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ALTERNATIVE GENERALIZED WOLFE TYPE AND MOND-WEIR TYPE VECTOR DUALITY

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Abstract. Considering a general vector optimization problem, we attach to it two new vector duals by means of perturbation theory. These vector duals are constructed with the help of the recent Wolfe and Mond-Weir scalar duals for general optimization problems proposed by R.I. Boţ and S.-M. Grad, by exploiting an idea due to W. Breckner and I. Kolumbán. Constrained and unconstrained vector optimization problems are seen as special cases of the initial primal vector optimization problem and from the general case we obtain vector dual problems of Wolfe type and Mond-Weir type for them by using different vector perturbation functions.

1. Introduction and preliminaries

Originally considered for scalar constrained optimization problems (see [12, 14]), the Wolfe and Mond-Weir duality approaches were quickly generalized for vector optimization problems, too. Usually this was done in a direct way, by retaining the objective functions of the primal problem into the objective function of the vector duals to it (cf. [13], for instance). This approach was recently extended for general vector optimization problems in [4], following the ideas from [2], where Wolfe and Mond-Weir type duals were introduced for general scalar optimization problems. However, in vector optimization it is possible to exploit the scalar duals from [2] in a different manner, too, by introducing new Wolfe and Mond-Weir type vector duals to a general vector optimization problem constructed following an idea considered in [8, 9, 11] for Fenchel and respectively Langrange type vector duals. For Wolfe or Mond-Weir type duality this way of constructing vector duals was considered mainly for fractional programming problems. To the best of our knowledge, the only paper where this approach was employed to construct a Mond-Weir type vector dual for a constrained vector minimization problem was [10], where the involved functions were considered quasidifferentiable.

The main scope of this paper is to introduce new Wolfe and Mond-Weir type vector duals achieved via the approach from [8, 9, 11] to a general vector minimization

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problem. They consist of vector maximizing a vector subject to some constraints which contain the generalized Wolfe and respectively Mond-Weir scalar duals of the scalarized problem attached to the primal vector optimization problem. We compare these new vector duals with the vector duals from [4] and we deliver weak and strong duality statements for them. Then we particularize the general problem to be constrained and unconstrained, respectively. For different vector perturbation functions we obtain new Wolfe and Mond-Weir type vector duals to these vector problems, extending thus the classes of problems for which these duality approaches can be applied to. We compare the image sets of the different vector duals attached to the same vector optimization problem, delivering either inclusion relations between them, or counterexamples that prove that in general neither of them is a subset of the other. We close the paper with a conclusive section, where some directions to continue the research began in [2, 4] and this paper are suggested.

Consider two separated locally convex spaces X and Y and their topological dual spaces X^* and Y^*, respectively, endowed with the corresponding weak* topologies and denote by $\langle x^*, x\rangle = x^*(x)$ the value at $x \in X$ of the linear continuous functional $x^* \in X^*$. A *cone* $K \subseteq X$ is a nonempty set which fulfills $\lambda K \subseteq K$ for all $\lambda \geq 0$. A *convex cone* is a cone which is a convex set. A cone $K \subseteq X$ is called *nontrivial* if $K \neq \{0\}$ and $K \neq X$ and *pointed* if $K \cap (-K) = \{0\}$.

On Y we consider the partial ordering "$\leqq_C$" induced by the convex cone $C \subseteq Y$, defined by $z \leqq_C y \Leftrightarrow y - z \in C$ when $z, y \in Y$. The notation $z \leq_C y$ is used to write more compactly that $z \leqq_C y$ and $z \neq y$, where $z, y \in Y$. The dual cone of C is $C^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \ \forall y \in C\}$. A greatest element with respect to "$\leqq_C$" which does not belong to Y denoted by ∞_C is attached to Y, and let $Y^\bullet = Y \cup \{\infty_C\}$. Then for any $y \in Y^\bullet$ one has $y \leqq_C \infty_C$ and we consider on $Y^\bullet$ the operations $y + \infty_C = \infty_C + y = \infty_C$ for all $y \in Y$ and $t \cdot \infty_C = \infty_C$ for all $t \geq 0$. Moreover, consider by convention $\langle v^*, \infty_C\rangle = +\infty$ for all $v^* \in C^*$.

For a subset U of X, by $\mathrm{cl}(U)$, $\mathrm{lin}(U)$, $\mathrm{ri}(U)$, $\dim(U)$, δ_U and $\mathrm{sqri}(U)$ we denote its *closure, linear hull, relative interior, dimension, indicator function* and *strong quasi relative interior,* respectively. In vector optimization it is used also the *quasi interior of the dual cone* of K, $K^{*0} = \{x^* \in K^* : \langle x^*, x\rangle > 0 \text{ for all } x \in K \setminus \{0\}\}$. We consider the *projection function* $\mathrm{Pr}_X : X \times Y \to X$ defined by $\mathrm{Pr}_X(x, y) = x$ for all $(x, y) \in X \times Y$, too.

For the function $f : X \to \overline{\mathbb{R}}$ we use the classical notations for *domain* $\mathrm{dom}\, f = \{x \in X : f(x) < +\infty\}$, *epigraph* $\mathrm{epi}\, f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ and *conjugate function* $f^* : X^* \to \overline{\mathbb{R}}$, $f^*(x^*) = \sup_{x\in X}\{\langle x^*, x\rangle - f(x)\}$. We call f *proper* if $f(x) > -\infty$ for all $x \in X$ and $\mathrm{dom}\, f \neq \emptyset$. If $f(x) \in \mathbb{R}$ the *(convex) subdifferential* of f at x is $\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle \ \forall y \in X\}$, while if $f(x) \notin \mathbb{R}$ we take by convention $\partial f(x) = \emptyset$. For $U \subseteq X$ we have for all $x \in U$ that $\partial \delta_U(x) = N_U(x)$, where $N_U(x)$ is the *normal cone* of U at x. Between a function and its conjugate there is the *Young-Fenchel inequality* $f^*(x^*) + f(x) \geq \langle x^*, x\rangle$ for all $x \in X$ and $x^* \in X^*$. This inequality is fulfilled as an equality if and only if $x^* \in \partial f(x)$. For a linear continuous mapping $A : X \to Y$ we have its *adjoint* $A^* : Y^* \to X^*$ given by $\langle A^*y^*, x\rangle = \langle y^*, Ax\rangle$ for any $(x, y^*) \in X \times Y^*$.

A vector function $F : X \to Y^\bullet$ is said to be *proper* if its *domain* $\operatorname{dom} F = \{x \in X : F(x) \in Y\}$ is nonempty. It is $C-$*convex* if $F(tx+(1-t)y) \leqq_C tF(x)+(1-t)F(y)$ for all $x, y \in X$ and all $t \in [0,1]$. The vector function F is said to be $C-$epi-*closed* if C is closed and its $C-$*epigraph* $\operatorname{epi}_C F = \{(x,y) \in X \times Y : y \in F(x) + C\}$ is closed, and it is called $C-$*semicontinuous* if for every $x \in X$, each neighborhood W of zero in Y and for any $b \in Y$ satisfying $b \leqq_C F(x)$, there exists a neighborhood U of x in X such that $F(U) \subseteq b + W + Y \cup \{\infty_C\}$.

For $v^* \in C^*$ the function $(v^*F) : X \to \overline{\mathbb{R}}$ is defined by $(v^*F)(x) = \langle v^*, F(x)\rangle$, $x \in X$. If F is $C-$lower semicontinuous then (v^*F) is lower semicontinuous whenever $v^* \in C^* \setminus \{0\}$ and if C is closed, then every $C-$ lower semicontinuous vector function is also $C-$epi-closed, but, not all $C-$epi-closed vector functions are $C-$lower semicontinuous. An example proving this can be found in [6].

In this paper the vector optimization problems consist of vector minimizing or vector maximizing a vector function with respect to the partial ordering induced in its image space by a pointed convex cone. For these vector optimization problems we use the classical notions of efficient and properly efficient solutions, the latter considered with respect to the linear scalarization.

2. New general Wolfe type and Mond-Weir type vector duals

Let X, Y and V be separated locally convex vector spaces, with V partially ordered by the nonempty pointed convex cone $K \subseteq V$. Let $F : X \to V^\bullet$ be a proper and $K-$ convex function and consider the general vector optimization problem

$$(PVG) \qquad \operatorname*{Min}_{x \in X} F(x).$$

For this vector optimization problem we consider the following solution concepts.

Definition 2.1. (i) An element $\overline{x} \in X$ is called *efficient solution* to the vector optimization problem (PVG) if $\overline{x} \in \operatorname{dom} F$ and for all $x \in \operatorname{dom} F$ from $F(x) \leqq_K F(\overline{x})$ follows $F(x) = F(\overline{x})$.

(ii) An element $\overline{x} \in X$ is called *properly efficient solution* to the vector optimization problem (PVG) if there exists $v^* \in K^{*0}$ such that $(v^*F)(\overline{x}) \leq (v^*F)(x)$ for all $x \in X$.

Remark 2.2. Every properly efficient solution to (PVG) belongs to $\operatorname{dom} F$ and it is also an efficient solution to the same vector optimization problem.

Using the K-convex vector perturbation function $\Phi : X \times Y \to V^\bullet$ which fulfills $0 \in \operatorname{Pr}_Y(\operatorname{dom}\Phi)$ and $\Phi(x, 0) = F(x)$ for all $x \in X$, the primal vector optimization problem introduced above can be reformulated as

$$(PVG) \qquad \operatorname*{Min}_{x \in X} \Phi(x, 0).$$

To (PVG) we attach two vector dual problems. To construct them, we used the scalar Wolfe type and Mond-Weir type duals introduced in [2], exploiting moreover the vector duality approach from [8, 9, 11].

The *Wolfe type vector dual* to (PVG) we consider is

$$(DVG^W) \qquad \underset{(v^*,y^*,v,u,y)\in\mathcal{B}_G^W}{\text{Max}} h_G^W(v^*,y^*,v,u,y)$$

where

$$\mathcal{B}_G^W = \{(v^*,y^*,v,u,y) \in K^{*0}\times Y^*\times V\times X\times Y : (0,y^*)\in\partial(v^*\Phi)(u,y), \\ \langle v^*,v\rangle \leq -(v^*\Phi)^*(0,y^*)\}$$

and

$$h_G^W(v^*,y^*,v,u,y) = v,$$

while the *Mond-Weir type vector dual* one is

$$(DVG^M) \qquad \underset{(v^*,y^*,v,u)\in\mathcal{B}_G^M}{\text{Max}} h_G^M(v^*,y^*,v,u)$$

where

$$\mathcal{B}_G^M = \{(v^*,y^*,v,u) \in K^{*0}\times Y^*\times V\times X : (0,y^*)\in\partial(v^*\Phi)(u,0), \\ \langle v^*,v\rangle \leq \langle v^*,\Phi(u,0)\rangle\}$$

and

$$h_G^M(v^*,y^*,v,u) = v.$$

We study the connections between the properly efficient solutions for the vector optimization problem (PVG) and the efficient solutions for these vector dual problems. The definition of the efficient solutions to the vector dual problem (DVG^W) is below and the one corresponding to the vector dual (DVG^M) follows similarly.

Definition 2.3. An element $(\overline{v}^*,\overline{y}^*,\overline{v},\overline{u},\overline{y}) \in \mathcal{B}_G^W$ is called *efficient solution* to the vector optimization problem (DVG^W) if $(\overline{v}^*,\overline{y}^*,\overline{v},\overline{u},\overline{y}) \in \mathcal{B}_G^W$ and for all $(v^*,y^*,v,u,y) \in \mathcal{B}_G^W$ from $h_G^W(\overline{v}^*,\overline{y}^*,\overline{v},\overline{u},\overline{y}) \leqq_K h_G^W(v^*,y^*,v,u,y)$ follows $h_G^W(\overline{v}^*,\overline{y}^*,\overline{v},\overline{u},\overline{y}) = h_G^W(v^*,y^*,v,u,y)$. The set containing all the values $h_G^W(\overline{v}^*,\overline{y}^*,\overline{v},\overline{u},\overline{y})$, when $(\overline{v}^*,\overline{y}^*,\overline{v},\overline{u},\overline{y})$ is an efficient solution to (DVG^W), is said to be the *maximal set* of (DVG^W).

Lemma 2.4. *One has* $h_G^M(\mathcal{B}_G^M) \subseteq h_G^W(\mathcal{B}_G^W)$.

Proof. Whenever $(v^*,y^*,v,u) \in \mathcal{B}_G^M$, it is easy to notice that $(v^*,y^*,v,u,0) \in \mathcal{B}_G^W$ and $h_G^M(v^*,y^*,v,u) = h_G^W(v^*,y^*,v,u,0)$. Therefore all the values taken by the objective function of (DVG^M) over its feasible set can be found also in $h_G^W(\mathcal{B}_G^W)$. $\square$

Remark 2.5. The sets $h_G^M(\mathcal{B}_G^M)$ and $h_G^W(\mathcal{B}_G^W)$ do not coincide in general. A situation like this will be given later in Example 3.5.

Now we investigate the connections between the duals to (PVG) considered here and other Wolfe and Mond-Weir type vector duals introduced in [4] to it, which are

$$(DVG_W) \qquad \underset{(v^*,y^*,u,y,r)\in\mathcal{B}_W^G}{\text{Max}} h_W^G(v^*,y^*,u,y,r)$$

where

$$\mathcal{B}_W^G = \{(v^*,y^*,u,y,r) \in K^{*0}\times Y^*\times X\times Y\times (K\setminus\{0\}) : \\ (0,y^*)\in\partial(v^*\Phi)(u,y)\}$$

and

$$h_W^G(v^*,y^*,u,y,r) = \Phi(u,y) - \frac{\langle y^*,y\rangle}{\langle v^*,r\rangle}r$$

and, respectively,

$$(DVG_M) \qquad \underset{(v^*,y^*,u)\in\mathcal{B}_M^G}{\operatorname{Max}} h_M^G(v^*,y^*,u)$$

where

$$\mathcal{B}_M^G = \{(v^*,y^*,u)\in K^{*0}\times Y^*\times X : (0,y^*)\in\partial(v^*\Phi)(u,0)\}$$

and

$$h_M^G(v^*,y^*,u)=\Phi(u,0).$$

Unlike these vector duals, the ones we introduced above do not have the objective function of (PVG) as objective functions. The newly introduced vector duals inherit all the constraints of the vector duals from [4], having an additional one which involves the vector that acts as an objective function and the corresponding dual problem of the scalarized primal. Moreover, the image sets of the vector duals introduced in this paper are larger than the ones of their counterparts of [4], as one can see below, and this can prove to be useful in practice.

Theorem 2.6. *One has* $h_W^G(\mathcal{B}_W^G)\subseteq h_G^W(\mathcal{B}_G^W)$ *and* $h_M^G(\mathcal{B}_M^G)\subseteq h_G^M(\mathcal{B}_G^M)$.

Proof. Whenever $(v^*,y^*,u,y,r)\in\mathcal{B}_W^G$, one has $(0,y^*)\in\partial(v^*\Phi)(u,y)$, which yields $(v^*\Phi)(u,y)+(v^*\Phi)^*(0,y^*)=\langle y^*,y\rangle$, $\Phi(u,y)\in V$ and

$$\langle v^*, h_W^G(v^*,y^*,u,y,r)-\Phi(u,y)\rangle=\left\langle v^*,-\frac{\langle y^*,y\rangle}{\langle v^*,r\rangle}r\right\rangle=-\langle y^*,y\rangle,$$

thus $\langle v^*,h_W^G(v^*,y^*,u,y,r)\rangle=\langle v^*,\Phi(u,y)\rangle-\langle y^*,y\rangle=-(v^*\Phi)^*(0,y^*)$. Then, it follows that $(v^*,y^*,h_W^G(v^*,y^*,u,y,r),u,y)\in\mathcal{B}_G^W$ and $h_G^W(v^*,y^*,h_W^G(v^*,y^*,u,y,r),u,y)=h_W^G(v^*,y^*,u,y,r)$, therefore $h_W^G(\mathcal{B}_W^G)\subseteq h_G^W(\mathcal{B}_G^W)$.

The inclusion $h_M^G(\mathcal{B}_M^G)\subseteq h_G^M(\mathcal{B}_G^M)$ can be proven analogously. □

Remark 2.7. The inclusions proven in Theorem 2.6 are in general strict, as the situation depicted in Example 4.4 shows.

Remark 2.8. If $\mathcal{B}_G^M\neq\emptyset$, then there exists some element (v^*,y^*,v,u) of this set. Then $(v^*,y^*,\Phi(u,0),u)\in\mathcal{B}_G^M$, too. Moreover, it follows immediately that $(v^*,y^*,u)\in\mathcal{B}_M^G$ and by [4, Proposition 2.11], (v^*,y^*,u) is efficient to (DVG_M) and u properly efficient to (PVG). Consequently, $(v^*,y^*,\Phi(u,0),u)$ turns out to be efficient to (DVG^M) and in this way it follows that the maximal set of (DVG_M) is a subset of the maximal set of (DVG^M). Moreover, $(v^*,y^*,u,0,r)\in\mathcal{B}_W^G$ for all $r\in K\setminus\{0\}$ is an efficient solution to (DVG_W) and $(v^*,y^*,\Phi(u,0),u,0)\in\mathcal{B}_G^W$ is one to (DVG^W).

For the newly introduced dual problems there is weak duality.

Theorem 2.9. *There are no* $x\in X$ *and* $(v^*,y^*,v,u,y)\in\mathcal{B}_G^W$ *such that* $F(x)\leq_K h_G^W(v^*,y^*,v,u,y)$.

Proof. Assume the contrary, namely that there are some $\overline{x}\in X$ and $(\overline{v}^*,\overline{y}^*,\overline{v},\overline{u},\overline{y})\in\mathcal{B}_G^W$ fulfilling $F(\overline{x})\leq_K h_G^W(\overline{v}^*,\overline{y}^*,\overline{v},\overline{u},\overline{y})$. Then $\overline{x}\in\operatorname{dom}F$ and it follows $\langle\overline{v}^*,\overline{v}-\Phi(\overline{x},0)\rangle>0$. On the other hand, from the feasibility of $(\overline{v}^*,\overline{y}^*,\overline{v},\overline{u},\overline{y})$ to (DVG^W), it follows $\langle\overline{v}^*,\overline{v}\rangle\leq-(\overline{v}^*\Phi)^*(0,\overline{y}^*)$ and since $-(\overline{v}^*\Phi)^*(0,\overline{y}^*)\leq(\overline{v}^*\Phi)(\overline{x},0)$, one gets $\langle\overline{v}^*,\overline{v}-\Phi(\overline{x},0)\rangle\leq 0$, which contradicts the strict inequality obtained above. □

Using Lemma 2.4 and the previous theorem, one can easily prove the following weak duality statement.

Theorem 2.10. *There are no* $x \in X$ *and* $(v^*, y^*, v, u) \in \mathcal{B}_G^M$ *such that* $F(x) \leq_K h_G^M(v^*, y^*, v, u)$.

Strong duality statements concerning the vector optimization problem (PVG) and its two newly introduced vector dual problems will follow. In order to give them, consider the following regularity conditions (cf. [1, 4, 7]). We begin with a classical one involving continuity

$$(RCV^1) \;\Big|\; \exists x' \in X \text{ such that } (x', 0) \in \operatorname{dom}\Phi \text{ and } \Phi(x', \cdot) \text{ is continuous at } 0;$$

followed by one that works when X and Y are Frechét spaces

$$(RCV^2) \;\left|\; \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces, } \Phi \text{ is } K- \text{ lower semicontinuous and} \\ 0 \in \operatorname{sqri}(\operatorname{Pr}_Y(\operatorname{dom}\Phi)); \end{array}\right.$$

then in finite dimensional case

$$(RCV^3) \;\Big|\; \dim(\operatorname{lin}(\operatorname{Pr}_Y(\operatorname{dom}\Phi))) < +\infty \text{ and } 0 \in \operatorname{ri}(\operatorname{Pr}_Y(\operatorname{dom}\Phi));$$

and the closedness type regularity condition

$$(RCV^4) \;\left|\; \begin{array}{l} \Phi \text{ is } K-\text{lower semicontinuous and } \operatorname{Pr}_{X^* \times \mathbb{R}}(\operatorname{epi}(v^*\Phi)^*) \text{ is closed in} \\ \text{the topology } w(X^*, X) \times \mathbb{R} \text{ for all } v^* \in K^{*0}. \end{array}\right.$$

Theorem 2.11. *Assume that one of* (RCV^i), $i \in \{1, 2, 3, 4\}$, *is fulfilled. If* $\overline{x} \in X$ *is a properly efficient solution to* (PVG), *then there are the efficient solutions* $(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u}, \overline{y})$ *to* (DVG^W) *and* $(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u})$ *to* (DVG^M) *such that* $F(\overline{x}) = h_G^W(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u}, \overline{y}) = h_G^M(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u})$.

Proof. Since $\overline{x} \in X$ is properly efficient to (PVG), there exists $\overline{v}^* \in K^{*0}$ such that

$$\langle \overline{v}^*, F(\overline{x}) \rangle \leq \langle \overline{v}^*, F(x) \rangle, \text{ for all } x \in X.$$

Let us consider the problem $\inf_{x \in X} (\overline{v}^* F)(x)$ and its Wolfe type dual (cf. [2]) $\sup\{-(\overline{v}^*\Phi)^*(0, y^*) : \; u \in X, \; y \in Y, \; y^* \in Y^*, \; (0, y^*) \in \partial(\overline{v}^*\Phi)(u, y)\}$. From the above inequality and the hypotheses we obtain via [2, Theorem 2] that there exists $\overline{y}^* \in Y^*$ such that

$$-(\overline{v}^*\Phi)^*(0, \overline{y}^*) = \sup_{y^* \in Y^*} \{-(\overline{v}^*\Phi)^*(0, y^*)\} = \inf_{x \in X} \langle \overline{v}^*, F(x) \rangle = \langle \overline{v}^*, F(\overline{x}) \rangle,$$

and $(0, \overline{y}^*) \in \partial(\overline{v}^*\Phi)(\overline{x}, 0)$. Letting $\overline{v} = F(\overline{x})$, one sees that $(v^*, \overline{y}^*, \overline{v}, \overline{x}, 0) \in \mathcal{B}_G^W$.

Moreover, $(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{x}, 0)$ is an efficient solution to (DVG^W). Indeed, if $(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{x}, 0)$ were not an efficient solution to problem (DVG^W) there would exist an element $(v^*, y^*, v, u, y) \in \mathcal{B}_G^W$ such that $h_G^W(v^*, y^*, v, u, y) = v \geq_K \overline{v} = F(\overline{x})$. But this contradicts the weak duality statement Theorem 2.9.

In order to deal with problem (DVG^M) we consider the Mond-Weir type dual to $\inf_{x \in X} (\overline{v}^*\Phi)(x, 0)$ given by $\sup\{\langle \overline{v}^*, \Phi(x, 0) \rangle : u \in X, y^* \in Y^*, (0, y^*) \in \partial(\overline{v}^*\Phi)(u, 0)\}$. In the same manner as for problem (DVG^W) we obtain that $(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{x})$ is an efficient solution to problem (DVG^M). $\square$

Remark 2.12. For the strong duality statement we can also use the regularity condition mentioned in [7, Remark 4.3.2], namely that for all $v^* \in K^{*0}$ the problem $\inf_{x \in X} \langle v^*, F(x) \rangle$ is stable.

Remark 2.13. In case $V = \mathbb{R}$ and $K = \mathbb{R}_+$, identifying $V^\bullet$ with $\mathbb{R} \cup \{+\infty\}$ and $\infty_{\mathbb{R}_+}$ with $+\infty$, for the function $F : X \to \mathbb{R} \cup \{+\infty\}$ proper and convex, we rediscover the Wolfe and Mond-Weir type scalar duality scheme from [7], as the problem (PVG) becomes the general scalar optimization problem (PG) and the vector duals (DVG^W) and (DVG^M) turn out to coincide with the scalar Wolfe and Mond-Weir type duals to (PG) introduced in that paper, i.e. (DG_W) and (DG_M), respectively.

In the next sections we consider some special instances of the vector optimization problem (PVG), more exactly, constrained and unconstrained vector optimization problems. To these problems we attach vector duals which are special cases of the vector duals (DVG^W) and (DVG^M), obtained by using different perturbation vector functions.

3. New Wolfe type and Mond-Weir type duals for particular classes of problems

3.1. **Constrained vector optimization problems.** We begin with vector duality of Wolfe type and Mond-Weir type via linear scalarization for constrained vector optimization problems obtained by using different vector perturbation functions. We use the same framework as in the previous section, with Y partially ordered by the nonempty convex cone $C \subseteq Y$, and we consider the nonempty convex set $S \subseteq X$, the proper $K-$ convex function $f : X \to V^\bullet$ and the proper $C-$ convex function $g : X \to Y^\bullet$ fulfilling $\operatorname{dom} f \cap S \cap g^{-1}(C) \neq \emptyset$. The primal vector optimization problem with geometric and cone constraints we work with is

$$(PV_C) \qquad \operatorname*{Min}_{x \in \mathcal{A}} f(x)$$

where

$$\mathcal{A} = \big\{x \in S : g(x) \in -C\big\}.$$

To it we attach different pairs of vector duals obtained by making use of some perturbation functions.

Consider the Lagrange vector perturbation function $\Phi^V_{C_L} : X \times Y \to V^\bullet$ given by

$$\Phi^V_{C_L}(x, y) = \begin{cases} f(x), & x \in S, g(x) \in y - C, \\ \infty_K, & \text{otherwise.} \end{cases}$$

By construction, it is K-convex. Using it, (PV_C) can be seen as a special case of (PVG). Let us see what happens with the vector duals attached to it in Section 2.

Let $v^* \in K^{*0}$, $y^* \in Y^*$, $v \in V$, $u \in X$ and $y \in Y$. We have $(0, y^*) \in \partial(v^*\Phi^V_{C_L})(u, y)$ if and only if $(v^*\Phi^V_{C_L})^*(0, y^*) + (v^*\Phi^V_{C_L})(u, y) = \langle y^*, y\rangle$, i.e. $((v^*f) - (y^*g) + \delta_S)^*(0) + \delta_{C^*}(-y^*) + (v^*f)(u) + \delta_S(u) + \delta_{-C}(g(u) - y) = \langle y^*, y\rangle$. Using that $\delta^*_{-C} = \delta_{C^*}$, we can rewrite the last relation as $((v^*f) - (y^*g) + \delta_S)^*(0) + ((v^*f) - (y^*g) + \delta_S)(u) + \delta^*_{-C}(-y^*) + \delta_{-C}(g(u) - y) - \langle -y^*, g(u) - y\rangle = 0$. Using the Young-Fenchel inequality and the characterization of the subdifferential by its equality case, it follows that $(0, y^*) \in \partial(v^*\Phi^V_{C_L})(u, y)$ is equivalent with $0 \in \partial((v^*f) - (y^*g) + \delta_S)(u)$, $y^* \in -C^*$ and $\delta_{-C}(g(u) - y) - \langle -y^*, g(u) - y\rangle = 0$. On the other hand $\langle v^*, v\rangle \leq -(v^*\Phi^V_{C_L})^*(0, y^*)$ can be rewritten as $\langle v^*, v\rangle \leq -((v^*f) + (y^*g) + \delta_S)^*(0, y^*)$ and $y^* \in -C^*$, which is equivalent, by using the definition of the conjugate, with

$\langle v^*, v - f(u)\rangle \leq -(y^*g)(u)$ and $y^* \in -C^*$. Thus from (DVG^W) we obtain the *Wolfe vector dual of Lagrange type*

$$(DV_{C_L}^{\widetilde{W}}) \qquad \underset{(v^*,y^*,v,u,y)\in \mathcal{B}_{C_L}^{\widetilde{W}}}{\text{Max}} h_{C_L}^{\widetilde{W}}(v^*, y^*, v, u, y)$$

where

$$\begin{aligned}\mathcal{B}_{C_L}^{\widetilde{W}} = \{&(v^*, y^*, v, u, y) \in K^{*0} \times (-C^*) \times V \times S \times Y : \langle v^*, v - f(u)\rangle \leq \\ &-(y^*g)(u), 0 \in \partial((v^*f) - (y^*g) + \delta_S)(u), \\ &\delta_{-C}(g(u) - y) - \langle y^*, g(u) - y\rangle = 0\}\end{aligned}$$

and

$$h_{C_L}^{\widetilde{W}}(v^*, y^*, v, u, y) = v,$$

which can be equivalently rewritten as (cf. [2, 4])

$$(DV_{C_L}^{W}) \qquad \underset{(v^*,y^*,v,u)\in \mathcal{B}_{C_L}^{W}}{\text{Max}} h_{C_L}^{W}(v^*, y^*, v, u)$$

where

$$\begin{aligned}\mathcal{B}_{C_L}^{W} = \{&(v^*, y^*, v, u) \in K^{*0} \times C^* \times V \times S : \langle v^*, v - f(u)\rangle \leq (y^*g)(u), \\ &0 \in \partial((v^*f) + (y^*g) + \delta_S)(u)\}\end{aligned}$$

and

$$h_{C_L}^{W}(v^*, y^*, v, u) = v.$$

As seen above, for $v^* \in K^{*0}$, $y^* \in Y^*$, $v \in V$ and $u \in X$ we have that $(0, y^*) \in \partial(v^*\Phi_{C_L}^V)(u, 0)$ means actually $0 \in \partial((v^*f) - (y^*g) + \delta_S)(u)$, $y^* \in -C^*$, $g(u) \in -C$ and $-(y^*g)(u) \geq 0$. On the other hand, $\langle v^*, v\rangle \leq \langle v^*, \Phi_{C_L}^V(u, 0)\rangle$ means $\langle v^*, v\rangle \leq (v^*f)(u)$, $u \in S$ and $g(u) \in -C$.

Consequently, the vector dual problem (DVG^M) turns into

$$(DV_{C_L}^{M}) \qquad \underset{(v^*,y^*,v,u)\in \mathcal{B}_{C_L}^{M}}{\text{Max}} h_{C_L}^{M}(v^*, y^*, v, u)$$

where

$$\begin{aligned}\mathcal{B}_{C_L}^{M} = \{&(v^*, y^*, v, u) \in K^{*0} \times C^* \times V \times S : (y^*g)(u) \geq 0, g(u) \in -C, \\ &\langle v^*, v\rangle \leq (v^*f)(u), 0 \in \partial((v^*f) + (y^*g) + \delta_S)(u)\}\end{aligned}$$

and

$$h_{C_L}^{M}(v^*, y^*, v, u) = v.$$

Note that in the constraints of this dual one can replace $(y^*g)(u) \geq 0$ by $(y^*g)(u) = 0$ without altering anything since $g(u) \in -C$ and $y^* \in C^*$. Like in [2, 4, 7], from $(DV_{C_L}^M)$ we remove the constraint $g(u) \in -C$, obtaining a new vector dual to (PV_C), further referred to as the *Mond-Weir vector dual of Lagrange type*

$$(DV_{C_L}^{MW}) \qquad \underset{(v^*,y^*,v,u)\in \mathcal{B}_{C_L}^{MW}}{\text{Max}} h_{C_L}^{MW}(v^*, y^*, v, u)$$

where

$$\begin{aligned}\mathcal{B}_{C_L}^{MW} = \{&(v^*, y^*, v, u) \in K^{*0} \times C^* \times V \times S : (y^*g)(u) \geq 0, \\ &\langle v^*, v\rangle \leq (v^*f)(u), 0 \in \partial((v^*f) + (y^*g) + \delta_S)(u)\}\end{aligned}$$

and

$$h_{C_L}^{MW}(v^*, y^*, v, u) = v.$$

Remark 3.1. Denote $\Delta_{X^3} = \{(x, x, x) : x \in X\}$. If one of the following conditions (see [7])

(i) f and g are continuous at a point in $\operatorname{dom} f \cap \operatorname{dom} g \cap S$;

(ii) $\operatorname{dom} f \cap \operatorname{int}(S) \cap \operatorname{dom} g \neq \emptyset$ and f or g is continuous at a point in $\operatorname{dom} f \cap \operatorname{dom} g$;

(iii) X is a Fréchet space, S is closed, f is $K-$lower semicontinuous, g is $C-$lower semicontinuous and $0 \in \operatorname{sqri}(\operatorname{dom} f \times S \times \operatorname{dom} g - \Delta_{X^3})$;

(iv) $\dim(\operatorname{lin}(\operatorname{dom} f \times S \times \operatorname{dom} g - \Delta_{X^3})) < +\infty$ and $\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri}(S) \cap \operatorname{ri}(\operatorname{dom} g) \neq \emptyset$; is satisfied, then, for all $v^* \in K^{*0}$ and all $y^* \in C^*$, it holds

$$\partial((v^* f) + (y^* g) + \delta_S)(x) = \partial(v^* f)(x) + \partial(y^* g)(x) + N_S(x) \text{ for all } x \in X.$$

Consequently, when one of these situations occurs the constraint involving the subdifferential in $(DV_{C_L}^W)$, $(DV_{C_L}^M)$ and $(DV_{C_L}^{MW})$ can be modified correspondingly.

Remark 3.2. A vector dual similar to $(DV_{C_L}^W)$, but with respect to weakly efficient solutions, was introduced in [10], under quasidifferentiability hypotheses for the functions involved. Later, it was mentioned also in [13], where the functions were taken differentiable.

Let us compare now the image sets of these vector duals.

Proposition 3.3. *One has* $h_{C_L}^M(\mathcal{B}_{C_L}^M) \subseteq h_{C_L}^{MW}(\mathcal{B}_{C_L}^{MW}) \subseteq h_{C_L}^W(\mathcal{B}_{C_L}^W)$.

Proof. The first inclusion is a consequence of the way $(DV_{C_L}^{MW})$ is constructed, while the second one follows since $\langle v^*, v - (v^* f)(u)\rangle \leq 0 \leq (y^* g)(u)$ yields $\langle v^*, v - (v^* f)(u)\rangle \leq (y^* g)(u)$. □

Remark 3.4. The inclusions in Proposition 3.3 are in general strict, as the following examples show.

Example 3.5. Let $X = \mathbb{R}$, $Y = \mathbb{R}$, $V = \mathbb{R}^2$, $C = \mathbb{R}_+$, $K = \mathbb{R}^2_+$, $S = \mathbb{R}_+$, $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x, x)^T$ and $g : X \to \mathbb{R} \cup \{+\infty\}$,

$$g(x) = \begin{cases} -x, & \text{if } x > 0 \\ 1, & \text{if } x = 0 \\ +\infty, & \text{if } x < 0. \end{cases}$$

Then $g(x) \neq 0$ for all $x \in \mathbb{R}$ and to obtain $(y^* g)(u) = 0$ for some feasible $u \geq 0$ it is binding to have $y^* = 0$. Since when $u > 0$ and $v^* = (v_1^*, v_2^*)^T$ the subdifferential of the function $(v^* f + 0g + \delta_S)(\cdot) = (v_1^* + v_2^*)(\cdot) + \delta_{\mathbb{R}_+}(\cdot)$ at u is the set $\{v_1^* + v_2^*\}$, the only eligible element for $\mathcal{B}_{C_L}^M$ would be $u = 0$, as $g(u) = +\infty$ when $u < 0$. But $g(0) = 1 \notin -C$, thus $\mathcal{B}_{C_L}^M = \emptyset$.

On the other hand, for $v^* = (1/2, 1/2)^T$ we have $0 \in \partial((v^* f) + (y^* g) + \delta_S)(0) = (-\infty, 1]$, $(y^* g)(u) = 0$ and for $v = (0, -1)$ we obtain that $\langle v^*, v\rangle - (v^* f)(u) = -1/2 < 0$. Thus $((1/2, 1/2)^T, 0, (0, -1), 0) \in \mathcal{B}_{C_L}^{MW}$ and $((1/2, 1/2)^T,\ 0, (0, -1), 0) \in \mathcal{B}_{C_L}^W$. Therefore $(0, -1) \in h_{C_L}^{MW}(\mathcal{B}_{C_L}^{MW}) \cap h_{C_L}^W(\mathcal{B}_{C_L}^W)$.

Consequently, $h_{C_L}^W(\mathcal{B}_{C_L}^W) \neq h_{C_L}^M(\mathcal{B}_{C_L}^M)$ and $h_{C_L}^{MW}(\mathcal{B}_{C_L}^{MW}) \neq h_{C_L}^M(\mathcal{B}_{C_L}^M)$.

Example 3.6. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $V = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $K = \mathbb{R}^2_+$, $S = \mathbb{R}_+$, $f : \mathbb{R} \to (\mathbb{R}^2)^\bullet$,

$$f(x) = \begin{cases} (1,1)^T x, & \text{if } x \leq 0, \\ \infty_{\mathbb{R}^2_+}, & \text{otherwise,} \end{cases}$$

and $g : \mathbb{R} \to \mathbb{R}^2$, $g(x) = (x-1, -x)^T$.

Like in [2, Example 2] one can show that $\mathcal{B}^{MW}_{C_L} = \emptyset$. On the other hand, for $v^* = (1/2, 1/2)^T$, $y^* = (0,0)$, $u = 0$ and $v = (0,0)^T$ the conditions involving them in $\mathcal{B}^W_{C_L}$ are satisfied, therefore $((1/2,1/2)^T, (0,0), (0,0)^T, 0) \in \mathcal{B}^W_{C_L}$ and $(0,0)^T \in h^W_{C_L}(\mathcal{B}^W_{C_L})$.

Consequently, $h^W_{C_L}(\mathcal{B}^W_{C_L}) \not\subseteq h^{MW}_{C_L}(\mathcal{B}^{MW}_{C_L})$.

In order to achieve strong duality for the vector duals of Lagrange type we attached to (PV_C), we need the fulfillment of some sufficient conditions. Particularizing (RCV^i), $i \in \{1,2,3,4\}$ one gets the following regularity conditions

$$(RCV^1_{C_L}) \;\Big|\; \exists x' \in \operatorname{dom} f \cap S \text{ such that } g(x') \in -\operatorname{int}(C);$$

$$(RCV^2_{C_L}) \;\left|\; \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces, } S \text{ is closed, } f \text{ is } K-\text{ lower} \\ \text{semicontinuous, } g \text{ is } C-\text{epi closed and} \\ 0 \in \operatorname{sqri}(g(\operatorname{dom} f \cap S \cap \operatorname{dom} g) + C); \end{array}\right.$$

$$(RCV^3_{C_L}) \;\left|\; \begin{array}{l} \dim\big(\operatorname{lin}(g(\operatorname{dom} f \cap S \cap \operatorname{dom} g) + C)\big) < +\infty \text{ and} \\ 0 \in \operatorname{ri}(g(\operatorname{dom} f \cap S \cap \operatorname{dom} g) + C); \end{array}\right.$$

and

$$(RCV^4_{C_L}) \;\left|\; \begin{array}{l} S \text{ is closed, } f \text{ is } K-\text{lower semicontinuous, } g \text{ is } C-\text{epi closed and} \\ \bigcup\limits_{y^* \in C^*} \operatorname{epi}((v^* f) + (y^* g) + \delta_S)^* \text{ is closed in the topology} \\ w(X^*, X) \times \mathbb{R} \text{ for all } v^* \in K^{*0}. \end{array}\right.$$

Particularizing the results from the general case, we obtain the following duality statements.

Theorem 3.7 (weak and strong duality for (PV_C) and $(DV^W_{C_L})$)**.** (a) *There are no* $x \in \mathcal{A}$ *and* $(v^*, y^*, v, u) \in \mathcal{B}^W_{C_L}$ *such that* $f(x) \leq_K h^W_{C_L}(v^*, y^*, v, u)$.

(b) *If* $\overline{x} \in \mathcal{A}$ *is a properly efficient solution to* (PV_C) *and one of the regularity conditions* $(RCV^i_{C_L})$, $i \in \{1,2,3,4\}$ *is fulfilled, then there exists* $(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u}) \in \mathcal{B}^W_{C_L}$ *efficient solution to* $(DV^W_{C_L})$ *such that* $f(\overline{x}) = h^W_{C_L}(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u})$.

Theorem 3.8 (weak and strong duality for (PV_C) and $(DV^M_{C_L})$)**.** (a) *There are no* $x \in \mathcal{A}$ *and* $(v^*, y^*, v, u) \in \mathcal{B}^M_{C_L}$ *such that* $f(x) \leq_K h^M_{C_L}(v^*, y^*, v, u)$.

(b) *If* $\overline{x} \in \mathcal{A}$ *is a properly efficient solution to* (PV_C) *and one of the regularity conditions* $(RCV^i_{C_L})$, $i \in \{1,2,3,4\}$ *is fulfilled, then there exists* $(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u}) \in \mathcal{B}^M_{C_L}$ *efficient solution to* $(DV^M_{C_L})$ *such that* $f(\overline{x}) = h^M_{C_L}(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u})$.

Analogously one can prove similar duality assertions for $(DV^{MW}_{C_L})$, too.

Theorem 3.9 (weak and strong duality for (PV_C) and $(DV^{MW}_{C_L})$)**.** (a) *There are no* $x \in \mathcal{A}$ *and* $(v^*, y^*, v, u) \in \mathcal{B}^{MW}_{C_L}$ *such that* $f(x) \leq_K h^{MW}_{C_L}(v^*, y^*, v, u)$.

(b) *If* $\overline{x} \in \mathcal{A}$ *is a properly efficient solution to* (PV_C) *and one of the regularity conditions* $(RCV^i_{C_L})$, $i \in \{1,2,3,4\}$ *is fulfilled, then there exists* $(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u}) \in \mathcal{B}^{MW}_{C_L}$ *efficient solution to* $(DV^{MW}_{C_L})$ *such that* $f(\overline{x}) = h^{MW}_{C_L}(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u})$.

Remark 3.10. The regularity condition in Theorems 3.7(b), 3.8(b) and 3.9(b) can be replaced by any condition which guarantees the stability of the optimization problem $\inf_{x\in\mathcal{A}}(\overline{v}^* f)(x)$ with respect to its Lagrange dual.

Remark 3.11. If $V = \mathbb{R}$ and $K = \mathbb{R}_+$, then the duals $(DV^W_{C_L})$, $(DV^M_{C_L})$ and $(DV^{MW}_{C_L})$ are nothing else than the scalar Wolfe and Mond-Weir dual problems of Lagrange type corresponding to (PV_C) considered in [2], respectively. They extend the classical Wolfe and Mond-Weir scalar dual problems from the literature (see [2, Remark 4]).

Another vector perturbation function we consider is the Fenchel-Lagrange type vector perturbation function $\Phi^V_{FL} : X \times X \times Y \to V^\bullet$ given by

$$\Phi^V_{C_{FL}}(x, t, y) = \begin{cases} f(x+t), & x \in S, g(x) \in y - C \\ \infty_K, & \text{otherwise.} \end{cases}$$

By construction, it is K-convex, too. Let $v^* \in K^{*0}$, $t^* \in X^*$, $y^* \in Y^*$, $v \in V$, $u \in S \subseteq X$, $t \in X$ and $y \in Y$. Following [4], we have $(0, t^*, y^*) \in \partial(v^*\Phi^V_{C_{FL}})(u, t, y)$ if and only if $u \in S$, $g(u) - y \in -C$, $y^* \in -C^*$, $t^* \in \partial(v^* f)(u + t) \cap (-\partial(-(y^* g) + \delta_S)(u))$ and $(y^* g)(u) = \langle t^*, t\rangle$. On the other hand, $\langle v^*, v\rangle \leq -(v^*\Phi^V_{C_{FL}})^*(0, t^*, y^*)$ is equivalent to $\langle v^*, v\rangle \leq \langle t^*, u\rangle - (v^* f)^*(t^*) - (-y^* g)(u)$ and $g(u) - y \in -C$. Noticing that the variable $y \in Y$ can be omitted and changing the sign of y^*, we obtain like in [2, 4] from (DVG^W) the following *Wolfe vector dual of Fenchel-Lagrange type* to (PV_C)

$$(DV^W_{C_{FL}}) \qquad \underset{(v^*,t^*,y^*,v,u,t)\in\mathcal{B}^W_{C_{FL}}}{\text{Max}} h^W_{C_{FL}}(v^*, t^*, y^*, v, u, t)$$

where

$$\begin{aligned} \mathcal{B}^W_{C_{FL}} = \{&(v^*, t^*, y^*, v, u, t) \in K^{*0} \times X^* \times C^* \times V \times S \times X : \langle v^*, v\rangle \leq \langle t^*, u\rangle \\ &-(v^* f)^*(t^*) + (y^* g)(u), t^* \in \partial(v^* f)(u + t) \cap (-\partial((y^* g) + \delta_S)(u))\} \end{aligned}$$

and

$$h^W_{C_{FL}}(v^*, t^*, y^*, v, u, t) = v.$$

Similarly with the Lagrange case, we obtain the vector dual problem that is a special case of (DVG^M), namely

$$(DV^M_{C_{FL}}) \qquad \underset{(v^*,y^*,v,u)\in\mathcal{B}^M_{C_{FL}}}{\text{Max}} h^M_{C_{FL}}(v^*, y^*, v, u)$$

where

$$\begin{aligned} \mathcal{B}^M_{C_{FL}} = \{&(v^*, y^*, v, u) \in K^{*0} \times C^* \times V \times S : (y^* g)(u) \geq 0, g(u) \in -C, \\ &\langle v^*, v\rangle \leq (v^* f)(u), 0 \in \partial(v^* f)(u) + \partial((y^* g) + \delta_S)(u)\} \end{aligned}$$

and

$$h^M_{C_{FL}}(v^*, y^*, v, u) = v.$$

Note that in its constraints one can replace $(y^* g)(u) \geq 0$ by $(y^* g)(u) = 0$ without altering anything since $g(u) \in -C$ and $y^* \in C^*$. Like in the other case, removing the constraint $g(u) \in -C$, one obtains another vector dual to (PV_C), further called the *Fenchel-Lagrange vector dual of Mond-Weir type*

$$(DV_{C_{FL}}^{MW}) \qquad \underset{(v^*,y^*,v,u)\in\mathcal{B}_{C_{FL}}^{MW}}{\text{Max}} h_{C_{FL}}^{MW}(v^*,y^*,v,u)$$

where

$$\begin{aligned}\mathcal{B}_{C_{FL}}^{MW} = & \{(v^*,y^*,v,u)\in K^{*0}\times C^*\times V\times S : (y^*g)(u)\geq 0, \\ & \langle v^*,v\rangle \leq (v^*f)(u), 0\in\partial(v^*f)(u)+\partial((y^*g)+\delta_S)(u)\}\end{aligned}$$

and

$$h_{C_{FL}}^{MW}(v^*,y^*,v,u)=v.$$

Remark 3.12. Like in Remark 3.1 we can formulate some conditions for separating the functions that appear together in the subdifferentials from the constraint sets of the Fenchel-Lagrange vector duals to (PV_C) (see [7, Section 3.5]).

Using the way these vector duals to (PV_C) are constructed, one gets the following inclusions, which turn out to be strict for the problems presented in Example 3.5 and, respectively, Example 3.6.

Proposition 3.13. *One has* $h_{C_{FL}}^{M}(\mathcal{B}_{C_{FL}}^{M})\subseteq h_{C_{FL}}^{MW}(\mathcal{B}_{C_{FL}}^{MW})\subseteq h_{C_{FL}}^{W}(\mathcal{B}_{C_{FL}}^{W})$.

In order to guarantee strong duality, one can particularize the regularity conditions (RCV^i), $i\in\{1,2,3,4\}$. For instance, (RCV^1) becomes

$$(RCV_{C_{FL}}^1) \left|\begin{array}{l}\exists x'\in\operatorname{dom} f\cap S \text{ such that } f \text{ is continuous at } x' \text{ and}\\ g(x')\in-\operatorname{int}(C)\end{array}\right.$$

and the others can be analogously obtained (see [4]).

From the general case we obtain the following weak and strong duality statements.

Theorem 3.14 (weak and strong duality for (PV_C) and $(DV_{C_{FL}}^{W})$)**.** (a) *There are no* $x\in\mathcal{A}$ *and* $(v^*,t^*,y^*,v,u,t)\in\mathcal{B}_{C_{FL}}^{W}$ *such that* $f(x)\leq_K h_{C_{FL}}^{W}(v^*,t^*,y^*,v,u,t)$.

(b) *If* $\overline{x}\in\mathcal{A}$ *is a properly efficient solution to* (PV_C) *and one of the regularity conditions* $(RCV_{C_{FL}}^i)$, $i\in\{1,2,3,4\}$ *is fulfilled, then there exists* $(\overline{v}^*,\overline{t}^*,\overline{y}^*,\overline{v},\overline{u},\overline{t})\in\mathcal{B}_{C_{FL}}^{W}$ *efficient solution to* $(DV_{C_{FL}}^{W})$ *such that* $f(\overline{x})=h_{C_{FL}}^{W}(\overline{v}^*,\overline{t}^*,\overline{y}^*,\overline{v},\overline{u},\overline{t})$.

Theorem 3.15 (weak and strong duality for (PV_C) and $(DV_{C_{FL}}^{M})$)**.** (a) *There are no* $x\in\mathcal{A}$ *and* $(v^*,y^*,v,u)\in\mathcal{B}_{C_{FL}}^{M}$ *such that* $f(x)\leq_K h_{C_{FL}}^{M}(v^*,y^*,v,u)$.

(b) *If* $\overline{x}\in\mathcal{A}$ *is a properly efficient solution to* (PV_C) *and one of the regularity conditions* $(RCV_{C_{FL}}^i)$, $i\in\{1,2,3,4\}$ *is fulfilled, then there exists* $(\overline{v}^*,\overline{y}^*,\overline{v},\overline{u})\in\mathcal{B}_{C_{FL}}^{M}$ *efficient solution to* $(DV_{C_{FL}}^{M})$ *such that* $f(\overline{x})=h_{C_{FL}}^{M}(\overline{v}^*,\overline{y}^*,\overline{v},\overline{u})$.

Analogously one can prove the following duality statements for $(DV_{C_{FL}}^{MW})$.

Theorem 3.16 (weak and strong duality for (PV_C) and $(DV_{C_{FL}}^{MW})$)**.** (a) *There are no* $x\in\mathcal{A}$ *and* $(v^*,y^*,v,u)\in\mathcal{B}_{C_{FL}}^{MW}$ *such that* $f(x)\leq_K h_{C_{FL}}^{MW}(v^*,y^*,v,u)$.

(b) *If* $\overline{x}\in\mathcal{A}$ *is a properly efficient solution to* (PV_C) *and one of the regularity conditions* $(RCV_{C_{FL}}^i)$, $i\in\{1,2,3,4\}$ *is fulfilled, then there exists* $(\overline{v}^*,\overline{y}^*,\overline{v},\overline{u})\in\mathcal{B}_{C_{FL}}^{MW}$ *efficient solution to* $(DV_{C_{FL}}^{MW})$ *such that* $f(\overline{x})=h_{C_{FL}}^{MW}(\overline{v}^*,\overline{y}^*,\overline{v},\overline{u})$.

Remark 3.17. The regularity condition in Theorems 3.14(b), 3.15(b) and 3.16(b) can be replaced by any condition which guarantees the stability of the optimization problem $\inf_{x\in\mathcal{A}}(\overline{v}^*f)(x)$ with respect to its Fenchel-Lagrange dual.

Remark 3.18. If $V = \mathbb{R}$ and $K = \mathbb{R}_+$, then the duals $(DV^W_{C_{FL}})$, $(DV^M_{C_{FL}})$ and $(DV^{MW}_{C_{FL}})$ are nothing else than the scalar Wolfe and Mond-Weir dual problems of Fenchel-Lagrange type corresponding to (PV_C) considered in [2], respectively.

3.2. **Unconstrained vector optimization problems.** Using the same framework as in Section 2, we consider the proper $K-$ convex vector functions $f : X \to V^\bullet$ and $h : Y \to V^\bullet$ and $A : X \to Y$ a linear continuous mapping such that $\operatorname{dom} f \cap A^{-1}(\operatorname{dom} h) \neq \emptyset$. The primal unconstrained vector optimization problem

$$(PV_A) \qquad \operatorname*{Min}_{x\in X}[f(x) + h(Ax)]$$

is a special case of (PVG) for $F = f + h \circ A$ and we consider the vector perturbation function $\Phi^V_A : X \times Y \to V^\bullet$ defined by $\Phi^V_A(x, y) = f(x) + h(Ax + y)$, which is K-convex by construction, too.

Using Φ^V_A, we obtain vector duals to (PV_A) which are special cases of (DVG^W) and (DVG^M), namely

$$(DV^W_A) \qquad \operatorname*{Max}_{(v^*,y^*,v,u,y)\in\mathcal{B}^W_A} h^W_A(v^*, y^*, v, u, y)$$

where

$$\begin{aligned}\mathcal{B}^W_A = \{&(v^*, y^*, v, u, y) \in K^{*0} \times Y^* \times V \times X \times Y : y^* \in (A^*)^{-1}(-\partial(v^*f)(u)) \\ &\cap\partial(v^*h)^*(Au + y) \text{ and } \langle v^*, v\rangle \leq -(v^*f)^*(-A^*y^*) + (v^*h)^*(y^*)\}\end{aligned}$$

and

$$h^W_A(v^*, y^*, v, u, y) = v$$

and, respectively,

$$(DV^M_A) \qquad \operatorname*{Max}_{(v^*,v,u)\in\mathcal{B}^M_A} h^M_A(v^*, v, u)$$

where

$$\begin{aligned}\mathcal{B}^M_A = \{&(v^*, v, u) \in K^{*0} \times V \times X : 0 \in (A^*)^{-1}(-\partial(v^*f)(u)) - \partial(v^*h)(Au) \\ &\text{and } \langle v^*, v\rangle \leq \langle v^*, f(u) + h(Au)\rangle\}\end{aligned}$$

and

$$h^M_A(v^*, v, u) = v.$$

For the primal vector problem (PV_A) and its Wolfe and Mond-Weir vector duals, namely (DV^W_A) and (DV^M_A) the weak and strong duality statements follow from the general case.

Back to (PV_C), seeing it as an unconstrained vector optimization problem, we can attach to it two vector dual problems generated by (DVG^W) and (DVG^M) by considering the Fenchel type vector perturbation function

$$\Phi^V_{C_F} : X \times Y \to V^\bullet,\ \Phi^V_{C_F}(x, y) = \begin{cases} f(x + y), & x \in \mathcal{A}, \\ \infty_K, & \text{otherwise,}\end{cases}$$

which turns out to be K-convex, too. Thus, we get a *Wolfe vector dual of Fenchel type* to (PV_C)

$$(DV^W_{C_F}) \qquad \underset{(v^*,y^*,v,u,y)\in\mathcal{B}^W_{C_F}}{\text{Max}} h^W_{C_F}(v^*,y^*,v,u,y)$$

where

$$\mathcal{B}^W_{C_F} = \{(v^*,y^*,v,u,y)\in K^{*0}\times Y^*\times V\times X\times X : \langle v^*,v\rangle \le \langle y^*,u\rangle - (v^*f)^*(y^*), y^*\in\partial(v^*f)(u+y)\cap(-N_{\mathcal{A}}(u))\}$$

and

$$h^W_{C_F}(v^*,y^*,v,u,y)=v,$$

and a *Mond-Weir vector dual of Fenchel type* to it

$$(DV^M_{C_F}) \qquad \underset{(v^*,v,u)\in\mathcal{B}^M_{C_F}}{\text{Max}} h^M_{C_F}(v^*,v,u)$$

where

$$\mathcal{B}^M_{C_F} = \{(v^*,v,u)\in K^{*0}\times V\times X : \langle v^*,v\rangle \le (v^*f)(u), 0\in\partial(v^*f)(u)+N_{\mathcal{A}}(u)\}$$

and

$$h^M_{C_F}(v^*,v,u)=v.$$

Remark 3.19. In the definition of $(DV^M_{C_F})$, the condition $g(u)\in -C$ does not appear explicitly. Thus we will not consider another Mond-Weir vector dual of Fenchel type to (PV_C) in this case.

From Lemma 2.4 one can derive the following statement.

Proposition 3.20. *One has* $h^M_{C_F}(\mathcal{B}^M_{C_F}) \subseteq h^W_{C_F}(\mathcal{B}^W_{C_F})$.

From the general case one can quickly obtain the weak and strong duality theorems.

Remark 3.21. If $V=\mathbb{R}$ and $K=\mathbb{R}_+$, then the duals $(DV^W_{C_F})$ and $(DV^M_{C_F})$ are nothing else than the scalar Wolfe and Mond-Weir dual problems of Fenchel type corresponding to (PV_C) considered in [2], respectively.

4. Comparisons between duals

In this section we compare the image sets of some of the Wolfe type and Mond-Weir type vector duals to (PV_C) with respect to the Lagrange, Fenchel and Fenchel-Lagrange type vector perturbation functions.

Theorem 4.1. *One has* $h^M_{C_{FL}}(\mathcal{B}^M_{C_{FL}}) \subseteq h^M_{C_L}(\mathcal{B}^M_{C_L})$ *and* $h^M_{C_{FL}}(\mathcal{B}^M_{C_{FL}}) \subseteq h^M_{C_F}(\mathcal{B}^M_{C_F})$.

Proof. As the objective functions of the three vector maximization dual problems are the same, being all equal to v, we show that the sets this vector can be taken from fulfill the desired inclusions.

Let (v^*,y^*,v,u) be feasible to $(DV^M_{C_{FL}})$. Then $0\in\partial(v^*f)(u)+\partial((y^*g)+\delta_S)(u)\subseteq \partial((v^*f)+(y^*g)+\delta_S)(u)$, $\langle v^*,v\rangle \le (v^*f)(u)$, $(y^*g)(u)\ge 0$ and $g(u)\in -C$. This yields that (v^*,y^*,v,u) is feasible to $(DV^M_{C_L})$, therefore the first inclusion we have to show works.

Since $\cup_{y^*\in C^*}\partial((y^*g)+\delta_S)(u)\subseteq N_{\mathcal{A}}(u)$, taking into consideration the way $\mathcal{A}$ is defined it also follows that (v^*,v,u) is feasible to $(DV^M_{C_F})$. Consequently, the second desired inclusion holds, too. □

Concerning the "MW" vector duals, one can easily prove, like above, the following statement.

Theorem 4.2. *One always has* $h^{MW}_{C_{FL}}(\mathcal{B}^{MW}_{C_{FL}}) \subseteq h^{MW}_{C_L}(\mathcal{B}^{MW}_{C_L})$.

However, the question if similar inclusions are valid also for the Wolfe type vector duals to (PV_C) has, like in the scalar case (see [2]), a negative answer, as the following examples show.

Example 4.3. Let $X = \mathbb{R}$, $Y = \mathbb{R}$, $V = \mathbb{R}^2$, $C = \mathbb{R}_+$, $K = \mathbb{R}^2_+$, $S = \mathbb{R}$, $f : \mathbb{R} \to (\mathbb{R}^2)^\bullet$,

$$f(x) = \begin{cases} (1,1)^T x, & \text{if } x > 0, \\ \infty_{\mathbb{R}^2_+}, & \text{otherwise,} \end{cases}$$

and

$$g : \mathbb{R} \to \mathbb{R},\ g(x) = \begin{cases} -x, & \text{if } x \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that for all $v^* = (v^*_1, v^*_2)^T \in \operatorname{int}(\mathbb{R}^2_+)$ and $y^* \geq 0$ one has

$$\partial((v^*f) + (y^*g) + \delta_S)(u) = \partial(v^*f)(u) = \begin{cases} \{v^*_1 + v^*_2\}, & \text{if } u > 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Consequently, $\mathcal{B}^W_{C_L} = \emptyset$.

On the other hand it can be shown that $((1/2,1/2)^T, 1, 1, (0,0)^T, 0, 1) \in \mathcal{B}^W_{C_{FL}}$, thus $(0,0)^T \in h^W_{C_{FL}}(\mathcal{B}^W_{C_{FL}})$. Indeed, for $v^* = (1/2,1/2)^T$, $t^* = 1$, $y^* = 1$, $v = (0,0)^T$, $u = 0$ and $t = 1$, the validity of the subdifferential constraint was proven in [4, Example 4.22], while the inequality constraint means $\langle v^*, v\rangle - \langle t^*, u\rangle + (v^*f)^*(t^*) + (y^*g)(u) = \langle (1/2,1/2)^T, (0,0)^T\rangle - \langle 1, 0\rangle + ((1/2,1/2)^T f)^*(1) + (1g)(0) = 0$, which is true. Consequently, $h^W_{C_{FL}}(\mathcal{B}^W_{C_{FL}}) \not\subseteq h^W_{C_L}(\mathcal{B}^W_{C_L})$.

Example 4.4. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $V = \mathbb{R}^2$, $C = \mathbb{R}_+$, $K = \mathbb{R}^2_+$,

$$S = \left\{ (x_1,x_2)^T \in \mathbb{R}^2 : 0 \leq x_1 \leq 2,\ \begin{array}{ll} 3 \leq x_2 \leq 4, & \text{if } x_1 = 0 \\ 1 \leq x_2 \leq 4, & \text{if } x_1 \in (0,2] \end{array} \right\},$$

$$f : \mathbb{R}^2 \to (\mathbb{R}^2)^\bullet,\ f(x_1,x_2) = \begin{cases} (1,1)^T x_2, & \text{if } x_1 \leq 0, \\ \infty_{\mathbb{R}^2_+}, & \text{otherwise,} \end{cases}$$

and $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x_1,x_2) = 0$.

For $v^* = (1/2,1/2)^T$, $y^* \in \mathbb{R}_+$, $v = (2,3)^T$ and $u = (0,3)$ we have $(0,0) \in \partial((v^*f) + (y^*g) + \delta_S)(u) \Leftrightarrow (0,0) \in \partial((((1/2,1/2)^T)f) + (y^*g) + \delta_S)(0,3) = \mathbb{R} \times (-\infty, 1]$ and $\langle v^*, v - f(u)\rangle + (y^*g)(u) = \langle (1/2,1/2)^T, (2,3)^T - f(0,3)\rangle + (y^*g)(0,3) < 0$. It follows that $((1/2,1/2)^T, y^*, (2,3)^T, (0,3)) \in \mathcal{B}^W_{C_L}$ and $(2,3)^T \in h^W_{C_L}(\mathcal{B}^W_{C_L})$. Analogously it can be shown that $(2,3)^T \in h^M_{C_L}(\mathcal{B}^M_{C_L})$, too.

Trying to find an element from $\mathcal{B}^W_{C_{FL}}$ such that $(2,3)^T \in h^W_{C_{FL}}(\mathcal{B}^W_{C_{FL}})$, leads to a contradiction (cf. [4, Example 4.21]). Consequently, $h^W_{C_L}(\mathcal{B}^W_{C_L}) \not\subseteq h^W_{C_{FL}}(\mathcal{B}^W_{C_{FL}})$.

On the other hand, due to the fact that g is everywhere equal to 0, the image sets of the the Lagrange type vector duals to (PV_C) obtained from (DVG_W) and (DVG_M) take as values only vectors with equal entries, so $(2,3)^T$ cannot belong there. Consequently, in general, $h^M_G(\mathcal{B}^M_G) \neq h^M_G(\mathcal{B}^M_G)$ and $h^W_G(\mathcal{B}^W_G) \neq h^G_W(\mathcal{B}^G_W)$.

Example 4.5. Consider again the situation from Example 3.5. Here $\mathcal{A} = (0, +\infty)$, $N_{\mathcal{A}}(u) = \{0\}$ for all $u \in \mathcal{A}$, $\partial(v^* f)(u) = \{v_1^* + v_2^*\}$ for all $v^* = (v_1^*, v_2^*)^T \in \operatorname{int}(\mathbb{R}_+^2)$ and $u \in \mathbb{R}$, thus $\partial(v^* f)(u + y) \cap (-N_{\mathcal{A}}(u)) = \emptyset$ for all $u \in S$ and all $y \in \mathbb{R}$. Consequently, $\mathcal{B}_{C_F}^W = \emptyset$.

On the other hand, it can be shown that $((1/2, 1/2)^T, 0, 0, (0, 0)^T, 0, 0) \in \mathcal{B}_{C_{FL}}^W$, thus $(0, 0)^T \in h_{C_{FL}}^W(\mathcal{B}_{C_{FL}}^W)$. Indeed, for $v^* = (1/2, 1/2)^T$, $t^* = 0$, $y^* = 0$, $v = (0, 0)^T$, $u = 0$ and $t = 0$, the subdifferential inclusion $t^* \in \partial(v^* f)(u + t) \cap (-\partial((y^* g) + \delta_S)(u)) = [0, 1]$ is fulfilled. Then, the inequality constraint means $\langle v^*, v\rangle - \langle t^*, u\rangle + (v^* f)^*(t^*) + (y^* g)(u) = \langle (1/2, 1/2)^T, (0, 0)^T\rangle - \langle 0, 0\rangle + ((1/2, 1/2)^T f)^*(0) + (0g)(0) = 0$, which is true. Therefore, $h_{C_{FL}}^W(\mathcal{B}_{C_{FL}}^W) \not\subseteq h_{C_F}^W(\mathcal{B}_{C_F}^W)$.

Example 4.6. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $V = \mathbb{R}^2$, $C = \mathbb{R}_+$, $K = \mathbb{R}_+^2$,

$$S = \left\{ (x_1, x_2)^T \in \mathbb{R}^2 : 0 \le x_1 \le 2, \begin{array}{ll} 3 \le x_2 \le 4, & \text{if } x_1 = 0 \\ 1 \le x_2 \le 4, & \text{if } x_1 \in (0, 2] \end{array} \right\},$$

$\mathcal{A} = \{0\} \times [3, 4]$, $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_2, x_2)^T$ and $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x_1, x_2) = x_1$.

For $v^* = (1/2, 1/2)^T$, $y^* = (0, 1)^T$, $v = (3, 3)^T$, $u = (0, 3)$ and $y = (0, 0)$ one can see that $y^* \in \partial(v^* f)(u + y) \cap (-N_{\mathcal{A}}(u)) \Leftrightarrow (0, 1)^T \in \partial((1/2, 1/2)^T f)(0, 3) \cap (-N_{\mathcal{A}}(0, 3)) = \{0\} \times (-\infty, 1]$ and $\langle v^*, v\rangle - \langle y^*, u\rangle + (v^* f)^*(y^*) = \langle (1/2, 1/2)^T, (3, 3)^T\rangle - \langle (0, 1)^T, (0, 3)\rangle + ((1/2, 1/2)^T f)^*(0, 1) = 0$. It follows that $((1/2, 1/2)^T, (0, 1)^T, (3, 3)^T, (0, 3), (0, 0)) \in \mathcal{B}_{C_F}^W$ and $(3, 3)^T \in h_{C_F}^W(\mathcal{B}_{C_F}^W)$. On the other hand, assuming that $(3, 3)^T \in h_{C_{FL}}^W(\mathcal{B}_{C_{FL}}^W)$ leads to a contradiction cf. [4, Example 4.23]. Consequently, $h_{C_F}^W(\mathcal{B}_{C_F}^W) \not\subseteq h_{C_{FL}}^W(\mathcal{B}_{C_{FL}}^W)$.

Remark 4.7. From the examples given above one can construct situations which demonstrate that in general no inclusion holds between the image sets of $(DV_{C_L}^W)$ and $(DV_{C_F}^W)$.

5. Conclusions and further challenges

Following the investigations from [2, 4], we propose two new duality schemes for general vector optimization problems for which properly efficient solutions are considered, based on the generalized scalar Wolfe and Mond-Weir duality schemes, by exploiting the vector duality approach from [8, 9, 11]. The new vector duals we propose to a general vector minimization problem have larger images than their counterparts introduced in [4] and we show for them weak and strong vector duality. Then, after particularizing the primal problem to be constrained, respectively unconstrained, we obtain as special cases new vector duals for each of these problems, via different vector perturbation functions. We also investigate which inclusions can be established between the vector duals assigned to the constrained vector minimization problem. Examples to illustrate that the newly introduced vector duals do not coincide and they are actually different to the ones from [4] are also given.

However, our investigations are far from being complete, some open questions remaining left for future research. For instance, it would be interesting to find weak sufficient conditions that guarantee the coincidence of some of the different vector dual problems we assigned to a primal vector optimization problem. The hypotheses that guarantee strong duality for the Fenchel-Lagrange type duals ensure their coincidence with the Fenchel and Lagrange corresponding duals, respectively, but they may be actually too strong. Another interesting question refers to the maximal sets of the vector duals we introduced in this paper. Do they always coincide with their counterparts from [4]? So far, we could not find any counterexamples to contradict this fact, but unfortunately neither a proof for it. Moreover, the approach we employ opens the gate towards further developing the Wolfe and Mond-Weir vector duality concepts towards using other scalarization functions than the linear ones, as has been done for the Fenchel-Lagrange duality in [3, 5, 7]. Last but not least, in the spirit of the classical Wolfe and Mond-Weir duality investigations, a legitimate question is how should one modify the hypotheses of the weak and strong duality statements when the convexity assumptions imposed on the involved functions are weakened to some generalized convexity notions like pseudoconvexity or quasiconvexity.

Analogously to the investigations in this paper one can introduce a similar vector duality scheme for a general vector optimization problem for which weakly efficient elements are considered and, consequently, for its special cases. We did not include them here since everything works in the same manner, the only changes consisting in reformulating the vector duals by taking the variable v^* from $K^* \setminus \{0\}$ and dealing only with weakly efficient solutions.

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