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# AN ABSTRACT POINT OF VIEW ON ITERATIVE APPROXIMATION OF FIXED POINTS OF NONSELF OPERATORS

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Abstract. In a previous paper by the third author [I. A. Rus, *An abstract point of view on iterative approximation of fixed points: impact on the theory of fixed point equations*, Fixed Point Theory **13** (2012), No. 1, 179–192], a new approach to fixed point iterative methods, based on the concept of admissible perturbation of a self operator, has been established. In continuation of that study, in the present paper we are concerned with the same problem but in the case of nonself operators.

## 1. Introduction

There exist a vast literature on the iterative approximation of fixed points, see for example [5] and [14] and the huge list of references therein. This literature includes explicit fixed point iterative methods as well as implicit fixed point iterative methods. The most important important explicit fixed point iterative methods are, see [5]: Picard iteration (also named sequence of successive approximations), Krasnoselskij iteration, Mann iteration and Ishikawa iteration. The main interest with respect to a certain fixed point iterative method is to obtain a (weak or strong) convergence theorem for a class of contractive operators defined in a concrete setting (generally, a Banach space with special geometric properties, see [14]). Starting with the paper by the third author [35], a new approach to fixed point iterative methods for self operators, based on the concept of admissible perturbation of an operator, has been established, see also [6].

The theory of admissible perturbations of an operator introduced in [35] opened a new direction of research and unified the most important aspects of the iterative approximation of fixed point for single valued self operators.

As the interest for nonself operators in fixed point theory is considerable and comes from their important applications in proving existence theorems for differential equations, see [30], in continuation of the study [35], in the present paper we are concerned with the same problem but in the case of nonself operators.

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## 2. Retractions. Retractible operators. Examples

In this section we shall present some notions, results and examples from the retraction theory (see , [1], [2], [10], [11], [12], [13], [20], [22], [36], ...).

Let $X$ be a nonempty set and $Y \subset X$ a nonempty subset of $X$. An operator $r : X \to Y$ is a *set retraction* if $r(X) = Y$ and $r\big|_Y = 1_Y$.
If $X$ is a structured set (partial ordered set, topological space,...) then $r : X \to Y$ is a *retraction* with respect to that structure if $r$ is a set retraction and $r$ is a morphism with respect to that structure (increasing, continuous,...). So, we can have set retractions, ordered set retractions, topological retractions etc.

An operator $f : Y \to X$ is *retractible* with respect to a retraction $r : X \to Y$ if $F_f = F_{r\circ f}$, where $F_f := \{x \in Y : f(x) = x\}$ denotes the set of fixed points of $f$.

**Example 2.1** (The radial retraction)**.** Let $(X, +, \mathbb{R}, \|\cdot\|)$ be a Banach space and $\overline{B}_R := \{x \in X : \|x\| \leq R\}$. The operator $r : X \to \overline{B}_R$ defined by

$$r(x) := \begin{cases} x & \text{if } \|x\| \leq R, \\ \dfrac{R}{\|x\|}x & \text{if } \|x\| \geq R, \end{cases}$$

is a set retraction.

Let $f : \overline{B}_R \to X$ be an operator such that:

$$x \in X,\ \|x\| = R,\ \lambda \in \mathbb{R}^*_+,\ f(x) = \lambda x \Rightarrow \lambda \leq 1. \tag{2.1}$$

Then $f$ is retractible with respect to the radial retraction $r$.

Condition (2.1) is called the Leray-Schauder boundary condition.

It is clear that the radial retraction is a continuous retraction, i.e., a topological retraction. If $X$ is a Hilbert space (see [19]), then the radial retraction is a nonexpansive operator.

**Example 2.2** (Ordered set retraction)**.** Let $(X, \leq)$ be a partially ordered set with the least element, 0. Let $Y \subset X$ be a nonempty subset such that:

(i) $0 \in Y$;

(ii) $(Y, \leq)$ is conditionally complete, i.e., if $Z \subset Y$ is a nonempty bounded subset, then there exist $\sup Z$ and $\inf Z$.

Let $r : X \to Y$ be defined by

$$r(x) := \begin{cases} x & \text{if } x \in Y \\ \sup_Y([0, x] \cap Y), & \text{if } x \in X \setminus Y. \end{cases}$$

The operator $r$ is an increasing set retraction, i.e., an ordered set retraction of $X$ onto $Y$.

Let $f : Y \to X$ be an operator such that:

$$x \in Y,\ f(x) \in X \backslash Y \Rightarrow \sup_Y([0, f(x)] \cap Y) \neq x.$$

Then the operator $f$ is retractible with respect to the retraction $r$.

**Example 2.3.** Let $(X, +, \mathbb{R})$ be a linear space and $Y \subset X$ a nonempty subset. A set retraction $r : X \to Y$ is by definition a *sunny retraction* if

$$r(r(x) + t(x - r(x))) = r(x), \forall x \in X \text{ and } \forall t \in \mathbb{R}_+.$$

(Note that this concept and term first appear in [12] and [29]). For example, if $X$ is a Banach space and $Y := \overline{B}_R$, then the radial retraction is a sunny retraction.

**Example 2.4.** If $(X, \to, \leq)$ is an ordered $L$-space and $f : X \to X$ is a weakly Picard operator, then $f^\infty : X \to F_f$, defined by

$$f^\infty(x) = x^*, \text{ where } x^* := \lim_{n\to\infty} f^n(x),$$

is a set retraction. Moreover, if $f$ is increasing, then $f^\infty$ is increasing, too, but if $f$ is continuous, then $f^\infty$ is not continuous, in general. (Recall, see [32], that $f$ is a weakly Picard operator if there exists $\lim\limits_{n\to\infty} f^n(x) \in F_f$, $f^\infty(x) := \lim\limits_{n\to\infty} f^n(x)$, for all $x \in X$).

**Problem 2.5.** Let $X$ be a structured set (e.g., partial ordered set, linear space, topological space etc.), $Y \subset X$ a nonempty subset of $X$ and $f : Y \to X$ an operator. Under which conditions there exists a retraction $r : X \to Y$ such that $f$ is retractible with respect to $r$ ?

## 3. Admissible perturbations of an non-self operator

Let $X$ be a nonempty set, $Y \subset X$ a subset of $X$, $G : Y \times Y \to Y$ an operator, $r : X \to Y$ a set retraction and $f : Y \to X$. We suppose that (see [35]):
$(A_1)$ $G(x, x) = x, \ \forall\, x \in Y$;
$(A_2)$ $G(x, y) = x \Rightarrow y = x$.
We consider the operator $f_{G,r} : Y \to Y$ defined by

$$f_{G,r}(x) := G(x, r \circ f(x)), \forall\, x \in Y.$$

We have

**Lemma 3.1.** *We suppose that:*
(i) *$G$ satisfies $(A_1)$ and $(A_2)$;*
(ii) *$f$ is retractible with respect to $r$.*
*Then*

$$F_{f_{G,r}} = F_f.$$

*Proof.* It is clear that $F_f \subset F_{f_{G,r}}$. Now, let $x \in F_{f_{G,r}}$. We have $G(x, r \circ f(x)) = x$. By $(A_2)$ it follows that $r \circ f(x) = x$. Since $f$ is retractible with respect to the retraction $r$, i.e., $F_f = F_{r \circ f}$ ., we have $f(x) = x$. □

**Definition 3.2.** If $G$ satisfies $(A_1)$ and $(A_2)$ and $f$ is retractible with respect to $r$, then by definition $f_{G,r}$ is called an *admissible perturbation* of $f$.

**Example 3.3.** Let $(X, +, \mathbb{R})$ be a linear space, $Y \subset X$ a nonempty convex subset, $\lambda \in ]0, 1[$, $f : Y \to X$ an operator, $r : X \to Y$ a set retraction and $G : Y \times Y \to Y$ be defined by

$$G(x, y) := (1 - \lambda)x + \lambda y.$$

If $f$ is retractible with respect to $r$, then $f_{G,r}$ is an admissible perturbation of the operator $f$.

**Example 3.4.** Let $(X, d)$ be a metric space and $W : X \times X \times [0,1] \to X$ an operator. If

$$d(u, W(x, y, \lambda) \leq \lambda d(u, x) + (1 - \lambda) d(u, y), \forall x, y, u \in X, \lambda \in [0,1],$$

then by definition $(X, d)$ is said to be endowed with the Takahashi's $W$-convex structure. Moreover, we suppose that

$$\lambda \in ]0,1[, W(x, y, \lambda) = x \Rightarrow y = x.$$

Let $\lambda \in ]0,1[$, $Y \subset X$ be a $W$-convex subset, $f : Y \to X$ and $G(x, y) := W(x, y, \lambda)$. Let also $r : Y \to X$ be a set retraction such that $f$ is retractible with respect to $r$. Then the operator $f_{G,r}$ is an admissible perturbation of the operator $f$.

## 4. Iterative algorithms in terms of admissible perturbations

Let $(X, \to)$ be an $L$-space, $Y \subset X$ a subset, $r : X \to Y$ a set retraction, and the operators $G$, $G_n : Y \times Y \to Y$, $n \in \mathbb{N}$. We suppose that $f_{G,r}$, $f_{G_{n,r}}$, $n \in \mathbb{N}$, are admissible perturbations of $f$ with respect to $r$.

In this section we present some examples of iterative algorithms in terms of admissible perturbations. For the case of self operators see [35].

**Example 4.1** (The $GK_r$-algorithm)**.** We consider the algorithm

$$x_0 \in Y,\ x_{n+1} = G(x_n, r \circ f(x_n)),\ n \in \mathbb{N}^*. \tag{4.1}$$

First of all, we note that

$$x_n = f^n_{G,r}(x_0),\ n \in \mathbb{N}.$$

By definition, the algorithm (4.1) is convergent if and only if

$$x_n \to x^*(x_0) \in F_f \quad \text{as } n \to \infty,\ \forall\, x_0 \in Y.$$

So, the algorithm (4.1) is convergent if and only if the operator $f_G$ is a WPO (see [32] and [33] for the theory of weakly Picard operators). If $f_{G,r}$ is an admissible perturbation of $f$ and $f_{G,r}$ is a WPO, then

$$f^\infty_{G,r} : X \to F_f,\ f^\infty_{G,r}(x) := x^*(x),\ \forall\, x \in Y,$$

is a set retraction.

The algorithm (4.1) is called the *Krasnoselskij algorithm corresponding to G and r or the $GK_r$-algorithm.*

**Remark 4.2.** For the classical Krasnoselskij algorithm see [5], [7], [14], [15], [24], [28], [34] and [42].

**Example 4.3** ($\mathbb{G}M_r$-algorithm)**.** Let us consider the following algorithm

$$x_0 \in Y,\ x_{n+1} := f_{G_{n,r}}(x_n) = G_n(x_n, r \circ f(x_n)),\ n \in \mathbb{N}. \tag{4.2}$$

By definition, the algorithm (4.2) is convergent if and only if

$$x_n \to x^*(x_0) \in F_f \quad \text{as } n \to \infty,\ \forall\, x_0 \in Y.$$

If the algorithm (4.2) is convergent, then as in the previous example, the operator

$$f^\infty_{\mathbb{G},r} : X \to F_f,\ f^\infty_{\mathbb{G},r}(x) := x^*(x),\ \forall\, x \in Y,$$

is a set retraction.
The algorithm (4.2) is called the *Mann algorithm corresponding to* $\mathbb{G} = (G_n)_{n\in\mathbb{N}^*}$ *and* $r$ or the $\mathbb{G}M_r$*-algorithm.*

**Remark 4.4.** For the classical Mann algorithm see [5], [14] and [15].

**Example 4.5** ($\mathbb{G}H_r$-algorithm)**.** Let $u \in X$. We consider the operator $f_{G_{n,r,u}} : Y \to Y$ defined by

$$f_{G_{n,r,u}}(x) := G_n(u, r\circ f(x)),\ x \in Y,\ n \in \mathbb{N}^*,$$

and the algorithm

$$x_0 \in Y,\ x_{n+1} = f_{G_{n,r,u}}(x_n),\ n \in \mathbb{N}^*. \tag{4.3}$$

We suppose that the algorithm (4.3) is convergent, i.e.,

$$x_n \to x^*(u, x_0) \in F_f,\ \text{as}\ \ n \to \infty,\ \forall\, u, x_0 \in Y.$$

In this case, we define the operator

$$f^{\infty}_{\mathbb{G}H_r} : Y \to F_f,\ f^{\infty}_{\mathbb{G}H_r}(x) := x^*(x, x).$$

We remark that $f^{\infty}_{\mathbb{G}H_r}$ is a set retraction.
The algorithm (4.3) is called *the Halpern algorithm corresponding to* $\mathbb{G} = (G_n)_{n\in\mathbb{N}}$ *and* $r$, or the $\mathbb{G}H_r$*-algorithm.*

**Remark 4.6.** *For the classical Halpern algorithm see* [5]*,* [14]*,* [15]*.*

**Example 4.7** ($\mathbb{G}_1\mathbb{G}_2I_r$-algorithm)**.** Let $G_{1n}, G_{2n} : Y \times Y \to Y$, $n \in \mathbb{N}$ be two sequences of operators and $r : X \to Y$ a set retraction. We suppose that $f_{G_{1n,r}}$, $f_{G_{2n,r}}$ are admissible perturbations with respect to $r$, for all $n \in \mathbb{N}^*$ and that the algorithm

$$x_0 \in Y,\ x_{n+1} = G_{2n}\left(x_n, r\circ f\left(G_{1n}(x_n, r\circ f(x_n)\right)\right),\ n \in \mathbb{N} \tag{4.4}$$

is convergent, i.e.,

$$x_n \to x^*(x_0) \in F_f\ \ \text{as}\ n \to \infty,\ \forall\, x_0 \in Y.$$

If we consider the operator

$$f^{\infty}_{\mathbb{G}_1\mathbb{G}_2r} : X \to F_f,\ f^{\infty}_{\mathbb{G}_1\mathbb{G}_2r}(x) := x^*(x),$$

then we note that the operator $f^{\infty}_{\mathbb{G}_1\mathbb{G}_2r}$ is a set retraction. Let us denote

$$f_{\mathbb{G}_1\mathbb{G}_2r}(x) := G_{2n}\left(x, r\circ f\left(G_{1n}(x, r\circ f(x)\right)\right).$$

The algorithm (4.4) is called the *Ishikawa algorithm corresponding to* $\mathbb{G}_i = (G_{in})_{n\in\mathbb{N}}$, $i = 1, 2$ *and to retraction* $r$, or the $\mathbb{G}_1\mathbb{G}_2I$*-algorithm.*

**Remark 4.8.** *For the classical Ishikawa algorithm see* [5]*,* [14]*,* [15]*.*

**Problem 4.9.** Study the convergence of the above algorithms, in terms of $Y \subset X$, $f : Y \to X$ and $G(G_n, G_{1n}, G_{2n})$, in the following cases for $X$:

- $X$ is a Hilbert space;
- $X$ is a Banach space;
- $X$ is a metric space.

In what follow we shall study the following problem:

**Problem 4.10.** *Which properties have the solutions of a fixed point equation for which a given iterative algorithm converges ?*

## 5. Data dependence

Let $(X,d)$ be a metric space, $Y \subset X$ a subset of $X$ and $f,g : Y \to X$ two operators. Let $f_{G,r}$ and $g_{G,r}$ be the corresponding admissible perturbations with respect to $r$ associated to $f$ and $g$, respectively.

Let us first remind, see [32], that a map $T : X \to X$ is a Picard operator (PO) if: (i) $F_T = \{x^*\}$; (ii) $T^n(x) \to x^*$, as $n \to \infty$ for all $x \in X$. The PO $T$ is called a $\psi$-PO with respect to $d$ if there exists an increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ continuous at 0 with $\psi(0) = 0$ and such that

$$d(x,x^*) \le \psi(d(x,T(x))),\ \forall x \in X. \tag{5.1}$$

We have

**Lemma 5.1.** *We suppose that:*

(i) $f_{G,r}$ *is a* $\psi - PO$ *with respect to* $d$*;*

(ii) *there exists* $\eta > 0$ *such that*

$$d(f(x),g(x)) \le \eta,\ \forall\, x \in Y;$$

(iii) *there exists* $l_2 > 0$ *such that*

$$d(G(x,y),G(x,z)) \le l_2 d(y,z), \forall\, x,y,z \in Y;$$

(iv) *there exists* $l_r > 0$ *such that*

$$d(r(x),r(y)) \le l_r d(x,y),\ \forall\, x,y \in Y;$$

(v) $F_g \neq \emptyset$.

*If we denote by* $x_f^*$ *the unique fixed point of* $f$*, then*

$$d(x_f^*,x_g^*) \le \psi(l_2 l_r \eta),\ \forall\, x_g^* \in F_g.$$

*Proof.* From (i) we have

$$d(x,x_f^*) \le \psi(d(x,f_{G,r}(x)),\ \forall\, x \in Y.$$

For $x = x_g^*$, we have

$$d(x_g,x_f^*) \le \psi(d(x_g^*,f_{G,r}(x_g^*))) = \psi(d(g_{G,r}(x_g^*),f_{G,r}(x_g^*))).$$

On the other hand, from (ii), (iii) and (iv) it follows that

$$\begin{aligned} d(g_{G,r}(x),f_{G,r}(x)) &= d(G(x,r\circ g(x)),G(x,r\circ f(x))) \\ &\le l_2 d(r\circ g(x),r\circ f(x)) \le l_2 l_r d(f(x),g(x)) \le l_2 l_r \eta. \end{aligned}$$

So,

$$d(x_g^*,x_f^*) \le \psi(l_2 l_r \eta).$$

□

In order to state and prove the next Lemma we shall need the following condition that appears in the WPO theory [32]: let $(X, d)$ be a metric space and let $f : X \to X$ be a $\psi$-WPO, i.e., a mapping $f$ which satisfies

$$d(x, f^{\infty}(x)) \leq \psi(d(x, f(x)), \forall x \in X, \tag{5.2}$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous at 0 and $\psi(0) = 0$.

**Lemma 5.2.** *We suppose that:*
(i) $f_{G,r}$ *and* $g_{G,r}$ *are* $\psi$*-WPOs;*
(ii) *there exists* $\eta > 0$ *such that*

$$d(f(x), g(x)) \leq \eta, \forall x \in Y;$$

(iii) *there exists* $l_2 > 0$ *such that*

$$d(G(x, y), G(x, z)) \leq l_2 d(y, z), \forall x, y, z \in Y;$$

(iv) *there exists* $l_r > 0$ *such that*

$$d(r(x), r(y)) \leq l_r d(x, y), \forall x, y \in Y.$$

*Then,*

$$H_d(F_f, F_g) \leq \psi(l_2 l_r \eta).$$

*Proof.* We have

$$d(g_{G,r}(x), f_{G,r}(x)) \leq l_2 l_r \eta.$$

Now the proof follows from the corresponding data dependence lemma for $\psi$-WPOs, see [32]. ☐

Now we give some applications of the above lemmas.

**Example 5.3.** *Let* $X$ *be a real Hilbert space,* $Y \subset X$ *a closed convex subset of* $X$ *and* $f : Y \to X$ *a contraction with contraction coefficient* $c$.

*Let* $r : X \to Y$ *be the metric projection of* $X$ *onto* $Y$*, defined by*

$$\|x - r(x)\| := \min_{y \in Y} \|x - y\|, \text{ for all } x \in X.$$

*It is known, see for example* [26]*, that* $r$ *is a nonexpansive set retraction. Consider now the Krasnoselskij averaged mapping*

$$G(x, y) = \lambda x + (1 - \lambda)y, \; x, y \in Y,$$

*with* $\lambda \in (0, 1)$*. Then the admissible perturbation associated to* $f$ *and* $G$ *with respect to* $r$*, given by*

$$f_{G,r}(x) = \lambda x + (1 - \lambda)r(f(x)), x \in X,$$

*is a* $\psi$*-PO with* $\psi(t) = \dfrac{1}{(1 - c)(1 - \lambda)} t, \; t \in \mathbb{R}_+$.

*Indeed, since* $r$ *is nonexpansive and* $f$ *is a* $c$*-contraction, we have*

$$\|f_{G,r}(x) - f_{G,r}(y)\| = \|\lambda(x - y) + (1 - \lambda)[r(f(x)) - r(f(y))]\|$$
$$\leq \lambda \|x - y\| + (1 - \lambda) \|f(x) - f(y)\| \leq (\lambda + c - c\lambda) \|x - y\|.$$

*If we also assume that*

$$f(\partial K) \subset K$$

*then $f$ is retractible with respect to $r$. Since $r \circ f$ is $c$-contraction, it follows that $f$ has a unique fixed point in $Y$, say $x_f^*$, and that the $GK_r$-algorithm $\{x_n\}$, $x_n = f_{G,r}^n(x_0)$, converges strongly to $x_f^*$, for any $x_0 \in Y$.*

*Now, let $g : Y \to X$ be a mapping with $F_g \neq \emptyset$ for which there exists $\eta > 0$ such that (ii) in Lemma 5.1 is satisfied.*

*Then we have the following data dependence result:*

$$\|x_f^* - x_g^*\| \leq \frac{\lambda \eta}{(1-c)(1-\lambda)}, \ \forall x_g^* \in F_g.$$

*To prove the previous assertion we apply Lemma 5.1 and use the fact that in this case $l_2 = \lambda$ and $l_r = 1$.*

**Example 5.4.** *Using the same setting and notations as in the previous example and based on a theorem due to Browder and Petryshyn (see* [5]*, Theorem 3.3), we can similarly deduce a data dependence result for the case of nonself mappings.*

*Here we shall have a bounded closed convex subset $Y \subset X$, $f : Y \to X$ is a nonexpansive operator with a unique fixed point in $Y$ and the convergence of the $GK_r$-algorithm is only weak.*

**Example 5.5.** *Let $X$ be a real Hilbert space, $Y \subset X$ a nonempty closed convex subset of $X$ and $f : Y \to X$ a generalized pseudo-contraction with contraction coefficient $c \in (0, 1)$, see* [4]*, that is, a mapping satisfying*

$$\|f(x) - f(y)\|^2 \leq c^2 \|x - y\|^2 + \|f(x) - f(y) - c(x - y)\|^2$$

*for all $x, y \in Y$. Also suppose that $f$ is Lipschitzian with constant $L > 0$, and $c \leq L$.*

*If we consider now the Krasnoselskij averaged mapping*

$$G(x, y) = \lambda x + (1 - \lambda)y, \ x, y \in Y,$$

*with $\lambda \in (0, 1)$, then the admissible perturbation associated to $f$ and $G$ with respect to the metric projection $r$ of $X$ onto $Y$,*

$$f_{G,r}(x) = \lambda x + (1 - \lambda)r(f(x)), x \in X,$$

*is a $\psi$-PO with $\psi(t) = \dfrac{1}{1-\theta}t$, $t \in \mathbb{R}_+$, where $\theta = \dfrac{2(1-c)}{1 - 2L + L^2}$ (see* [4] *for details). If we also assume that*

$$f(\partial K) \subset K$$

*then by the main theorem in* [18]*, it follows that $f$ has a unique fixed point in $Y$, say $x_f^*$, and that the $GK_r$-algorithm $\{x_n\}$, $x_n = f_{G,r}^n(x_0)$, converges strongly to $x_f^*$, for any $x_0 \in Y$ and any $\lambda \in \left(0, \dfrac{2(1-c)}{1 - 2L + L^2}\right)$.*

*Let $g : Y \to X$ be a mapping with $F_g \neq \emptyset$ for which there exists $\eta > 0$ such that (ii) in Lemma 5.1 is satisfied.*

*Then, similarly to Example 5.3, we have the following data dependence result:*

$$\|x_f^* - x_g^*\| \leq \frac{\lambda \eta}{(1-c)(1-\lambda)}, \ \forall x_g^* \in F_g.$$

**Remark 5.6.** In the iterative approximation of fixed point the following condition appears (named condition $(D)$ in [5], condition $(A)$ in [40], [16], and condition $I$ in [34]; see also [39]).

Condition $(D)$ : $F_f \neq \emptyset$ and there exists an increasing function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\theta(0) = 0$, $\theta(t) > 0$ for $t > 0$ such that

$$d(x, f(x)) \geq \theta(d(x, F_f)),\ \forall\, x \in X. \tag{5.3}$$

Clearly, (5.2) is related to the $\psi$-PO and $\psi$-WPO conditions (5.1) and (5.2). For example, if the function $\theta$ is a bijection and $F_f = \{x^*\}$, then by $(D)$ we get exactly the $\psi$-PO condition (5.1) with $\psi := \theta^{-1}$.

**Problem 5.7.** Study the data dependence in the case of: (i) $\mathbb{G}M_r$-algorithm; (ii) $\mathbb{G}H_r$-algorithm; (iii) $\mathbb{G}_1\mathbb{G}_2 I_r$-algorithm.

## 6. Stability of an iterative algorithm

Taking into account the various notions of stability in Difference equations, Dynamical systems, Differential equations, Operator theory and Numerical analysis, in this section we try to unify these notions (see [4], [25], [35], ...).

Let $(X, d)$ be a metric space, $Y \subset X$ a subset of $X$, $f : Y \to X$ an operator, $r : X \to Y$ a set retraction and $G, G_n, G_{1n}, G_{2n} : Y \times Y \to Y$ be such that $f_G$, $f_{G_n}$, $f_{G_{1n}}$ and $f_{G_{2n}}$, $n \in \mathbb{N}$, are admissible perturbations of $f$.

**Definition 6.1.** The operator $f$ has a stable $GK_r$-sequence at $x_0 \in Y$ if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that:

$$x_0 \in Y,\ x_{n+1} = G(x_n, r \circ f(x_n)),\ n \in \mathbb{N},$$
$$y_0 \in Y,\ y_{n+1} = G(y_n, r \circ f(y_n)),\ n \in \mathbb{N},$$

and

$$d(x_0, y_0) < \delta(\varepsilon) \text{ implies that } d(x_n, y_n) < \varepsilon, \text{ for all } n \in \mathbb{N}.$$

The operator $f$ has stable $GK_r$-sequences on $Y$ if it has a stable $GK_r$-sequence at each $x_0 \in Y$.

The operator $f$ has attractive $GK_r$-sequence at $x_0$ if there exists $\delta > 0$ such that

$$d(x_0, y_0) < \delta \Rightarrow d(x_n, y_n) \to 0 \text{ as } n \to \infty.$$

**Definition 6.2.** The operator $f$ has a stable $\mathbb{G}M_r$-sequence at $x_0 \in Y$ if for every $\varepsilon < 0$, there exists $\delta(\varepsilon) > 0$ such that:

$$x_0 \in Y,\ x_{n+1} = f(x_n),\ n \in \mathbb{N}; \qquad y_0 \in Y,\ y_{n+1} = f_{G_n, r}(y_n),\ n \in \mathbb{N},$$

and

$$d(x_0, y_0) < \delta(\varepsilon) \text{ implies that } d(x_n, y_n) < \varepsilon, \text{ for all } n \in \mathbb{N}.$$

The operator $f$ has stable $\mathbb{G}M_r$-sequences on $Y$ if it has stable $\mathbb{G}M_r$-sequence at each $x_0 \in Y$.

The operator $f$ has attractive $\mathbb{G}M_r$-sequence at $x_0$ if there exists $\delta > 0$ such that

$$d(x_0, y_0) < \delta \Rightarrow d(x_n, y_n) \to 0 \text{ as } n \to \infty.$$

**Definition 6.3.** The operator $f$ has a stable $\mathbb{G}H_r$-sequence at $x_0 \in Y$ if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that:

$$x_0 \in Y,\ x_{n+1} = G_n(x_0, f(x_n)),\ n \in \mathbb{N},$$
$$y_0 \in Y,\ y_{n+1} = G_n(y_0, f(y_n)),\ n \in \mathbb{N},$$

and

$$d(x_0, y_0) < \delta(\varepsilon) \text{ implies that } d(x_n, y_n) < \varepsilon, \text{ for all } n \in \mathbb{N}.$$

The operator $f$ has stable $\mathbb{G}H_r$-sequences on $Y$ if is has a stable $\mathbb{G}H_r$-sequence at each $x_0 \in Y$.

The operator $f$ has attractive $\mathbb{G}H_r$-sequence at $x_0$ if there exists $\delta > 0$ such that

$$d(x_0, y_0) < \delta \Rightarrow d(x_n, y_n) \to 0 \text{ as } n \to \infty.$$

**Definition 6.4.** The operator $f$ has a stable $\mathbb{G}_1\mathbb{G}_2I_r$-sequence at $x_0 \in Y$ for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that:

$$x_0 \in Y,\ x_{n+1} = f_{G_{2n,r}} \circ f_{G_{1n,r}}(x_n),\ n \in \mathbb{N},$$
$$y_0 \in Y,\ y_{n+1} = f_{G_{2n,r}} \circ f_{G_{1n,r}}(y_n),\ n \in \mathbb{N},$$

and

$$d(x_0, y_0) < \delta(\varepsilon),\ \text{imply that } d(x_n, y_n) < \varepsilon,\ \text{for all } n \in \mathbb{N}.$$

The operator $f$ has stable $\mathbb{G}_1\mathbb{G}_2I_r$-sequences on $Y$ if it has a stable $\mathbb{G}_1\mathbb{G}_2I_r$-sequence at each $x_0 \in Y$.

The operator $f$ has attractive $\mathbb{G}_1\mathbb{G}_2I_r$-sequence at $x_0$ if there exists $\delta > 0$ such that

$$d(x_0, y_0) < \delta \Rightarrow d(x_n, y_n) \to 0 \text{ as } n \to \infty.$$

By definition, the operator $f$ has asymptotically stable sequence at $x_0$ generated by an algorithm if this sequence is both stable and attractive.

**Definition 6.5.** The operator $f$ has the limit shadowing property with respect to $GK_r$-algorithm if

$$y_n \in Y,\ n \in \mathbb{N},\ d(y_{n+1}, f_{G_r}(y_n)) \to 0 \text{ as } n \to \infty$$

implies that there exists $x_0 \in Y$ such that

$$d(y_n, f^n_{G_r}(x_0)) \to 0 \text{ as } n \to \infty.$$

**Definition 6.6.** The operator $f$ has the limit shadowing property with respect to $\mathbb{G}M_r$-algorithm if

$$y_n \in Y,\ n \in \mathbb{N},\ d(y_{n+1}, f_{G_{n,r}}(y_n)) \to 0 \text{ as } n \to \infty$$

implies that there exists $x_0 \in Y$ such that

$$d(y_n, f_{G_{n,r}} \circ f_{G_{n-1,r}} \circ \cdots \circ f_{G_{0,r}}(x_0)) \to 0 \text{ as } n \to \infty.$$

**Definition 6.7.** The operator $f$ has the limit shadowing property with respect to $\mathbb{G}H_r$– algorithm if

$$y_n \in Y,\ n \in \mathbb{N},\ d(y_{n+1}, G_n(y_0, r \circ f(y_n)) \to 0 \text{ as } n \to \infty$$

implies that there exists $x_0 \in Y$ such that

$$d(y_n, G_n(x_0, r \circ f(\cdot)) \circ \ldots G_0(x_0, r \circ f(x_0)) \to 0 \text{ as } n \to \infty.$$

**Definition 6.8.** The operator $f$ has the limit shadowing property with respect to $\mathbb{G}_1\mathbb{G}_2I_r$-algorithm if

$$y_n \in Y,\ n \in \mathbb{N},\ d(y_{n+1}, f_{G_{2n}G_{1n}r}(y_n)) \to 0 \text{ as } n \to \infty$$

implies that there exists $x_0 \in Y$ such that

$$d(y_n, f_{G_{2n}G_{1n}r} \circ \cdots \circ f_{G_{2n}G_{1n}r}(x_0)) \to 0 \text{ as } n \to \infty.$$

**Definition 6.9.** An iterative algorithm (Picard-retraction algorithm, $GK_r$-algorithm, $\mathbb{G}M_r$-algorithm, $\mathbb{G}H_r$-algorithm, $\mathbb{G}\mathbb{G}_2I_r$- algorithm) is stable with respect to an operator $f : Y \to X$ if it is convergent with respect to $f$ and the operator $f$ has the limit shadowing property with respect to this algorithm.

For a better understanding of the above definitions see the remarks and examples in section 6 of the paper [35].

**Problem 6.10.** Study the stability of: (i) $GK_r$-algorithm; (ii) $\mathbb{G}M_r$-algorithm; (iii) $\mathbb{G}H_r$-algorithm; (iv) $\mathbb{G}_1\mathbb{G}_2I_r$-algorithm.

## 7. Gronwall lemmas

We begin our considerations with the $GK_r$-algorithm.

Let $(X, \to, \leq)$ be an ordered $L$-space, $Y \subset X$ a subset of $X$, $f : Y \to X$ an operator and $f_{G,r}$ an admissible perturbation of $f$ with respect to the set retraction $r : X \to Y$, where . We have

**Lemma 7.1.** *We suppose that:*
(i) $f_{G,r}$ *is PO;*
(ii) $G, r$ *and* $f$ *are increasing.*
*Then,* $F_f = \{x^*\}$ *and*
(a) $x \in Y,\ x \leq f(x) \Rightarrow x \leq x^*$;
(b) $x \in Y,\ x \geq f(x) \Rightarrow x \geq x^*$.

*Proof.* The fact that $f_{G,r}$ is an admissible perturbation of $f$ implies that $f_{f_{G,r}} = F_f$. From (i) it follows that $f_{f_{G,r}} = \{x^*\}$. So, $F_f = \{x^*\}$. Now, let us prove (a). The condition (ii) implies that $f_{G,r}$ is increasing.

Let $x \in Y$ be such that $x \leq f(x)$. From (ii) and $(A_1)$ we have $x \leq r \circ f(x)$ and hence $x = G(x, x) \leq G(x, r \circ f(x)) = f_{G,r}(x)$.

The proof follows from Gronwall lemma for POs ([32], [33]). □

In a similar way we have

**Lemma 7.2.** *We suppose that:*
(i) $f_{G,r}$ *is a WPO;*
(ii) $G, r$ *and* $f$ *are increasing.*
*Then:*
(a) $x \in Y,\ x \leq f(x) \Rightarrow x \leq f^\infty_{G,r}(x)$;
(b) $x \in Y,\ x \geq f(x) \Rightarrow x \geq f^\infty_{G,r}(x)$.

**Remark 7.3.** In the above lemmas, instead of condition (ii) we can put the condition:
(ii') $G$, $f$ and $r \circ f$ are increasing.

**Remark 7.4.** The above results are partial answers to Problem 1 and Problem 2 in [33], which are presented for convenience in the following.

Let $(X, \leq)$ be an ordered set and let $f : X \to X$ be an operator.

**Problem 1** ([33]). *If $F_f = \{x^*\}$, in which conditions we have that:*

(a) $x \in X$, $x \leq f(x) \Rightarrow x \leq x^*$?

(b) $x \in X$, $x \geq f(x) \Rightarrow x \geq x^*$?

**Problem 2** ([33]). *If $F_f \neq \emptyset$, in which conditions there exists a set retraction $r : X \to F_f$ such that*

(a) $x \in X$, $x \leq f(x) \Rightarrow x \leq r(x)$?

(b) $x \in X$, $x \geq f(x) \Rightarrow x \geq r(x)$?

**Remark 7.5.** We have similar results in the cases of the algorithms: $\mathbb{G}M_r$, $\mathbb{G}H_r$ and $\mathbb{G}_1\mathbb{G}_2I$.

**Remark 7.6.** In some particular cases there are conditions which imply condition (i) in Lemma 7.1 and Lemma 7.2 (see [5], [14], [15], [3], [16], [21], [39], [44], [31], ...).

**Example 7.7.** *The following convergence result is well known:*

**Theorem 7.8** (Browder-Petryshyn [9])**.** *Let $X$ be a Hilbert space, $Y := \{x \in X : \|x\| \leq 1\}$, $G(x, y) := \lambda x + (1 - \lambda)y$, with $0 < \lambda < 1$, $r : X \to Y$ the radial retraction and $f : Y \to X$ an operator. We suppose that:*

(i) *$f$ is nonexpansive;*

(ii) *$f$ satisfies the Leray-Schauder boundary condition.*

*Then the $GK_r$-algorithm associated to $f$ is convergent.*

Corresponding to this convergence result we have the following Gronwall type result.

**Lemma 7.9.** *Let $X$ be an ordered Hilbert space in the sense that the ordered relation, $\leq$, and $X$ are such that the set $\{(x, y) \in X \times X : x \leq y\}$ is weakly closed. Let $f : Y \to X$ be such that:*

(i) *$f$ is as in Theorem 7.8;*

(ii) *$G$, $f$ and $r \circ f$ are increasing.*

*Then*

(a) *$x \in X$, $x \leq f(x) \Rightarrow x \leq f^{\infty}_{G,r}(x)$;*

(b) *$x \in X$, $x \geq f(x) \Rightarrow x \geq f^{\infty}_{G,r}(x)$.*

## 8. Comparison lemmas

Let $(X, \to, \leq)$ be an ordered $L$-space, $Y \subset X$, $f, g, h : Y \to X$ be three operators and $r : X \to Y$ a set retraction. Let $f_{G,r}$, $g_{G,r}$ and $h_{G,r}$ be the corresponding admissible perturbations associated to $f$, $g$ and $h$, respectively.

We have

**Lemma 8.1.** *We suppose that:*

(i) *$f_{G,r}$, $g_{G,r}$ and $h_{G,r}$ are POs;*

(ii) *$G, r$ and $g$ are increasing;*

(iii) $f \leq g \leq h$.
*Then:*
(a) $F_f = \{x_f^*\}$, $F_g = \{x_g^*\}$ *and* $F_h = \{x_h^*\}$*;*
(b) $x_f^* \leq x_g^* \leq x_h^*$.

*Proof.* From (i) it follows that

$$F_{f_{G,r}} = \{x_f^*\},\ F_{g_{G,r}} = \{x_g^*\} \text{ and } F_{h_{G,r}} = \{x_h^*\},$$

so,

$$F_f = \{x_f^*\},\ F_g = \{x_g^*\} \text{ and } F_h = \{x_h^*\}.$$

The condition implies that $g_{G,r}$ is increasing. From condition (iii) and (ii) we have that

$$f_{G,r} \leq g_{G,r} \leq h_{G,r}.$$

Now, the proof follows from the comparison lemma for WPOs ([32]). □

In a similar way we have

**Lemma 8.2.** *We suppose that:*
(i) $f_{G,r}$*,* $g_{G,r}$ *and* $h_{G,r}$ *are WPOs;*
(ii) $G, r$ *and* $g$ *are increasing;*
(iii) $f \leq g \leq h$.
*Then*

$$x, y, z \in Y,\ x \leq y \leq z \Rightarrow f_{G,r}^\infty(x) \leq g_{G,r}^\infty(y) \leq h_{G,r}^\infty(z).$$

**Remark 8.3.** In the above lemmas, instead of condition (ii) we can put the following condition:
(ii') $G$, $g$ and $r \circ g$ are increasing.

**Remark 8.4.** In the case of the convergence result of Browder-Petryshyn (Theorem 7.8) we have the following comparison result:

**Lemma 8.5.** *Let* $X$*,* $Y$ *and* $f, g, h : Y \to X$ *be as* $f$ *in Theorem 7.8. We suppose that:*
(i) $G$*,* $g$ *and* $r \circ g$ *are increasing;*
(ii) $f \leq g \leq h$.
*Then*
$x, y, z \in Y$, $x \leq y \leq z \Rightarrow f_{G,r}^\infty(x) \leq g_{G,r}^\infty(y) \leq h_{G,r}^\infty(z)$.

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## References

[1] Y. Alber, S. Reich and J.-C. Yao, *Iterative methods for solving fixed point problems with nonself-mappings in Banach spaces*, Abstr. Appl. Anal. **4** (2003), 193–216
[2] Y. Alber and J. L. Li, *The connection between the metric and generalized projection operators in Banach spaces*, Acta Math. Sinica **23** (2006), 1109–1120

[3] K. Aoyama, K. Eshita and W. Takahashi, *Iteration processes for nonexpansive mappings in convex metric spaces*, Proceed. Intern. Conf. Nonlinear Anal. Convex Anal. (Okinawa, 2005), W. Takahashi, T. Tanaka (eds). Yokohama Publishers, Yokohama, 2007, pp. 31–39
[4] V. Berinde, *Approximating fixed points of Lipschitzian pseudo-contractions*, in Mathematics & mathematics education(Bethlehem, 2000), World Sci. Publishing, River Edge, 2002, NJ, pp. 73–81
[5] V. Berinde, *Iterative Approximation of Fixed Points*, Lectures Notes in Math., 1912, Springer, Berlin, 2007.
[6] V. Berinde, *Convergence theorems for fixed point iterative methods defined as admissible perturbations of a nonlinear operator*, Carpathian J. Math. **29** (2013), 9–18
[7] V. Berinde and G. Kovacs, *Stabilizing discrete dynamical systems by monotone Krasnoselskij type iterative schemes*, Creat. Math. Inform. **17** (2008), 298–307
[8] F. E. Browder, *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Zeitschr. **100** (1967), 201–225
[9] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl. **20** (1967), 197–228
[10] R. F. Brown, *Retraction methods in Nielsen fixed point theory*, Pacific J. Math. **115** (1984), 277–297
[11] R. E. Bruck, *Nonexpansive retracts of Banach spaces*, Bull. Amer. Math. Soc. **76** (1970), 384–386
[12] R. E. Bruck, *Nonexpansive projections on subsets of Banach spaces*, Pacific J. Math. **47** (1973), 341–355
[13] L.-C. Ceng and J.-L. Ho, *Convergence criteria of generalized hybrid proximal point algorithms*, Fixed Point Theory **12** (2011), 29–48
[14] C. Chidume,*Geometric Properties of Banach Spaces and Nonlinear Iteration*, Lecture Notes in Mathematics, 1965, Springer, London, 2009.
[15] C. E. Chidume and Şt. Măsuşter, *Iterative methods for the computation of fixed points of demicontractive mappings*, J. Comput. Appl. Math. **234** (2010), 861–882
[16] C. E. Chidume, N. Shahzad and H. Zegeye, *Convergence theorems for mappings which are asymptotically nonexpansive in the intermediate sense*, Num. Funct. Anal. Optim. **25** (2004), 239–257
[17] A. Chiş-Novac, R. Precup and I. A. Rus, I. *Data dependence of fixed points for non-self generalized contractions*, Fixed Point Theory **10** (2009), 73–87
[18] Lj. Ćirić, *Quasi contraction non-self mappings on Banach spaces.* Bull. Cl. Sci. Math. Nat. Sci. Math. **23** (1998), 25–31
[19] D. G. de Figueiredo and L. A. Karlovitz, *On the radial projection in normed spaces*, Bull. Amer. Math. Soc. **73** (1967), 364–368
[20] D. Grundmane, *Fixed points for nonself mappings in metric spaces*, Math. Morav. **9** (2005), 5–11
[21] Y. Hao, *Iterative approximation of common fixed points for two nonexpansive non-self mappings*, Nonlinear Funct. Anal. Appl. **13** (2008), 775–780
[22] E. Kopecka and S. Reich, *Nonexpansive retracts in Banach spaces*, Banach Center Publications 77 (2007), 161–174
[23] Z. Liu, J. S. Ume and S. M. Kang, *Strong convergence and pseudostability for operators of the $\phi$-accretive type in uniformly smooth Banach spaces*, Rostock. Math. Kolloq. **59** (2005), 29–40
[24] D. St. Marinescu and M. Monea, *About Krasnoselskij iterative method*, Creat. Math. Inform. **22** (2013), (in print).
[25] Şt. Măruşter, *The stability of gradient-like methods*, Appl. Math. Comput. **177** (2001), 103–115.
[26] S. Matsushita and W. Takahashi, *Strong convergence of averaging iterations of nonexpansive nonself-mappings*, J. Math. Anal. Appl. **294** (2004), 206–214
[27] K. Nakajo and W. Takahashi, *A nonlinear strong ergodic theorem for asymptotically nonexpansive mappings with compact domains*, Dynamics of Continuous, Discrete and Impulsive Systems, Series A, **9** (2002), 257–270

[28] M. Păcurar, *Common fixed points for almost Presić type operators*, Carpathian J. Math. **28** (2012), 117–126.
[29] S. Reich, *Asymptotic behavior of contractions in Banach spaces*, J. Math. Anal. Appl. **44** (1973), 57–70
[30] S. Reich, *On fixed point theorems obtained from existence theorems for differential equations*, J. Math. Anal. Appl. **54** (1976), 26–36
[31] S. Reich and L. Shemen, *Two algorithms for nonexpansive mappings*, Fixed Point Theory **12** (2011), 443–448
[32] I. A. Rus, *Picard operators and applications*, Sci. Math. Jpn. **58** (2003), 191–219
[33] I. A. Rus, *Gronwall lemmas: ten open problems*, Sci. Math. Jpn. **70** (2009), 221–228
[34] I. A. Rus, *Some properties of the solutions of those equations for which the Krasnoselskii iteration converges*, Carpathian J. Math. **28** (2012), 329–336
[35] I. A. Rus, *An abstract point of view on iterative approximation of fixed points*, Fixed Point Theory **13** (2012), 179–192
[36] I. A. Rus, *Fixed Point Structure Theory*, Cluj Univ. Press, Cluj-Napoca, 2006
[37] H. F. Senter and W. G. Dotson, *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc. **44** (1974), 375–380
[38] T. Suzuki, *Browder's type convergence theorems for one-parameter semigroups of nonexpansive mappings in Hilbert spaces*, Proceed. Inter. Conf. Nonlinear Anal. Convex Anal. (Okinawa, 2005), W. Takahashi, T. Tanaka (eds.), Yokohama Publishers, Yokohama, 2007, pp. 599–607
[39] T. Suzuki, *Moudafi's viscosity approximations with Meir-Keeler contractions*, J. Math. Anal. Appl. **325** (2007), 342–352
[40] T. Suzuki, *Browder convergence and Mosco convergence for families of nonexpansive mappings*, CUBO **10** (2008), 101–108
[41] R. L. Thele, *Iterative techniques for approximation of fixed point of certain nonlinear mappings in Banach spaces*, Pacific J. Math. **53** (1974), 259–266
[42] F. Vetro, *On approximating curves associated with nonexpansive mappings*, Carpathian J. Math. **27** (2011), 142–147
[43] R. Wangkeeree, *Noor iterations with errors for non-Lipschitzian mappings in Banach spaces*, Proceed. Int. Conf. Nonlinear Anal. Convex Anal. (Okinawa, 2005), W. Takahashi, T. Tanaka (eds.), Yokohama Publishers, Yokohama, 2007, pp. 669–676
[44] Q. Ye and X. Xu, *A class of Newton-like methods with cubic convergence for nonlinear equations*, Fixed Point Theory **11** (2010), 161–168

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