cases} \rho < \frac{b(\delta-\tau)}{e\tau-\alpha b}, \\ e\tau > \alpha b, \quad \text{where} \quad e = \sigma_1\mu_1 + \sigma_2\mu_2, \\ b = 1 - \sqrt{1 - 2\gamma + \xi^2} > 0, \\ 2\gamma \leq 1 + \xi^2, \end{cases}$$

then the sequences $\{z_n\}$, $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ generated by Algorithm 3.1 in [20] strongly converge to $z \in H$, $x \in H$, $u \in T(x)$, $v \in B(x)$, respectively, and (z, x, u, v) is a solution of GWHE *(3.3).*

Remark 5.3. In view of the above arguments, Theorems 4.1 and 4.2 generalize and improve Theorem 5.2.

Kazmi and Khan [20] presented the following definition which can be viewed as an extension of the concept of stability of the iteration procedure given by Harder and Hicks [15].

Definition 5.4 ([20, Definition 4.1])**.** Let $G : H \to 2^H$ be a set-valued mapping and $x_0 \in H$. Assume that $x_{n+1} \in f(G, x_n)$ defines an iteration procedure which yields a sequence of points $\{x_n\}$ in H. Suppose that $\mathrm{F}(G) = \{x \in H : x \in G(x)\} \neq \emptyset$ and $\{x_n\}$ converges to some $x \in G(x)$. Let $\{y_n\}$ be an arbitrary sequence in H, and $\epsilon_n = \|y_{n+1} - x_{n+1}\|$, $x_{n+1} \in f(G, x_n)$.

(a) If $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = x$, then the iteration procedure $x_{n+1} \in f(G, x_n)$ is said to be *G-stable*.
(b) If $\sum_{n=0}^{\infty} \epsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = x$, then the iteration procedure $x_{n+1} \in f(G, x_n)$ is said to be *almost G-stable*.

They have stated the following stability result for Algorithm 3.1 in [20].

Theorem 5.5 ([20, Theorem 4.2])**.** *Let the mappings η, F, N, T, B, g be the same as in Theorem 4.1 in [20] and conditions (4.1) and (4.2) in Theorem 4.1 hold with $e = (1 + \epsilon)(\sigma_1\mu_1 + \sigma_2\mu_2)$. Let $\{q_n\}$ be any sequence in $\mathcal{H}$ and define $\{a_n\} \subset [0, \infty)$*

by

$$(5.3)\qquad \begin{aligned} &g(y_n) = J_\rho^F(q_n),\\ &\bar{u}_n \in T(y_n):\ \|\bar{u}_{n+1} - \bar{u}_n\| \le (1 + (1+n)^{-1})\mathcal{H}(T(y_{n+1}), T(y_n)),\\ &\bar{v}_n \in B(y_n):\ \|\bar{v}_{n+1} - \bar{v}_n\| \le (1 + (1+n)^{-1})\mathcal{H}(B(y_{n+1}), B(y_n)),\\ &a_n = \|q_{n+1} - (1-\lambda)q_n - \lambda[g(y_n) - \rho N(\bar{u}_n, \bar{v}_n)]\|, \end{aligned}$$

where $n = 0, 1, 2, \dots$; $\rho > 0$ *is a constant and* $0 < \lambda < 1$ *is a relaxation parameter. Then* $\lim_{n\to\infty}(q_n, y_n, \bar{u}_n, \bar{v}_n) = (z, x, u, v)$ *if and only if* $\lim_{n\to\infty} a_n = 0$*, where* (z, x, u, v) *is a solution of* GWHE *(3.3) in [20].*

By a careful reading, we found that there is a fatal in the proof of Theorem 4.2 in [20] (Page 1320 line 9 from the bottom). In the proof of Theorem 4.2, authors deduced the following inequality by using the assumptions of Theorem 4.2:

$$\begin{aligned} \|q_{n+1} - z\| &\le \|(1-\lambda)q_n + \lambda[g(y_n) - \rho N(\bar{u}_n, \bar{v}_n)] - z\|\\ &\quad + \|q_{n+1} - (1-\lambda)q_n - \lambda[g(y_n) - \rho N(\bar{u}_n, \bar{v}_n)]\|\\ &\le (1-\lambda)\|q_n - z\| + \lambda\|g(y_n) - g(x)\|\\ &\quad + \rho\lambda\|N(\bar{u}_n, \bar{v}_n) - N(u, v)\| + a_n. \end{aligned}$$

Then by using Theorems 2.1 and 3.1 in [20], they claimed that the preceding inequality reduces to the following inequality (see inequality (4.13) in [20]):

$$(5.4)\qquad \begin{aligned} \|q_{n+1} - z\| &\le (1-\lambda)\|q_n - z\| + \lambda\frac{\tau}{\delta + \rho\alpha}\|q_n - z\| + \rho\lambda(\sigma_1(1+\epsilon)\mathcal{H}(T(y_n), T(x))\\ &\quad + \sigma_2(1+\epsilon)\mathcal{H}(B(y_n), B(x)))\\ &\le (1-\lambda)\|q_n - z\| + \lambda\frac{\tau}{\delta + \rho\alpha}\|q_n - z\|\\ &\quad + \lambda\rho(1+\epsilon)(\sigma_1\mu_1 + \sigma_2\mu_2)\|y_n - x\| + a_n. \end{aligned}$$

In fact, they claimed that Lemma 3.1 (Theorem 2.1 in [20]) implies that

$$\begin{aligned} \|N(\bar{u}_n, \bar{v}_n) - N(u, v)\| &\le \sigma_1\|\bar{u}_n - u\| + \sigma_2\|\bar{v}_n - v\|\\ &\le \sigma_1(1+\epsilon)\mathcal{H}(T(y_n), T(x)) + \sigma_2(1+\epsilon)\mathcal{H}(B(y_n), B(x))\\ &\le (1+\epsilon)(\sigma_1\mu_1 + \sigma_2\mu_2)\|y_n - x\|. \end{aligned}$$

There is a fatal error in the above inequalities. Since (z, x, u, v) is a solution GWHE (3.3) in [20], we have $u \in T(x)$ and $v \in B(x)$. On the other hand, in view of (5.3), for each $n \in \mathbb{N} \cup \{0\}$, we have $\bar{u}_n \in T(y_n)$ and $\bar{v}_n \in B(y_n)$. But, for any given $\epsilon > 0$, we cannot deduce that $\|\bar{u}_n - u\| \le (1+\epsilon)\mathcal{H}(T(y_n), T(x))$ and $\|\bar{v}_n - v\| \le (1+\epsilon)\mathcal{H}(B(y_n), B(x))$. In other words, under the assumptions of Theorem 4.2, one cannot deduce inequality (4.13) in [20]. However, we cannot verify Theorem 4.2 without the inequality (5.4) (inequality (4.13) of [20]).

The following example shows that for any given $x, y \in X$, $u \in T(x)$, $v \in T(y)$ and $\epsilon > 0$, inequality (2.3) in Lemma 2.2 does not hold necessarily.

Example 5.6. Let $X = \mathbb{R}$ be the set of real numbers with the usual metric and let $T : X \to CB(X)$ be a set-valued mapping defined as follows:

$$T(x) = \begin{cases} \left\{\frac{1}{\gamma}, \frac{1}{\delta}\right\}, & \text{if } x \leq 0, \\ \left\{0, \frac{1}{\kappa}\right\}, & \text{if } x > 0, \end{cases}$$

where $\gamma, \delta, \kappa \in \mathbb{R}$ are arbitrary and fixed such that $0 < \gamma < \delta < \kappa < \frac{\delta\gamma}{\delta-\gamma}$. Take $x = \alpha \leq 0$, $y = \beta > 0$, $u = \frac{1}{\gamma}$, $v = 0$ and $\epsilon < \frac{\delta-\gamma}{\gamma}$. If $a = \frac{1}{\gamma}$, then from $\gamma > 0$, it follows that

$$d(a, T(y)) = \inf\left\{d(\frac{1}{\gamma}, 0), d(\frac{1}{\gamma}, \frac{1}{\kappa})\right\} = \inf\left\{\frac{1}{\gamma}, \frac{\kappa-\gamma}{\gamma\kappa}\right\} = \frac{\kappa-\gamma}{\gamma\kappa}.$$

For the case, $a = \frac{1}{\delta}$, by using $\delta > 0$, we have

$$d(a, T(y)) = \inf\left\{d\left(\frac{1}{\delta}, 0\right), d\left(\frac{1}{\delta}, \frac{1}{\kappa}\right)\right\} = \inf\left\{\frac{1}{\delta}, \frac{\kappa-\delta}{\delta\kappa}\right\} = \frac{\kappa-\delta}{\delta\kappa}.$$

Since $\gamma < \delta$, it follows that

$$\sup_{a\in T(x)} d(a, T(y)) = \max\left\{\frac{\kappa-\gamma}{\gamma\kappa}, \frac{\kappa-\delta}{\delta\kappa}\right\} = \frac{\kappa-\gamma}{\gamma\kappa}.$$

Taking $b = 0$ and using the fact that $\gamma < \delta$, we obtain

$$d(T(x), b) = \inf\left\{d\left(\frac{1}{\gamma}, 0\right), d\left(\frac{1}{\delta}, 0\right)\right\} = \inf\left\{\frac{1}{\gamma}, \frac{1}{\delta}\right\} = \frac{1}{\delta}.$$

If $b = \frac{1}{\kappa}$, then from $\gamma < \delta$, it follows that

$$d(T(x), b) = \inf\left\{d\left(\frac{1}{\gamma}, \frac{1}{\kappa}\right), d\left(\frac{1}{\delta}, \frac{1}{\kappa}\right)\right\} = \inf\left\{\frac{\kappa-\gamma}{\gamma\kappa}, \frac{\kappa-\delta}{\delta\kappa}\right\} = \frac{\kappa-\delta}{\delta\kappa}.$$

Since $\delta > 0$, we deduce that

$$\sup_{b\in T(y)} d(T(x), b) = \max\left\{\frac{1}{\delta}, \frac{\kappa-\delta}{\delta\kappa}\right\} = \frac{1}{\delta}.$$

Because $\kappa < \frac{\delta\gamma}{\delta-\gamma}$, one has

$$\begin{aligned} \mathcal{H}(T(x), T(y)) &= \max\left\{\sup_{a\in T(x)} d(a, T(y)), \sup_{b\in T(y)} d(T(x), b)\right\} \\ &= \max\left\{\frac{\kappa-\gamma}{\gamma\kappa}, \frac{1}{\delta}\right\} = \frac{1}{\delta}. \end{aligned}$$

Now, $\epsilon < \frac{\delta-\gamma}{\gamma}$ implies that

$$(1+\epsilon)\mathcal{H}(T(x), T(y)) = \frac{1+\epsilon}{\delta} < \frac{1}{\delta} + \frac{\delta-\gamma}{\delta\gamma} = \frac{1}{\gamma} = |u-v| = d(u, v).$$

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