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UNIQUE FIXED POINT THEOREMS FOR NONLINEAR MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we prove unique fixed point theorems for nonlinear mappings in Hilbert spaces. Using these results, we prove unique fixed point theorems for strict pseudo-contractions in Hilbert spaces. In particular, we obtain an extension of the famous strong convergence theorem with implicit iteration which was proved by Browder [4].

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A mapping $U : C \to H$ is called a *widely strict pseudo-contraction* [22] if there exists $r \in \mathbb{R}$ with $r < 1$ such that

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + r\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C.$$

We call such U a *widely r-strict pseudo-contraction.* If $0 \leq r < 1$, then U is a *strict pseudo-contraction* [5]. Furthermore, if $r = 0$, then U is *nonexpansive.* In 1967, Browder [4] proved the famous strong convergence theorem with implicit iteration in a Hilbert space.

Theorem 1.1 ([4]). *Let H be a Hilbert space, let C be a bounded closed convex subset of H and let T be a nonexpansive mapping of C into C. Let $u \in C$ and define a sequence $\{y_{\alpha_n}\}$ in C by*

$$y_{\alpha_n} = \alpha_n u + (1 - \alpha_n)Ty_{\alpha_n}, \quad \forall \alpha_n \in (0, 1).$$

Then $\{y_{\alpha_n}\}$ converges strongly to Pu as $\alpha_n \to 0$, where P is the metric projection of H onto $F(T)$.

If we replace a nonexpansive mapping T in Theorem 1.1 by a strict pseudo-contraction, does such a theorem hold?

In 2010, Kocourek, Takahashi and Yao [12] defined a class of nonlinear mappings in a Hilbert space. A mapping T from C into H is said to be *generalized hybrid* if there exist real numbers α and β such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for any $x, y \in C$. We call such a mapping (α, β)-*generalized hybrid.* Kawasaki and Takahashi [11] defined the following class of nonlinear mappings in a Hilbert space which covers contractive mappings and generalized hybrid mappings. A mapping

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T from C into H is said to be *widely generalized hybrid* if there exist real numbers $\alpha, \beta, \gamma, \delta, \varepsilon$ and ζ such that

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \leq 0$$

for any $x, y \in C$. Motivated by Kawasaki and Takahashi [11], Takahashi, Wong and Yao [23] introduced a more broad class of nonlinear mappings than the class of widely generalized hybrid mappings in a Hilbert space. A mapping $T : C \to C$ is said to be *symmetric generalized hybrid* [23] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + \beta(\|x - Ty\|^2 + \|Tx - y\|^2) + \gamma\|x - y\|^2 \\ + \delta(\|x - Tx\|^2 + \|y - Ty\|^2) \leq 0 \tag{1.1}$$

for all $x, y \in C$. Such a mapping T is also called *$(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid.* If $\alpha = 1$, $\beta = \delta = 0$ and $\gamma = -1$ in (1.1), then the mapping T is nonexpansive [19], i.e.,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

If $\alpha = 2$, $\beta = -1$ and $\gamma = \delta = 0$ in (1.1), then the mapping T is *nonspreading* [14], i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Furthermore, if $\alpha = 3$, $\beta = \gamma = -1$ and $\delta = 0$ in (1.1), then the mapping T is *hybrid* [20], i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

They proved the following fixed point theorem for symmetric generalized hybrid mappings in a Hilbert space.

Theorem 1.2 ([23]). *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta > 0$ *and* (3) $\delta \geq 0$ *hold. Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, \ldots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition* (1).

Furthermore, they introduced the following class of nonlinear mappings which contains the class of symmetric generalized hybrid mappings. A mapping T from C into C is called *symmetric more generalized hybrid* [23] if there exist $\alpha, \beta, \gamma, \delta, \zeta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + \beta(\|x - Ty\|^2 + \|Tx - y\|^2) + \gamma\|x - y\|^2 \\ + \delta(\|x - Tx\|^2 + \|y - Ty\|^2) + \zeta\|x - y - (Tx - Ty)\|^2 \leq 0 \tag{1.2}$$

for all $x, y \in C$. Such a mapping T is also called *$(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid.* They also proved the following fixed point theorem.

Theorem 1.3 ([23]). *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ *and* (3) $\delta + \zeta \geq 0$ *hold. Then T has a fixed point if and only if there exists*

$z \in C$ such that $\{T^n z : n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition (1).

In the case when the mappings in Theorems 1.2 and 1.3 have unique fixed points, what kind of iterations can we use to find such unique fixed points? This question is natural.

In this paper, motivated by Theorems 1.2 and 1.3, we prove unique fixed point theorems for symmetric generalized hybrid mappings and symmetric more generalized hybrid mappings in Hilbert spaces. Using these results, we prove unique fixed point theorems for strict pseudo-contractions in Hilbert spaces. In particular, we obtain an extension of the famous strong convergence theorem with implicit iteration which was proved by Browder [4].

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let A be a nonempty subset of H. We denote by $\overline{co}A$ the closure of the convex hull of A. In a Hilbert space, it is known that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$; see [19]. Furthermore, in a Hilbert space, we have that

$$2\langle x - y, z - w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2 \tag{2.2}$$

for all $x, y, z, w \in H$. Let C be a nonempty subset of H and let T be a mapping from C into H. We denote by $F(T)$ the set of fixed points of T. A mapping T from C into H with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|Tx - u\| \leq \|x - u\|$ for any $x \in C$ and $u \in F(T)$. A nonexpansive mapping with a fixed point is quasi-nonexpansive. It is well-known that if $T : C \to H$ is quasi-nonexpansive and C is closed and convex, then $F(T)$ is closed and convex; see Itoh and Takahashi [10]. It is not difficult to prove such a result in a Hilbert space. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \to z$. Since C is closed, we have $z \in C$. Furthermore, from

$$\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \to 0,$$

we have that z is a fixed point of T and hence $F(T)$ is closed. Let us show that $F(T)$ is convex. For $x, y \in F(T)$ and $\alpha \in [0, 1]$, put $z = \alpha x + (1-\alpha)y$. Then we have from (2.1) that

$$\begin{aligned}
\|z - Tz\|^2 &= \|\alpha x + (1-\alpha)y - Tz\|^2 \\
&= \alpha\|x - Tz\|^2 + (1-\alpha)\|y - Tz\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&\leq \alpha\|x - z\|^2 + (1-\alpha)\|y - z\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&= \alpha(1-\alpha)^2\|x-y\|^2 + (1-\alpha)\alpha^2\|x-y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&= \alpha(1-\alpha)(1-\alpha+\alpha-1)\|x-y\|^2 \\
&= 0.
\end{aligned}$$

This implies $Tz = z$. Thus $F(T)$ is convex. Let D be a nonempty closed convex subset of H and $x \in H$. We know that there exists a unique nearest point $z \in D$ such that $\|x - z\| = \inf_{y \in D} \|x - y\|$. We denote such a correspondence by $z = P_D x$. The mapping P_D is called the *metric projection* of H onto D. It is known that P_D is nonexpansive and

$$\langle x - P_D x, P_D x - u \rangle \geq 0$$

for all $x \in H$ and $u \in D$; see [19] for more details.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ or $\mu_n x_n$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n \to \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \to \infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [18] for the proof of existence of a Banach limit and its other elementary properties. Using means and the Riesz theorem, we can obtain the following result; see [16], [17] and [18].

Lemma 2.1. *Let H be a Hilbert space, let $\{x_n\}$ be a bounded sequence in H and let μ be a mean on l^∞. Then there exists a unique point $z_0 \in \overline{co}\{x_n : n \in \mathbb{N}\}$ such that*

$$\mu_n \langle x_n, y \rangle = \langle z_0, y \rangle, \quad \forall y \in H.$$

Let H be a Hilbert space and let C be a nonempty closed convex subset of H. A mapping $U : C \to H$ is called *extended hybrid* [8] if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} \alpha(1+\gamma)\|Ux - Uy\|^2 &+ (1 - \alpha(1+\gamma))\|x - Uy\|^2 \\ &\leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ &\quad - (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2 \end{aligned} \tag{2.3}$$

for all $x, y \in C$. Such a mapping U is called (α, β, γ)-*extended hybrid.* We know the following fixed point result for strict pseudo-contractions in a Hilbert space.

Lemma 2.2 ([21])**.** *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let k be a real number with $0 \leq k < 1$ and let $U : C \to H$ be a k-strict pseudo-contraction. Then, U is a (1,0,-k)-extended hybrid mapping and $F(U)$ is closed and convex. If, in addition, C is bounded and U is of C into itself, then $F(U)$ is nonempty.*

The following lemma was proved by Takahashi, Wong and Yao [22].

Lemma 2.3 ([22])**.** *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\alpha > 0$ and let A, U and T be mappings of C into H such that $U = I - A$ and $T = 2\alpha\, U + (1 - 2\alpha)I$. Then, the following are equivalent:*

(a) *A is an α-inverse-strongly monotone mapping, i.e.,*

$$\alpha\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay\rangle, \quad \forall x, y \in C;$$

(b) *U is a widely $(1-2\alpha)$-strict pseudo-contraction, i.e.,*

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + (1-2\alpha)\|(I-U)x - (I-U)y\|^2, \quad \forall x, y \in C;$$

(c) *U is a $(1, 0, 2\alpha - 1)$-extended hybrid mapping, i.e.,*

$$\begin{aligned} &2\alpha\|Ux - Uy\|^2 + (1-2\alpha)\|x - Uy\|^2 \\ &\quad \leq (2\alpha - 1)\|Ux - y\|^2 + 2(1-\alpha)\|x - y\|^2 \\ &\qquad - (2\alpha - 1)\|x - Ux\|^2 - (2\alpha - 1)\|y - Uy\|^2, \quad \forall x, y \in C; \end{aligned}$$

(d) *T is a nonexpansive mapping.*

Using Lemma 2.3, we obtain the following result.

Lemma 2.4 ([22]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let k be a real number with $k < 1$ and let A, U and T be mappings of C into H such that $U = I - A$ and $T = (1-k)U + kI$. Then, the following are equivalent:*

(a) *A is a $\frac{1-k}{2}$-inverse-strongly monotone mapping;*
(b) *U is a widely k-strict pseudo-contraction;*
(c) *U is a $(1, 0, -k)$-extended hybrid mapping;*
(d) *T is a nonexpansive mapping.*

The following lemma was also proved by Takahashi, Wong and Yao [21].

Lemma 2.5 ([21]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let α, β, γ be real numbers and let $U : C \to H$ be an (α, β, γ)-extended hybrid mapping with $1 + \gamma > 0$. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Using Lemmas 2.2 and 2.5, we have the following result obtained by Marino and Xu [15]; see also [1].

Lemma 2.6 ([15]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let k be a real number with $0 \leq k < 1$ and $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

3. Unique fixed point theorems without boundedness

We first prove the following unique fixed point theorem for symmetric generalized hybrid mappings in a HIlbert space.

Theorem 3.1. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions (1) $\alpha + 2\beta + \gamma > 0$, (2) $\beta \leq 0$, (3) $\beta + \gamma \leq 0$, and (4) $\beta + \delta \geq 0$ hold. Then*

(i) *T has a unique fixed point u in C;*
(ii) *for every $z \in C$, the sequence $\{T^n z\}$ converges to u.*

Proof. Let T be an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping of C into itself satisfying four conditions (1), (2), (3) and (4). Take $x \in C$. Replacing x by $T^n x$ and y by $T^{n+1}x$ in (1.1), we have that

$$\begin{aligned}\alpha\|T^{n+1}x - T^{n+2}x\|^2 + \beta(\|T^n x - T^{n+2}x\|^2 + \|T^{n+1}x - T^{n+1}x\|^2) \\ + \gamma\|T^n x - T^{n+1}x\|^2 + \delta(\|T^n x - T^{n+1}x\|^2 + \|T^{n+1}x - T^{n+2}x\|^2) \le 0\end{aligned} \tag{3.1}$$

for all $n \in \mathbb{N} \cup \{0\}$. From

$$\begin{aligned}\|T^n x - T^{n+2}x\|^2 &= \|T^n x - T^{n+1}x\|^2 + \|T^{n+1}x - T^{n+2}x\|^2 \\ &\quad + 2\langle T^n x - T^{n+1}x, T^{n+1}x - T^{n+2}x\rangle \\ &\le \|T^n x - T^{n+1}x\|^2 + \|T^{n+1}x - T^{n+2}x\|^2 \\ &\quad + 2\|T^n x - T^{n+1}x\|\|T^{n+1}x - T^{n+2}x\|\end{aligned}$$

and (2) $\beta \le 0$, we have that

$$\begin{aligned}\beta\|T^n x - T^{n+2}x\|^2 \ge \beta\|T^n x - T^{n+1}x\|^2 + \beta\|T^{n+1}x - T^{n+2}x\|^2 \\ + 2\beta\|T^n x - T^{n+1}x\|\|T^{n+1}x - T^{n+2}x\|.\end{aligned} \tag{3.2}$$

From (3.1) and (3.2) we have that

$$\begin{aligned}(\alpha + \beta)\|T^{n+1}x - T^{n+2}x\|^2 + (\beta + \gamma)\|T^n x - T^{n+1}x\|^2 \\ + 2\beta\|T^n x - T^{n+1}x\|\|T^{n+1}x - T^{n+2}x\| \\ + \delta(\|T^n x - T^{n+1}x\|^2 + \|T^{n+1}x - T^{n+2}x\|^2) \le 0.\end{aligned} \tag{3.3}$$

From (4) $\beta + \delta \ge 0$ we have that

$$2\beta\|T^n x - T^{n+1}x\|\|T^{n+1}x - T^{n+2}x\| \ge -2\delta\|T^n x - T^{n+1}x\|\|T^{n+1}x - T^{n+2}x\|.$$

From (3.3) we have that

$$\begin{aligned}(\alpha + \beta)\|T^{n+1}x - T^{n+2}x\|^2 + (\beta + \gamma)\|T^n x - T^{n+1}x\|^2 \\ - 2\delta\|T^n x - T^{n+1}x\|\|T^{n+1}x - T^{n+2}x\| \\ + \delta(\|T^n x - T^{n+1}x\|^2 + \|T^{n+1}x - T^{n+2}x\|^2) \le 0\end{aligned}$$

and hence

$$\begin{aligned}(\alpha + \beta)\|T^{n+1}x - T^{n+2}x\|^2 + (\beta + \gamma)\|T^n x - T^{n+1}x\|^2 \\ + \delta(\|T^n x - T^{n+1}x\| - \|T^{n+1}x - T^{n+2}x\|)^2 \le 0.\end{aligned}$$

Since $\delta \ge 0$ from (4), we obtain that

$$(\alpha + \beta)\|T^{n+1}x - T^{n+2}x\|^2 + (\beta + \gamma)\|T^n x - T^{n+1}x\|^2 \le 0. \tag{3.4}$$

Using (1) $\alpha + 2\beta + \gamma > 0$ and (3) $\beta + \gamma \le 0$, we obtain that $\alpha + \beta > -(\beta + \gamma) \ge 0$. Then we have from (3.4) that

$$\|T^{n+1}x - T^{n+2}x\|^2 \le \frac{-(\beta + \gamma)}{\alpha + \beta}\|T^n x - T^{n+1}x\|^2 \tag{3.5}$$

and

$$0 \le \frac{-(\beta + \gamma)}{\alpha + \beta} < 1. \tag{3.6}$$

Putting $\lambda = (\frac{-(\beta+\gamma)}{\alpha+\beta})^{\frac{1}{2}}$, we have that for any $n, m \in \mathbb{N}$ with $n \geq m$,

$$\begin{aligned}\|T^m x - T^n x\| &\leq \|T^m x - T^{m+1}x\| + \|T^{m+1}x - T^{m+2}x\| + \dots + \|T^{n-1}x - T^n x\| \\ &\leq \lambda^m \|x - Tx\| + \lambda^{m+1}\|x - Tx\| + \dots + \lambda^{n-1}\|x - Tx\| \\ &\leq \lambda^m \|x - Tx\| + \lambda^{m+1}\|x - Tx\| + \dots + \lambda^{n-1}\|x - Tx\| + \dots \\ &= \lambda^m \|x - Tx\|(1 + \lambda + \dots + \lambda^{n-1} + \dots) \\ &= \lambda^m \|x - Tx\| \frac{1}{1-\lambda}.\end{aligned}$$

Thus the sequence $\{T^n x\}$ is a Cauchy sequence. Since C is complete, the sequence $\{T^n x\}$ converges. Let $T^n x \to u$. We have from (1.1) that for every $x, u \in C$,

$$\begin{aligned}\alpha\|T^{n+1}x - Tu\|^2 &+ \beta(\|T^n x - Tu\|^2 + \|T^{n+1}x - u\|^2) \\ &+ \gamma\|T^n x - u\|^2 + \delta(\|T^n x - T^{n+1}x\|^2 + \|u - Tu\|^2) \leq 0\end{aligned} \tag{3.7}$$

for all $n \in \mathbb{N} \cup \{0\}$. Since $T^n x \to u$, we have that

$$\begin{aligned}\alpha\|u - Tu\|^2 &+ \beta(\|u - Tu\|^2 + \|u - u\|^2) \\ &+ \gamma\|u - u\|^2 + \delta(\|u - u\|^2 + \|u - Tu\|^2) \leq 0\end{aligned}$$

and hence

$$(\alpha + \beta + \delta)\|u - Tu\|^2 \leq 0.$$

From $\alpha + \beta > 0$ and $\delta \geq -\beta \geq 0$, we have that $\alpha + \beta + \delta > 0$. Thus we have that $\|u - Tu\|^2 \leq 0$ and hence $Tu = u$. Let p_1 and p_2 be fixed points of T. Then we have that

$$\begin{aligned}\alpha\|Tp_1 - Tp_2\|^2 + \beta(\|p_1 - Tp_2\|^2 + \|Tp_1 - p_2\|^2) + \gamma\|p_1 - p_2\|^2 \\ + \delta(\|p_1 - Tp_1\|^2 + \|p_2 - Tp_2\|^2) \leq 0\end{aligned}$$

and hence $(\alpha + 2\beta + \gamma)\|p_1 - p_2\|^2 \leq 0$. We have from $\alpha + 2\beta + \gamma > 0$ that $p_1 = p_2$. Therefore a fixed point of T is unique. This completes the proof. $\square$

Using Theorem 3.1, we prove the following fixed point theorem.

Theorem 3.2. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma > 0$, (2) $\beta \leq \zeta$, (3) $\beta + \gamma \leq 0$, *and* (4) $\beta + \delta \geq 0$ *hold. Then*

(i) *T has a unique fixed point u in C;*
(ii) *for every $z \in C$, the sequence $\{T^n z\}$ converges to u.*

Proof. Since $T : C \to C$ is an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping, there exist $\alpha, \beta, \gamma, \delta, \zeta \in \mathbb{R}$ satisfying (1.2). We also have that

$$\begin{aligned}\|x - y - (Tx - Ty)\|^2 &= \|x - Tx\|^2 + \|y - Ty\|^2 \\ &- \|x - Ty\|^2 - \|y - Tx\|^2 + \|x - y\|^2 + \|Tx - Ty\|^2\end{aligned} \tag{3.8}$$

for all $x, y \in C$. Thus we obtain from (1.2) that

$$(\alpha + \zeta)\|Tx - Ty\|^2 + (\beta - \zeta)(\|x - Ty\|^2 + \|Tx - y\|^2) \tag{3.9}$$

$$+ (\gamma+\zeta)\|x-y\|^2 + (\delta+\zeta)(\|x-Tx\|^2 + \|y-Ty\|^2) \leq 0.$$

The conditions $(\alpha+\zeta)+2(\beta-\zeta)+(\gamma+\zeta) > 0$ and $\beta-\zeta \leq 0$ in Theorem 3.1 are equivalent to (1) $\alpha+2\beta+\gamma > 0$ and (2) $\beta \leq \zeta$, respectively. Furthermore, the conditions $(\beta-\zeta)+(\gamma+\zeta) \leq 0$ and $(\beta-\zeta)+(\delta+\zeta) \geq 0$ in Theorem 3.1 are equivalent to (3) $\beta+\gamma \leq 0$ and (4) $\beta+\delta \geq 0$, respectively. Thus we have the desired result from Theorem 3.1. □

The following is an extension of Theorem 3.2.

Theorem 3.3. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\zeta)$-symmetric more generalized hybrid mapping from C into itself which satisfies the conditions* (1) $\alpha+2\beta+\gamma > 0$, (2) *there exists $\lambda \in [0,1)$ such that $(\alpha+\beta)\lambda+\zeta-\beta \geq 0$,* (3) $\beta+\gamma \leq 0$ *and* (4) $\beta+\delta \geq 0$. *Then*

(i) *T has a unique fixed point u in C;*
(ii) *for every $z \in C$, the sequence $\{(\lambda I+(1-\lambda)T)^n z\}$ converges to u.*

Proof. Let $\lambda \in [0,1) \cap \{\lambda : (\alpha+\beta)\lambda+\zeta-\beta \geq 0\}$ and define $S = (1-\lambda)T+\lambda I$. Since C is convex, S is a mapping from C into itself. Since $\lambda \neq 1$, we obtain that $F(S) = F(T)$. Moreover, from $T = \frac{1}{1-\lambda}S - \frac{\lambda}{1-\lambda}I$ and (2.1), we have that

$$\begin{aligned}
&\alpha \left\| \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x\right) - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right) \right\|^2 \\
&\quad + \beta \left\| x - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right) \right\|^2 + \beta \left\| \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x\right) - y \right\|^2 \\
&\quad + \gamma\|x-y\|^2 \\
&\quad + \delta \left\| x - \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x\right) \right\|^2 + \delta \left\| y - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right) \right\|^2 \\
&\quad + \zeta \left\| \left(x - \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x\right)\right) - \left(y - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right)\right) \right\|^2 \\
&= \alpha \left\| \frac{1}{1-\lambda}(Sx-Sy) - \frac{\lambda}{1-\lambda}(x-y) \right\|^2 \\
&\quad + \beta \left\| \frac{1}{1-\lambda}(x-Sy) - \frac{\lambda}{1-\lambda}(x-y) \right\|^2 \\
&+ \beta \left\| \frac{1}{1-\lambda}(Sx-y) - \frac{\lambda}{1-\lambda}(x-y) \right\|^2 + \gamma\|x-y\|^2 \\
&\quad + \delta \left\| \frac{1}{1-\lambda}(x-Sx) \right\|^2 + \delta \left\| \frac{1}{1-\lambda}(y-Sy) \right\|^2 \\
&\quad + \zeta \left\| \frac{1}{1-\lambda}(x-Sx) - \frac{1}{1-\lambda}(y-Sy) \right\|^2 \\
&= \frac{\alpha}{1-\lambda}\|Sx-Sy\|^2 + \frac{\beta}{1-\lambda}\|x-Sy\|^2
\end{aligned}$$

$$+\frac{\beta}{1-\lambda}\|Sx-y\|^2+\left(-\frac{\lambda}{1-\lambda}(\alpha+2\beta)+\gamma\right)\|x-y\|^2$$
$$+\frac{\delta+\beta\lambda}{(1-\lambda)^2}\|x-Sx\|^2+\frac{\delta+\beta\lambda}{(1-\lambda)^2}\|y-Sy\|^2$$
$$+\frac{\zeta+\alpha\lambda}{(1-\lambda)^2}\|(x-Sx)-(y-Sy)\|^2\le 0.$$

Therefore S is an $\left(\frac{\alpha}{1-\lambda},\frac{\beta}{1-\lambda},-\frac{\lambda}{1-\lambda}(\alpha+2\beta)+\gamma,\frac{\delta+\beta\lambda}{(1-\lambda)^2},\frac{\zeta+\alpha\lambda}{(1-\lambda)^2}\right)$-symmetric more generalized hybrid mapping. Furthermore, we obtain that

$$\frac{\alpha}{1-\lambda}+\frac{2\beta}{1-\lambda}-\frac{\lambda}{1-\lambda}(\alpha+2\beta)+\gamma=\alpha+2\beta+\gamma>0,$$
$$\frac{\zeta+\alpha\lambda}{(1-\lambda)^2}-\frac{\beta}{1-\lambda}=\frac{\lambda(\alpha+\beta)+\zeta-\beta}{(1-\lambda)^2}\ge 0,$$
$$\frac{\beta}{1-\lambda}-\frac{\lambda}{1-\lambda}(\alpha+2\beta)+\gamma=\frac{\beta+\gamma-\lambda(\alpha+2\beta+\gamma)}{1-\lambda}\le 0,$$
$$\frac{\beta}{1-\lambda}+\frac{\delta+\beta\lambda}{(1-\lambda)^2}=\frac{\beta+\delta}{(1-\lambda)^2}\ge 0.$$

Therefore by Theorem 3.2 we obtain the desired result. □

4. Unique fixed point theorems with boundedness

In this section, we first obtain a unique fixed point theorem with boundedness for symmetric generalized hybrid mappings in a Hilbert space.

Theorem 4.1. *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha+2\beta+\gamma>0$, (2) $\alpha+\beta+\delta>0$ *and* (3) $\delta\ge 0$ *hold. Then*

(i) *T has a unique fixed point u in C;*
(ii) *for every $z\in C$, a subsequence $\{T^{n_i}z\}$ of $\{T^n z\}$ converges to u.*

In particular, if $\beta+\gamma\le 0$, then $\{T^n z\}$ for all $z\in C$ converges to u.

Proof. Since C is bounded, $\{T^n z : n=0,1,\dots\}$ is bounded for all $z\in C$. For an $(\alpha,\beta,\gamma,\delta)$-symmetric generalized hybrid mapping T of C into itself, we have that

$$\alpha\|Tx-T^{n+1}z\|^2+\beta(\|x-T^{n+1}z\|^2+\|Tx-T^n z\|^2)+\gamma\|x-T^n z\|^2$$
$$+\delta(\|x-Tx\|^2+\|T^n z-T^{n+1}z\|^2)\le 0$$

for all $n\in\mathbb{N}\cup\{0\}$ and $x\in C$. Since $\{T^n z\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. Since $\mu_n\|Tx-T^n z\|^2=\mu_n\|Tx-T^{n+1}z\|^2$ and $\mu_n\|x-T^n z\|^2=\mu_n\|x-T^{n+1}z\|^2$, we have that

$$(\alpha+\beta)\mu_n\|Tx-T^n z\|^2+(\beta+\gamma)\mu_n\|x-T^n z\|^2$$
$$+\delta(\|x-Tx\|^2+\mu_n\|T^n z-T^{n+1}z\|^2)\le 0.$$

Furthermore, since

$$\mu_n\|Tx-T^n z\|^2=\|Tx-x\|^2+2\mu_n\langle Tx-x,x-T^n z\rangle+\mu_n\|x-T^n z\|^2,$$

we have that

$$(\alpha+\beta+\delta)\|Tx-x\|^2+2(\alpha+\beta)\mu_n\langle Tx-x,x-T^nz\rangle \\ +(\alpha+2\beta+\gamma)\mu_n\|x-T^nz\|^2+\delta\mu_n\|T^nz-T^{n+1}z\|^2\leq 0.$$

From (3) $\delta \geq 0$, we have that

$$(\alpha+\beta+\delta)\|Tx-x\|^2+2(\alpha+\beta)\mu_n\langle Tx-x,x-T^nz\rangle \\ +(\alpha+2\beta+\gamma)\mu_n\|x-T^nz\|^2\leq 0. \tag{4.1}$$

There exists $p \in H$ from Lemma 2.1 such that

$$\mu_n\langle y, T^nz\rangle = \langle y,p\rangle$$

for all $y \in H$. From (4.1) we have that

$$(\alpha+\beta+\delta)\|Tx-x\|^2+2(\alpha+\beta)\langle Tx-x,x-p\rangle \\ +(\alpha+2\beta+\gamma)\mu_n\|x-T^nz\|^2\leq 0.$$

Since C is closed and convex, we have that

$$p\in\overline{co}\{T^nz : n\in\mathbb{N}\}\subset C.$$

Putting $x = p$, we have from (4.1) that

$$(\alpha+\beta+\delta)\|Tp-p\|^2+(\alpha+2\beta+\gamma)\mu_n\|p-T^nz\|^2\leq 0. \tag{4.2}$$

We have from (2) $\alpha+\beta+\delta > 0$ and (1) $\alpha+2\beta+\gamma > 0$ that $\|Tp-p\|^2 = 0$ and $\mu_n\|p-T^nz\|^2 = 0$. This implies that p is a fixed point of T. Furthermore, from

$$\liminf_{n\to\infty}\|p-T^nz\|^2\leq\mu_n\|p-T^nz\|^2\leq\limsup_{n\to\infty}\|p-T^nz\|^2,$$

we have that a subsequence $\{T^{n_i}z\}$ of $\{T^nz\}$ converges to p.

Let p_1 and p_2 be fixed points of T. Then we have that

$$\alpha\|Tp_1-Tp_2\|^2+\beta(\|p_1-Tp_2\|^2+\|Tp_1-p_2\|^2)+\gamma\|p_1-p_2\|^2 \\ +\delta(\|p_1-Tp_1\|^2+\|p_2-Tp_2\|^2)\leq 0$$

and hence $(\alpha+2\beta+\gamma)\|p_1-p_2\|^2 \leq 0$. We have from $\alpha+2\beta+\gamma > 0$ that $p_1 = p_2$. Therefore a fixed point of T is unique. In particular, if $\beta+\gamma \leq 0$, then we have that $\alpha+\beta > -(\beta+\gamma) \geq 0$. We also have from (1.1) that

$$\alpha\|Tz-p\|^2+\beta(\|z-p\|^2+\|Tz-p\|^2)+\gamma\|z-p\|^2+\delta\|z-Tz\|^2\leq 0.$$

Since $\delta \geq 0$, we have that

$$\alpha\|Tz-p\|^2+\beta(\|z-p\|^2+\|Tz-p\|^2)+\gamma\|z-p\|^2\leq 0$$

and hence

$$(\alpha+\beta)\|Tz-p\|^2\leq-(\beta+\gamma)\|z-p\|^2.$$

From $\alpha+\beta > -(\beta+\gamma) \geq 0$, we have that

$$\|Tz-p\|^2\leq-\frac{\beta+\gamma}{\alpha+\beta}\|z-p\|^2.$$

Since $0 \leq -\frac{\beta+\gamma}{\alpha+\beta} < 1$, the sequence $\{T^nz\}$ for all $z \in C$ converges to p. This completes the proof. $\square$

Using Theorem 4.1, we prove the following fixed point theorem.

Theorem 4.2. *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma > 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ *and* (3) $\delta + \zeta \geq 0$ *hold. Then*

(i) *T has a unique fixed point u in C;*
(ii) *for every $z \in C$, a subsequence $\{T^{n_i} z\}$ of $\{T^n z\}$ converges to u.*

In particular, if $\beta + \gamma \leq 0$, then $\{T^n z\}$ for all $z \in C$ converges to u.

Proof. As in the proof of Theorem 3.2, we have that

$$(\alpha + \zeta)\|Tx - Ty\|^2 + (\beta - \zeta)(\|x - Ty\|^2 + \|Tx - y\|^2) + (\gamma + \zeta)\|x - y\|^2 + (\delta + \zeta)(\|x - Tx\|^2 + \|y - Ty\|^2) \leq 0. \tag{4.3}$$

The conditions (1) $\alpha + 2\beta + \gamma \geq 0$ and (2) $\alpha + \beta + \delta + \zeta > 0$ are equivalent to $(\alpha + \zeta) + 2(\beta - \zeta) + (\gamma + \zeta) \geq 0$ and $(\alpha + \zeta) + (\beta - \zeta) + (\delta + \zeta) > 0$, respectively. Furthermore, since (3) $\delta + \zeta \geq 0$ holds, we have the desired result from Theorem 4.1. Furthermore, since $\beta + \gamma \leq 0$ is equivalent to $(\beta - \zeta) + (\gamma + \zeta) \leq 0$, we have that $\{T^n z\}$ for all $z \in C$ converges to u. □

The following theorem is an extension of Theorem 4.2.

Theorem 4.3. *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping from C into itself which satisfies the conditions* (1) $\alpha + 2\beta + \gamma > 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ *and* (3) *there exists $\lambda \in [0, 1)$ such that $(\alpha + \beta)\lambda + \delta + \zeta \geq 0$. Then*

(i) *T has a unique fixed point u in C;*
(ii) *for every $z \in C$, a subsequence $\{(\lambda I + (1 - \lambda)T)^{n_i} z\}$ of $\{(\lambda I + (1 - \lambda)T)^n z\}$ converges to u.*

In particular, if $\beta + \gamma \leq 0$, then $\{(\lambda I + (1 - \lambda)T)^n z\}$ for all $z \in C$ converges to u.

Proof. Let $\lambda \in [0, 1) \cap \{\lambda : (\alpha + \beta)\lambda + \zeta + \eta \geq 0\}$ and define $S = (1 - \lambda)T + \lambda I$. Since C is convex, S is a mapping from C into itself. Since C is bounded, $\{S^n z : n = 0, 1, \ldots\}$ is bounded for any $z \in C$. Since $\lambda \neq 1$, we obtain that $F(S) = F(T)$. Moreover, as in the proof of Theorem 3.3, we have that S is an $\left(\frac{\alpha}{1-\lambda}, \frac{\beta}{1-\lambda}, -\frac{\lambda}{1-\lambda}(\alpha + 2\beta) + \gamma, \frac{\delta + \beta\lambda}{(1-\lambda)^2}, \frac{\zeta + \alpha\lambda}{(1-\lambda)^2}\right)$-symmetric more generalized hybrid mapping. Furthermore, we obtain that

$$\frac{\alpha}{1 - \lambda} + \frac{2\beta}{1 - \lambda} - \frac{\lambda}{1 - \lambda}(\alpha + 2\beta) + \gamma = \alpha + 2\beta + \gamma > 0,$$

$$\frac{\alpha}{1 - \lambda} + \frac{\beta}{1 - \lambda} + \frac{\delta + \beta\lambda}{(1 - \lambda)^2} + \frac{\zeta + \alpha\lambda}{(1 - \lambda)^2} = \frac{\alpha + \beta + \delta + \zeta}{(1 - \lambda)^2} > 0,$$

$$\frac{\delta + \beta\lambda}{(1 - \lambda)^2} + \frac{\zeta + \alpha\lambda}{(1 - \lambda)^2} = \frac{(\alpha + \beta)\lambda + \delta + \zeta}{(1 - \lambda)^2} \geq 0.$$

Therefore by Theorem 4.2 we obtain the desired result. Furthermore, if $\beta+\gamma \le 0$, then

$$\begin{aligned}\frac{\beta}{1-\lambda}-\frac{\lambda}{1-\lambda}(\alpha+2\beta)+\gamma &= \beta+\gamma-\frac{\lambda}{1-\lambda}(\alpha+\beta)\\ &\le \beta+\gamma+\frac{\lambda}{1-\lambda}(\beta+\gamma)\\ &\le 0.\end{aligned}$$

Thus $\{(\lambda I+(1-\lambda)T)^n z\}$ for all $z \in C$ converges to a unique fixed point u of T. $\square$

For the case $\beta+\delta=0$ in Theorem 4.3, we have the following theorem.

Theorem 4.4. *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be an $(\alpha,\beta,\gamma,-\beta,\zeta)$-symmetric more generalized hybrid mapping from C into itself, i.e., there exist $\alpha,\beta,\gamma,\zeta \in \mathbb{R}$ such that*

$$\begin{aligned}\alpha\|Tx-Ty\|^2+\beta(\|x-Ty\|^2+\|Tx-y\|^2)+\gamma\|x-y\|^2\\ -\beta(\|x-Tx\|^2+\|y-Ty\|^2)+\zeta\|x-y-(Tx-Ty)\|^2 \le 0\end{aligned} \tag{4.4}$$

for all $x,y \in C$. Furthermore, suppose that T satisfies the following conditions: (1) *$\alpha+2\beta+\gamma \ge 0$,* (2) *$\alpha+\zeta > 0$ and* (3) *there exists $\lambda \in [0,1)$ such that $(\alpha+\beta)\lambda-\beta+\zeta \ge 0$. Then*

(i) *T has a unique fixed point u in C;*
(ii) *for every $z \in C$, a subsequence $\{(\lambda I+(1-\lambda)T)^{n_i}z\}$ of $\{(\lambda I+(1-\lambda)T)^n z\}$ converges to u.*

In particular, if $\beta+\gamma \le 0$, then $\{(\lambda I+(1-\lambda)T)^n z\}$ for all $z \in C$ converges to u.

5. Applications

Using Theorem 3.1, we can first prove the following fixed point theorem.

Theorem 5.1. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let $T : C \to C$ be a contractive mapping, i.e., there exists a real number r with $0 \le r < 1$ such that*

$$\|Tx-Ty\| \le r\|x-y\| \tag{5.1}$$

for all $x,y \in C$. Then the following hold:

(i) *T has a unique fixed point u in C;*
(ii) *for every $z \in C$, the sequence $\{T^n z\}$ converges to u.*

Proof. We have from (5.1) that

$$\|Tx-Ty\|^2-r^2\|x-y\|^2 \le 0$$

for all $x,y \in C$. This implies that T is $(1,0,-r^2,0)$-symmetric generalized hybrid. For α,β,γ and δ in Theorem 3.1, we have that

$$\alpha+2\beta+\gamma = 1-r^2 > 0,\ \beta = 0 \le 0,\ \beta+\gamma = -r^2 \le 0 \text{ and } \beta+\delta = 0 \ge 0.$$

From Theorem 3.1, we have the desired result. $\square$

Let H be a real Hilbert space and let C be a nonempty subset of H. Then $U : C \to H$ is called a *contractively strict pseudo-contraction* if there exist $s \in [0,1)$ and $r \in \mathbb{R}$ with $0 \le r < 1$ such that

$$\|Ux - Uy\|^2 \le s\|x - y\|^2 + r\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C.$$

Using Theorem 3.3, we prove the following unique fixed point theorem.

Theorem 5.2. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let U be a contractively strict pseudo-contraction from C into itself, i.e., there exist $s \in [0,1)$ and $r \in \mathbb{R}$ with $0 \le r < 1$ such that*

$$\|Ux - Uy\|^2 \le s\|x - y\|^2 + r\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C. \tag{5.2}$$

Then the following hold:

(i) *U has a unique fixed point u in C;*
(ii) *for every $z \in C$, the sequence $\{(\lambda I + (1 - \lambda)U)^n z\}$ converges to u, where $r \le \lambda < 1$.*

Proof. In Theorem 3.3, we have that (1) $\alpha + 2\beta + \gamma = 1 - s > 0$, (2) $(\alpha + \beta)\lambda + \zeta - \beta = \lambda - r \ge 0$ for $\lambda \in [0,1)$ with $r \le \lambda < 1$, (3) $\beta + \gamma = -r \le 0$ and (4) $\beta + \delta = 0$. Thus we have desired result from Theorem 3.3. □

Using Theorem 3.1, we have the following theorem for strict pseudo-contractions in a Hilbert space.

Theorem 5.3. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be a strict pseudo-contraction from C into itself, i.e., there exists $r \in \mathbb{R}$ with $0 \le r < 1$ such that*

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + r\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{5.3}$$

Let $u \in C$ and $s \in (0,1)$ with $r \le s < 1$. Define a mapping $U : C \to C$ as follows:

$$Ux = su + (1 - s)Tx, \quad \forall x \in C.$$

Then U has a unique fixed point z in C. Furthermore, define a mapping $S : C \to C$ as follows:

$$Sx = rx + (1 - r)\big(su + (1 - s)Tx\big), \quad \forall x \in C.$$

Then, for all $x \in C$, the sequence $\{S^n x\}$ converges to a unique fixed point z.

Proof. From (5.3), we have that for any $x, y \in C$,

$$\|Tx - Ty\|^2 - \|x - y\|^2 - r\|x - y - (Tx - Ty)\|^2 \le 0. \tag{5.4}$$

For $u \in C$ and $s \in (0,1)$, define a mapping $S : C \to C$ as follows:

$$Sx = rx + (1 - r)\big(su + (1 - s)Tx\big), \quad \forall x \in C.$$

Since $Sx = rx + s(1 - r)u + (1 - r)(1 - s)Tx$, we have that for any $x \in C$,

$$Tx - Ty = \frac{Sx - Sy}{(1 - r)(1 - s)} - \frac{r(x - y)}{(1 - r)(1 - s)}$$

and

$$x - y - (Tx - Ty) = x - y - \left(\frac{Sx - Sy}{(1 - r)(1 - s)} - \frac{r(x - y)}{(1 - r)(1 - s)}\right)$$

$$= \left(1 + \frac{r}{(1-r)(1-s)}\right)(x-y) - \frac{Sx - Sy}{(1-r)(1-s)}.$$

Thus we have from (5.4) that

$$\frac{\|Sx-Sy\|^2}{(1-r)^2(1-s)^2} + \frac{r^2\|x-y\|^2}{(1-r)^2(1-s)^2} - \frac{2r}{(1-r)^2(1-s)^2}\langle x-y, Sx-Sy\rangle$$
$$-\|x-y\|^2 - r\left(1+\frac{r}{(1-r)(1-s)}\right)^2\|x-y\|^2 - \frac{r\|Sx-Sy\|^2}{(1-r)^2(1-s)^2}$$
$$+ 2r\left(1+\frac{r}{(1-r)(1-s)}\right)\frac{1}{(1-r)(1-s)}\langle x-y, Sx-Sy\rangle \le 0.$$

Then we have that

$$\frac{1}{(1-r)(1-s)^2}\|Sx-Sy\|^2 + \frac{r^2}{(1-r)^2(1-s)^2}\|x-y\|^2$$
$$-\|x-y\|^2 - r\left(1+\frac{r}{(1-r)(1-s)}\right)^2\|x-y\|^2$$
$$+ 2\left(\frac{r}{(1-r)(1-s)} - \frac{r(1-r)}{(1-r)^2(1-s)^2}\right)\langle x-y, Sx-Sy\rangle \le 0$$

and hence

$$\frac{1}{(1-r)(1-s)^2}\|Sx-Sy\|^2 + \frac{r^2}{(1-r)^2(1-s)^2}\|x-y\|^2$$
$$-\|x-y\|^2 - r\left(1+\frac{r}{(1-r)(1-s)}\right)^2\|x-y\|^2$$
$$-\frac{2rs}{(1-r)(1-s)^2}\langle x-y, Sx-Sy\rangle \le 0.$$

Since $2\langle x-y, Sx-Sy\rangle = \|x-Sy\|^2 + \|y-Sx\|^2 - (\|x-Sx\|^2 + \|y-Sy\|^2)$ and

$$\frac{r^2}{(1-r)^2(1-s)^2} - 1 - r\left(1+\frac{r}{(1-r)(1-s)}\right)^2$$
$$= \frac{r^2(1-r)}{(1-r)^2(1-s)^2} - 1 - r\left(1+\frac{2r}{(1-r)(1-s)}\right)$$
$$= \frac{r^2}{(1-r)(1-s)^2} - \frac{1-s+r^2(1+s)}{(1-r)(1-s)},$$

we have that

$$\frac{1}{(1-r)(1-s)^2}\|Sx-Sy\|^2 - \frac{rs}{(1-r)(1-s)^2}(\|x-Sy\|^2 + \|y-Sx\|^2)$$
$$+\left(\frac{r^2}{(1-r)(1-s)^2} - \frac{1-s+r^2(1+s)}{(1-r)(1-s)}\right)\|x-y\|^2$$
$$+\frac{rs}{(1-r)(1-s)^2}(\|x-Sx\|^2 + \|y-Sy\|^2) \le 0.$$

For this inequality, we apply Theorem 3.1. We first obtain that

$$\begin{aligned}\frac{1}{(1-r)(1-s)^2} - \frac{2rs}{(1-r)(1-s)^2} + \frac{r^2}{(1-r)(1-s)^2} - \frac{1-s+r^2(1+s)}{(1-r)(1-s)}\\ = \frac{1-2rs+r^2-(1-s)^2-r^2(1-s^2)}{(1-r)(1-s)^2}\\ = \frac{s(1-r)(2-s(1+r))}{(1-r)(1-s)^2} > 0.\end{aligned}$$

Furthermore, we have that $-\frac{rs}{(1-r)(1-s)^2} \leq 0$. From $r \leq s$, we also have that

$$-\frac{rs}{(1-r)(1-s)^2} + \frac{r^2}{(1-r)(1-s)^2} - \frac{1-s+r^2(1+s)}{(1-r)(1-s)} \leq 0.$$

Finally, we have that

$$-\frac{rs}{(1-r)(1-s)^2} + \frac{rs}{(1-r)(1-s)^2} = 0.$$

Thus S has a unique fixed point z in C from Theorem 3.1. Since z is a fixed point of S, we have $z = rz + (1-r)(su + (1-s)Tz)$. From $1 - r \neq 0$, we have that

$$z = su + (1-s)Tz.$$

From Theorem 3.1, we also have that for all $x \in C$, $\{S^n x\}$ converges strongly to a unique fixed point z. This completes the proof. □

Using Theorem 4.1, we can prove the following fixed point theorems.

Theorem 5.4. *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let $T : C \to C$ be contractively nonspreading, i.e., there exists a real number s with $0 \leq s < \frac{1}{2}$ such that*

$$\|Tx - Ty\|^2 \leq s\{\|Tx - y\|^2 + \|Ty - x\|^2\} \tag{5.5}$$

for all $x, y \in C$. Then

(i) *T has a unique fixed point u in C;*
(ii) *for every $z \in C$, the sequence $\{T^n z\}$ converges to u.*

Proof. From (5.5) we have that

$$\|Tx - Ty\|^2 - s(\|x - Ty\|^2 + \|Tx - y\|^2) \leq 0$$

for all $x, y \in C$. That is, T is a $(1, -s, 0, 0)$-symmetric generalized hybrid mapping. For α, β, γ and δ in Theorem 4.1, we also have that

$$\alpha + 2\beta + \gamma = 1 - 2s > 0 \ , \alpha + \beta + \delta = 1 - s > 0 \text{ and } \delta = 0 \geq 0.$$

From Theorem 4.1, we have the desired result. Furthermore, since $\beta + \gamma = -s \leq 0$, we have that for every $z \in C$, the sequence $\{T^n z\}$ converges to u. □

Theorem 5.5. *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let $T : C \to C$ be contractively hybrid, i.e., there exists a real number s with $0 \leq s < \frac{1}{3}$ such that*

$$\|Tx - Ty\|^2 \leq s\{\|Tx - y\|^2 + \|Ty - x\|^2 + \|x - y\|^2\} \tag{5.6}$$

for all $x, y \in C$. Then

(i) *T has a unique fixed point u in C;*
(ii) *for every $z \in C$, the sequence $\{T^n z\}$ converges to u.*

Proof. From (5.6) we have that

$$\|Tx - Ty\|^2 - s(\|x - Ty\|^2 + \|Tx - y\|^2) - s\|x - y\|^2 \leq 0$$

for all $x, y \in C$. Thus T is a $(1, -s, -s, 0)$-symmetric generalized hybrid mapping. For α, β, γ and δ in Theorem 4.1, we also have that

$$\alpha + 2\beta + \gamma = 1 - 3s > 0 \ , \alpha + \beta + \delta = 1 - s > 0 \text{ and } \delta = 0 \geq 0.$$

From Theorem 4.1, we have the desired result. Furthermore, since $\beta + \gamma = -2s \leq 0$, we have that for every $z \in C$, the sequence $\{T^n z\}$ converges to u. □

Using Theorem 4.1, we obtain an extension of Theorem 1.1 which was proved by Browder [4].

Theorem 5.6. *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be a strict pseudo-contraction from C into itself, i.e., there exists $r \in \mathbb{R}$ with $0 \leq r < 1$ such that*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + r\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{5.7}$$

Let $u \in C$ and $s_n \in (0, 1)$ for all $n \in \mathbb{N}$. Define a mapping $U_n : C \to C$ as follows:

$$U_n x = s_n u + (1 - s_n)Tx, \quad \forall x \in C, \ n \in \mathbb{N}.$$

Then the following hold:

(i) *U_n has a unique fixed point z_n in C;*
(ii) *if $s_n \to 0$, then the sequence $\{z_n\}$ converges to $P_{F(T)}u$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$.*

Proof. We first note from Lemma 2.2 that $F(T)$ is nonempty, closed and convex. Then there exists the metric projection $P_{F(T)}$ of H onto $F(T)$. For the proof of (i), see [23]. However, for the sake of completeness, we give the proof as in the proof of Theorem 5.3. For $u \in C$ and $s_n \in (0, 1)$ for all $n \in \mathbb{N}$, define a mapping $S_n : C \to C$ as follows:

$$S_n x = rx + (1 - r)\big(s_n u + (1 - s_n)Tx\big), \quad \forall x \in C.$$

As in the proof of Theorem 5.3, from (5.7) we have that

$$\begin{aligned}
\frac{1}{(1-r)(1-s_n)^2}&\|S_n x - S_n y\|^2 - \frac{rs_n}{(1-r)(1-s_n)^2}(\|x - S_n y\|^2 + \|y - S_n x\|^2) \\
&+ \left(\frac{r^2}{(1-r)(1-s_n)^2} - \frac{1 - s_n + r^2(1 + s_n)}{(1-r)(1-s_n)}\right)\|x - y\|^2 \\
&\qquad + \frac{rs_n}{(1-r)(1-s_n)^2}(\|x - S_n x\|^2 + \|y - S_n y\|^2) \leq 0.
\end{aligned}$$

For this mapping S_n, we apply Theorem 4.1. We first have that

$$\begin{aligned}
\frac{1}{(1-r)(1-s_n)^2} &- \frac{2rs_n}{(1-r)(1-s_n)^2} + \frac{r^2}{(1-r)(1-s_n)^2} - \frac{1 - s_n + r^2(1 + s_n)}{(1-r)(1-s_n)} \\
&= \frac{s_n(1-r)(2 - s_n(1+r))}{(1-r)(1-s_n)^2} > 0.
\end{aligned}$$

Furthermore, we have that

$$\frac{1}{(1-r)(1-s_n)^2} - \frac{rs_n}{(1-r)(1-s_n)^2} + \frac{rs_n}{(1-r)(1-s_n)^2} = \frac{1}{(1-r)(1-s_n)^2} > 0,$$

$$\frac{rs_n}{(1-r)(1-s_n)^2} \geq 0.$$

Thus S_n has a unique fixed point z_n in C from Theorem 4.1. Since z_n is a fixed point of S_n, we have $z_n = rz_n + (1-r)(s_n u + (1-s_n)Tz_n)$. From $1 - r \neq 0$, we have that

$$z_n = s_n u + (1-s_n)Tz_n = U_n z_n.$$

To show that $\{z_n\}$ converges strongly to $u_0 = P_{F(T)}u$, we may show that each subsequence $\{z_{n_i}\}$ of $\{z_n\}$ has a subsequence $\{z_{n_{ij}}\}$ of $\{z_{n_i}\}$ such that $z_{n_{ij}} \to u_0$. To show this, put $v_i = z_{n_i}$. Without loss of generality, we may assume that $\{v_i\}$ converges weakly to $v \in C$. Let us show $v \in F(T)$. From $s_n \to 0$, we get that $z_n - Tz_n \to 0$. In fact, from

$$z_n = U_n z_n = (1-s_n)Tz_n + s_n u,$$

we get

$$z_n - Tz_n = s_n(u - Tz_n).$$

Since $\{Tz_n\}$ is bounded, from $s_n \to 0$ we obtain that $z_n - Tz_n \to 0$. Since $v_i = z_{n_i} \rightharpoonup v$, from Lemma 2.6 we get $v = Tv$. Using $v \in F(T)$, we show that $\{v_i\}$ converges strongly to $u_0 = P_{F(T)}u$. Since v_i is a fixed point of U_{n_i}, we get

$$v_i = (1-s_{n_i})Tv_i + s_{n_i}u$$

and hence

$$s_{n_i}v_i + (1-s_{n_i})(v_i - Tv_i) = s_{n_i}u. \tag{5.8}$$

From $u_0 \in F(T)$ we also have that

$$s_{n_i}u_0 + (1-s_{n_i})(u_0 - Tu_0) = s_{n_i}u_0. \tag{5.9}$$

Setting $A = I - T$, where I is the identity mapping, from (5.8) and (5.9) we obtain that

$$\begin{aligned} s_{n_i}\langle v_i - u_0, v_i - u_0\rangle &+ (1-s_{n_i})\langle Av_i - Au_0, v_i - u_0\rangle \\ &= s_{n_i}\langle u - u_0, v_i - u_0\rangle. \end{aligned}$$

We know from Lemma 2.4 that

$$\langle Av_i - Au_0, v_i - u_0\rangle \geq \frac{1-r}{2}\|Av_i - Au_0\|^2.$$

Thus we get that

$$s_{n_i}\|v_i - u_0\|^2 \leq s_{n_i}\langle u - u_0, v_i - u_0\rangle.$$

Then we obtain that

$$\begin{aligned} \|v_i - u_0\|^2 &\leq \langle u - u_0, v_i - u_0\rangle \\ &= \langle u - u_0, v_i - v\rangle + \langle u - u_0, v - u_0\rangle. \end{aligned}$$

From $u_0 = P_{F(T)}u$ and $v \in F(T)$, we get that

$$\langle u - u_0, v - u_0\rangle \leq 0.$$

Using this inequality, we obtain that

$$\|v_i - u_0\|^2 \leq \langle u - u_0, v_i - v \rangle.$$

From $v_i \rightharpoonup v$, we get that $v_i \to u_0$. $\square$

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