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SOME PROBLEMS IN OPTIMAL CONTROL GOVERNED BY THE SWEEPING PROCESS

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Abstract. We present several results on Bolza, Relaxation and Viscosity problems governed by the sweeping process with Young measure controls in the framework of Optimal Control Theory.

1. Introduction

Pioneering work concerning the existence of continuous and bounded variation (BVC) solutions for the perturbation of sweeping process (Moreau's process)[22] of the form

$$-\frac{dDu}{d|Du|}(t) \in N_{C(t)}(u(t)) + F(t, u(t)) \tag{1.1}$$

where C is a closed convex valued continuous multifunction from $[0, T]$ to $\mathbf{R}^d$ and $F : [0, T] \times \mathbf{R}^d \to \mathbf{R}^d$ is a convex compact valued upper semicontinuous multifunction, goes back to [7]. In a series of papers [5, 9, 10] the authors study some Bolza type and viscosity problems governed by ordinary differential equations (ODE) and evolution inclusions with fixed domain governed by a subdifferential operator with perturbations containing Young measure controls.
In the present paper we treat a Bolza and a Relaxation problem for the perturbation of the sweeping process $(\mathcal{PSW})$ of the form (1.1) in a new setting. As an application we show the viscosity property of the value function associated with the problem $(\mathcal{PSW})$. We refer to [8, 9, 10] for Young measures and their applications. The paper is organized as follows. In Section 2, we recall some basic results on sweeping processes and Young measures, in particular, we focus on some stability results for the sweeping process with application to some bang-bang type theorems. Several results on the dependence of the solution with respect to the control measure are stated here, in both the BV and the Lipschitz case for the $(\mathcal{PSW})$ with application to Bolza and Relaxation problems. In this new framework we also prove that the value function related to the $(\mathcal{PSW})$ satisfies the dynamic programming principle. Section 3 is devoted to the viscosity property of the value function associated with the $(\mathcal{PSW})$. In section 4 we present further extensions and applications. This paper sheds a new light in the theory of Optimal Control and leads to further developments in differential games theory involving the sweeping process with various control spaces and also to the stochastic perturbation of the sweeping process.

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2. Relaxation problem in the sweeping process: The BVC case

For the sake of completeness, we recall the notation and summarize some useful facts concerning Young measures. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let X be a Polish space and let $\mathcal{C}^b(X)$ be the space of all bounded continuous functions defined on X. Let $\mathcal{M}^1_+(X)$ be the set of all Borel probability measures on X equipped with the narrow topology. A Young measure $\lambda : \Omega \to \mathcal{M}^1_+(X)$ is, by definition, a *scalarly measurable* mapping from Ω into $\mathcal{M}^1_+(X)$, that is, for every $f \in \mathcal{C}^b(X)$, the mapping $\omega \mapsto \langle f, \lambda_\omega \rangle := \int_X f(x)\, d\lambda_\omega(x)$ is $\mathcal{F}$-measurable. A sequence (λ^n) in the space of Young measures $\mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}^1_+(X))$ *stably converges* to a Young measure $\lambda \in \mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}^1_+(X))$ if the following holds:

$$\lim_{n\to\infty} \int_A [\int_X f(x)\, d\lambda^n_\omega(x)]\, dP(\omega) = \int_A [\int_X f(x)\, d\lambda_\omega(x)]\, dP(\omega)$$

for every $A \in \mathcal{F}$ and for every $f \in \mathcal{C}^b(X)$.

Theorem 2.1 ([10, Theorem 3.3.1]). *Assume that S and T are Polish spaces. Let (μ^n) be a sequence in $\mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}^1_+(S))$ and let (ν^n) be a sequence in $\mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}^1_+(T))$. Assume that*

(i) *(μ^n) converges in probability to $\mu^\infty \in \mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}^1_+(S))$,*
(ii) *(ν^n) stably converges to $\nu^\infty \in \mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}^1_+(T))$.*

Then $(\mu^n \otimes \nu^n)$ stably converges to $\mu^\infty \otimes \nu^\infty$ in $\mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}^1_+(S \times T))$.

Theorem 2.2 ([10, Theorem 6.3.5]). *Assume that X and Z are Polish spaces. Let (u^n) be sequence of $\mathcal{F}$-measurable mappings from Ω into X such that (u^n) converges in probability to a $\mathcal{F}$-measurable mapping u^∞ from Ω into X, and let (v^n) be a sequence of $\mathcal{F}$-measurable mappings from Ω into Z such that (v^n) stably converges to $\nu^\infty \in \mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}^1_+(Z))$. Let $h : \Omega \times X \times Z \to \mathbf{R}$ be a Carathéodory integrand such that the sequence $(h(., u_n(.), v_n(.))$ is uniformly integrable. Then the following holds:*

$$\lim_{n\to\infty} \int_\Omega h(\omega, u^n(\omega), v^n(\omega))\, dP(\omega) = \int_\Omega [\int_Z h(\omega, u^\infty(\omega), z)\, d\nu^\infty_\omega(z)]\, dP(\omega).$$

In the remainder Z is a compact metric space, $\mathcal{M}^1_+(Z)$ is the space of all probability Radon measures on Z. We will endow $\mathcal{M}^1_+(Z)$ with the narrow topology so that $\mathcal{M}^1_+(Z)$ is a compact metrizable space. Let us denote by $\mathcal{Y}([0,T]; \mathcal{M}^1_+(Z))$ the space of all Young measures (alias *relaxed controls*) defined on $[0,T]$ endowed with the stable topology so that $\mathcal{Y}([0,T]; \mathcal{M}^1_+(Z))$ is a compact metrizable space with respect to this topology. By the Portmanteau Theorem for Young measures [10, Theorem 2.1.3], a sequence (ν^n) in $\mathcal{Y}([0,T]; \mathcal{M}^1_+(Z))$ stably converges to $\nu \in \mathcal{Y}([0,T]; \mathcal{M}^1_+(Z))$ if

$$\lim_{n\to\infty} \int_0^T [\int_Z h_t(z) d\nu^n_t(z)]\, dt = \int_0^T [\int_Z h_t(z) d\nu_t(z)]\, dt$$

for all $h \in L^1_{\mathcal{C}(Z)}([0,T])$, here $\mathcal{C}(Z)$ denotes the space of all continuous real valued functions defined on Z endowed with the norm of uniform convergence. Finally let us denote by $\mathcal{Z}$ the set of all Lebesgue measurable mappings (alias *original controls*)

$z : [0, T] \to Z$ and $\mathcal{R} := \mathcal{Y}([0, T]; \mathcal{M}^1_+(Z))$ the set of all relaxed controls (alias Young measures) associated with Z.

Let us recall a fundamental result on sweeping process [21, 24].

Proposition 2.3. *Let $C : [0, T] \to \mathbf{R}^d$ be closed convex valued lower semicontinuous multifunction. Assume that*

(i) *there exist $x_0 \in \mathbf{R}^d$ and $r_0 > 0$ such that $\overline{B}_{\mathbf{R}^d}(x_0, r_0) \subset C(t), \forall t \in [0, T]$,*

(ii) *the graph of C is closed with respect to the left topology on $[0, T]$ and the usual one on $\mathbf{R}^d$.*

Let $a \in C(0)$. Then there exists a unique BVC solution to $\mathcal{SW}(C, a)$, that is,

$$\begin{cases} -\dfrac{dDu}{d|Du|}(t) \in N_{C(t)}(u(t)), \ |Du|\text{-a.e.}, \\ u(0) = a \in C(0). \end{cases}$$

Here Du is the differential vector measure associated with the BV function u and $|Du|$ is its variation measure, and $N_{C(t)}(u(t))$ denotes the normal cone of $C(t)$ at the point $u(t) \in C(t)$. Further, the solution u satisfies

$$||Du|| := \int_0^T |Du| \le l(r_0, ||a - x_0||)$$

where the function $l : \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ is given by

$$l(s, S) := \begin{cases} \max\{0, \frac{S^2 - s^2}{2s}\} & \textit{if} \ \ d \ge 2, \\ \max\{0, S - s\} & \textit{if} \ \ d = 1. \end{cases}$$

Assumptions (*) **and** (**) **:**

We will consider a closed convex valued multifunction $C : [0, T] \to \mathbf{R}^d$ and we assume that C is continuous for the Hausdorff distance d_{H}, that is,

$$(*) \qquad \forall s \in [0, T], \ \lim_{t \to s} d_{\mathrm{H}}(C(t), C(s)) = 0$$

and that there exists $r > 0$ such that

$$(**) \qquad \forall t \in [0, T], \ \ 2r\overline{B}_{\mathbf{R}^d}(0, 1) \subset C(t).$$

Also $E = \mathbf{R}^d$, in the sequel.

Assumptions on f :

Let us consider a mapping $f : [0, T] \times E \times Z \to E$ satisfying:

(i) for every fixed $t \in [0, T], f(t, ., .)$ is continuous on $E \times Z$,

(ii) for every $(x, z) \in E \times Z, f(., x, z)$ is Lebesgue-measurable on $[0, T]$,

(iii) there is a nonnegative Lebesgue integrable function g such that

$$f(t, x, z) \in g(t)(1 + ||x||)\bar{B}_E(0, 1)$$

for all (t, x, z) in $[0, T] \times E \times Z$,

(iv) there exists $\alpha \in L^1_{\mathbf{R}^+}[0, T])$ such that

$$||f(t, x_1, z) - f(t, x_2, z)|| \le \alpha(t)||x_1 - x_2||$$

for all $(t, x_1, z), (t, x_2, z) \in [0, T] \times E \times Z$.

Let $C : [0,T] \to ck(E)$ be a convex compact valued continuous mapping satisfying (*) and (**). We consider the sets of BVC solutions of the following two "perturbed sweeping process" problems $(\mathcal{PSW})(C;\zeta,x)$ and $(\mathcal{PSW})(C;\lambda,x)$, namely

$$\begin{cases} -\dfrac{dDu_{x,\zeta}}{|dDu_{x,\zeta}|}(t) \in N_{C(t)}(u_{x,\zeta}(t)) + f(t,u_{x,\zeta}(t),\zeta(t)) \\ u_{x,\zeta}(0) = x \in C(0) \end{cases}$$

and

$$\begin{cases} -\dfrac{dDu_{x,\lambda}}{|dDu_{x,\lambda}|}(t) \in N_{C(t)}(u_{x,\lambda}(t)) + \displaystyle\int_Z f(t,u_{x,\lambda}(t),z)\,d\lambda_t(z) \\ u_{x,\lambda}(0) = x \in C(0) \end{cases}$$

where ζ belongs to the set $\mathcal{Z}$ of all original controls and λ belongs to the set $\mathcal{R}$ of all relaxed controls. Taking [7, Theorem 4.4] into account, for each $(x,\zeta) \in C(0) \times \mathcal{Z}$ (resp. $(x,\lambda) \in C(0) \times \mathcal{R}$), there exists a unique BVC solution $u_{x,\zeta}$ (resp. $u_{x,\lambda}$) to $(\mathcal{PSW})(C;\zeta,x)$ (resp. $(\mathcal{PSW})(C;\lambda,x)$), that is, for every positive Radon measure $d\nu$ on $[0,T]$ such that $|Du_{x,\zeta}| << d\nu$, $|Du_{x,\lambda}| << d\nu$ and $dt << d\nu$, the densities $Du_{x,\zeta}/d\nu$, $Du_{x,\lambda}/d\nu$ and $dt/d\nu$ satisfy

$$-\frac{dDu_{x,\zeta}}{d\nu}(t) - f(t,u_{x,\zeta}(t),\zeta(t))\frac{dt}{d\nu}(t) \in N_{C(t)}(u_{x,\zeta}(t)),\ \nu\text{-a.e.}$$

and respectively

$$-\frac{dDu_{x,\lambda}}{d\nu}(t) - \int_Z f(t,u_{x,\lambda}(t),z)\,d\lambda_t(z)\frac{dt}{d\nu}(t) \in N_{C(t)}(u_{x,\lambda}(t)),\ \nu\text{-a.e.}$$

We aim to present some relaxation problems in the framework of Optimal Control Theory for the above $(\mathcal{PSW})$. In particular, we state a viscosity property of the value function associated with these evolution inclusions. Similar problems governed by evolution inclusion with perturbation containing Young measures were initiated by [3, 4, 9, 10]. Before going further, we recall first a fundamental result in [7, Theorem 4.4] for the existence of BVC solutions to $(\mathcal{PSW})$.

Proposition 2.4. *Let $C : [0,T] \to ck(E)$ be a convex compact valued continuous mapping satisfying (*) and (**). Let us consider a mapping $f : [0,T] \times E \to E$ satisfying:*

(i) *for every fixed $t \in [0,T], f(t,.)$ is continuous on E,*
(ii) *for every $x \in E, f(.,x)$ is Lebesgue-measurable on $[0,T]$,*
(iii) *there is a nonnegative Lebesgue integrable function g such that $f(t,x) \in g(t)(1+||x||)\bar{B}_E(0,1)$ for all (t,x) in $[0,T] \times E$,*
(iv) *there exists $\alpha \in L^1_{\mathbf{R}^+}[0,T])$ such that*

$$||f(t,x_1) - f(t,x_2)|| \le \alpha(t)\,||x_1 - x_2||$$

for all $(t,x_1),(t,x_2) \in [0,T] \times E$.

Then there exists a unique BVC solution $u_x : [0,T] \to E$ with $u_x(0) = x \in C(0)$ to the $(\mathcal{PSW})(C;f;x)$, that is, for every positive Radon measure ν on $[0,T]$ such that Du_x and dt are absolutely continuous with respect to ν, we have

$$-\frac{dDu_x}{d\nu}(t) - f(t,u_x(t))\frac{dt}{d\nu}(t) \in N(C(t);u_x(t)),\ \nu\text{-a.e.}, t \in [0,T].$$

Briefly u_x *is the unique BVC solution of the inclusion*

$$-Du_x \in N_{C(t)}(u_x(t)) + f(t, u_x(t)),\ u_x(0) = x.$$

For the convenience of the reader we recall and summarize some results.

Proposition 2.5. *The set of original controls* $\mathcal{Z}$ *is dense in the set* $\mathcal{R}$ *with respect to the stable topology.*

Proof. See [10, Lemma 7.1.1, page 197]. □

Let us recall the following denseness result based on Lyapunov's theorem. See e.g. [12, 25].

Proposition 2.6. *Let* $\Gamma : [0, T] \to ck(E)$ *be a convex compact valued measurable and integrably bounded mapping. Let* $ext(\Gamma) : t \mapsto ext(\Gamma(t))$ *where* $ext(\Gamma(t))$ *is the set of extreme points of* $\Gamma(t)(t \in [0, T])$*. Then the set* S^1_Γ *of all integrable selections of* Γ *is convex and* $\sigma(L^1_E, L^\infty_E)$ *compact and the set of all integrable selections* $S^1_{ext(\Gamma)}$ *of* $ext(\Gamma)$ *is dense in* S^1_Γ *with respect to this topology.*

The following stability result in the sweeping process is useful for our purposes.

Proposition 2.7. *Let* $(C_n)_{n\in\mathbf{N}\cup\{\infty\}}$ *be a sequence of lower semicontinuous multifunctions from* $[0, T]$ *to the set* $cc(E)$ *of nonempty closed convex subsets of* E*. Assume that the following assumptions are fulfilled:*

(i) *for any sequence* (t_n) *in* $[0, T]$ *converging to* t*,* $C_\infty(t) \subset Li(C_n(t_n))$*,*
(ii) *for any* $t \in [0, T]$*,* $Ls(C_n(t)) \subset C_\infty(t)$*,*
(iii) *for every* $n \in \mathbf{N} \cup \{\infty\}$*, the graph of* C_n *is left-closed,*
(iv) *there exists* $r > 0$ *such that* $\overline{B}_E(0, r) \subset C_n(t), \forall t \in [0, T], \forall n \in \mathbf{N}$*.*

Let $(u_n)_{n\in\mathbf{N}}$ *be the BVC solution of the sweeping process*

$$-\frac{dDu_n}{d|Du_n|}(t) \in N_{C_n(t)}(u_n(t)),\ |Du_n|\text{-a.e.},\ u_n(0) = a_n \in C_n(0).$$

If $(a_n)_{n\in\mathbf{N}}$ *converges to* $a_\infty \in E$*, then* $(u_n)_{n\in\mathbf{N}}$ *pointwise converges to the unique BVC continuous solution* u_∞ *of the sweeping process*

$$-\frac{dDu_\infty}{d|Du_\infty|}(t) \in N_{C_\infty(t)}(u_\infty(t)),\ |Du_\infty|\text{-a.e.},\ u_\infty(0) = a_\infty \in C_\infty(0).$$

Proof. See [2, Theorem 4.4]. □

For simplicity we begin with some particular results on relaxation associated with the above sweeping process which are based on the continuity of the solution with respect to the control and the stability result for sweeping process.

Theorem 2.8. *Let* $C : [0, T] \to cc(E)$ *be a closed convex valued continuous mapping satisfying* (*) *and* (**)*. Let* $h_n, h \in L^1_E([0, T]$ *with* $||h_n(t)|| \leq g(t)$ *for all* $n \in \mathbf{N}$ *and for all* $t \in [0, T]$*, for some positive integrable function* g*, and let us consider the two following* $(\mathcal{PSW})$

$$\begin{cases} -\dfrac{dDu_{x,h_n}}{d|Du_{x,h_n}|}(t) - h_n(t) \in N_{C(t)}(u_{x,h_n}(t)) \\ u_{x,h_n}(0) = x \in C(0) \end{cases}$$

$$\begin{cases} -\dfrac{dDu_{x,h}}{d|Du_{x,h}|}(t) - h(t) \in N_{C(t)}(u_{x,h}(t)) \\ u_{x,h}(0) = x \in C(0) \end{cases}$$

where u_{x,h_n} *and* $u_{x,h}$ *are the BVC solutions to* $(\mathcal{PSW})(C; h_n; x)$ *and* $(\mathcal{PSW})(C; h; x)$*, respectively. Assuming that* (h_n) *converges* $\sigma(L^1_E, L^\infty_E)$ *to* h*, then* u_{x,h_n} *pointwise converges to* $u_{x,h}$*.*

Proof. Let us consider the closed convex valued continuous mappings

$$C_h(t) := C(t) + \int_0^t h(s)\,ds,\ \forall t \in [0,T]$$

$$C_{h_n}(t) := C(t) + \int_0^t h_n(s)\,ds,\ \forall t \in [0,T].$$

Then the sweeping processes $(\mathcal{SW})(C_h; x)$ and $(\mathcal{SW})(C_{h_n}; x)$ with initial value x have unique BVC solutions respectively $w_{x,h}$ and w_{x,h_n} with $w_{x,h}(0) = w_{x,h_n}(0) = x$, $w_{x,h}(t) \in C_h(t)$, and $w_{x,h_n}(t) \in C_{h_n}(t)$ for all $t \in [0,T]$. Since h_n converges $\sigma(L^1_E, L^\infty_E)$ to h, we have

$$S_{h_n}(t) := \int_0^t h_n(s)\,ds \to S_h(t) := \int_0^t h(s)\,ds,\ \forall t \in [0,T]$$

so that

$$\lim_{n\to\infty} d_{\mathrm{H}}(C_{h_n}(t), C_h(t)) = \lim_{n\to\infty} ||S_{h_n}(t) - S_h(t))|| = 0,\ \forall t \in [0,T]. \tag{2.1}$$

Further if $t_n \to t$, then using (2.1)

$$\begin{aligned} \lim_{n\to\infty} d_{\mathrm{H}}(C_{h_n}(t_n), C_h(t)) &\le \lim_{n\to\infty} [d_{\mathrm{H}}(C(t_n), C(t)) + |\int_t^{t_n} ||h_n(s)||\,ds|] \\ &\le \lim_{n\to\infty} |\int_t^{t_n} g(s)\,ds| = 0. \end{aligned} \tag{2.2}$$

Then (2.1) and (2.2) imply the following conditions involving the well-known Kuratowski limits of sets:

(i) for any sequence (t_n) in $[0,T]$ converging to t, $C_h(t) \subset Li\, C_{h_n}(t_n)$
(ii) for any $t \in [0,T]$, $Ls\, C_{h_n}(t) \subset C_h(t)$.

Since for every n, the graph of C_{h_n} is left closed, then by (i) and (ii) we may apply the stability result for sweeping process in Proposition 2.7 which shows that the solutions w_{x,h_n} to the sweeping process $(\mathcal{SW})(C_{h_n}; x)$ pointwise converge to the unique solution $w_{x,h}$ to the sweeping process $(\mathcal{SW})(C_h; x)$. Hence we deduce that, for all $t \in [0,T]$

$$\lim_{n\to\infty} u_{x,h_n}(t) = \lim_{n\to\infty} w_{x,h_n}(t) - Sh_n(t) = w_{x,h}(t) - Sh(t) = u_{x,h}(t).$$

□

Remark. In further applications, we will consider the BVC solution u_{x_n,h_n} to $(\mathcal{PSW})(C; h_n; x_n)$ and $u_{x,h}$ to $(\mathcal{PSW})(C; h; x)$. Then if $(h_n)\,\sigma(L^1_E, L^\infty_E)$ converges to h and (x_n) converges to x in $C(0)$, then u_{x_n,h_n} pointwise converges to $u_{x,h}$.

Now we provide some applications of the sweeping process in Control Theory.

Theorem 2.9. *Let $C : [0,T] \to ck(E)$ be a convex compact valued continuous mapping satisfying (*) and (**). Let $\Gamma : [0,T] \to ck(E)$ be a convex compact valued measurable and integrably bounded mapping. Let $ext(\Gamma) : t \mapsto ext(\Gamma(t))$ where $ext(\Gamma(t))$ is the set of extreme points of $\Gamma(t)(t \in [0,T])$. Let S^1_Γ and $S^1_{ext(\Gamma)}$ be the set of all integrable selections of Γ and $ext(\Gamma)$, respectively. Let us consider the control problems governed by the two following sweeping processes*

$$\begin{cases} -\dfrac{dDu_{x,h}}{d|Du_{x,h}|}(t) - h(t) \in N_{C(t)}(u_{x,h}(t)), h \in S^1_\Gamma \\ u_{x,h}(0) = x \in C(0) \end{cases}$$

$$\begin{cases} -\dfrac{dDu_{x,h}}{d|Du_{x,h}|}(t) - h(t) \in N_{C(t)}(u_{x,h}(t)), h \in S^1_{ext(\Gamma)} \\ u_{x,h}(0) = x \in C(0). \end{cases}$$

Then the following hold:

(a) *For each $t \in [0,T]$, the mapping $h \mapsto u_{x,h}(t) \in E$ is continuous on S^1_Γ; here S^1_Γ is endowed with the $\sigma(L^1_E, L^\infty_E)$ topology.*

(b) *For each $t \in [0,T]$, the set $\{u_{x,h}(t) : h \in S^1_{ext(\Gamma)}\}$ is dense in the compact set $\{u_{x,h}(t) : h \in S^1_\Gamma\}$.*

Proof. (a) Observe that S^1_Γ is compact metrizable with respect to the $\sigma(L^1_E, L^\infty_E)$ topology. Let $h_n \in S^1_\Gamma$ $\sigma(L^1_E, L^\infty_E)$ converge to $h \in S^1_\Gamma$. Then by Theorem 2.8, u_{x,h_n} pointwise converges to $u_{x,h}$ which proves (a).
(b) is consequence of this continuity property and the denseness property in Proposition 2.6. □

The following shows the continuous dependence of the solution with respect to the control in the sweeping process $(\mathcal{PSW})$.

Theorem 2.10. *Let $C : [0,T] \to ck(E)$ be a convex compact valued continuous mapping satisfying (*) and (**) and let Z be a compact subset of E. Let us consider the control problem governed by the sweeping process $(\mathcal{PSW})(C;\nu;x)$*

$$\begin{cases} -\dfrac{dDu_{x,\nu}}{d|Du_{x,\nu}|}(t) - \mathrm{bar}(\nu_t) \in N_{C(t)}(u_{x,\nu}(t)), \nu \in \mathcal{R} \\ u_{x,\nu}(0) = x \in C(0) \end{cases}$$

where $\mathrm{bar}(\nu_t)$ denotes the barycenter of the measure $\nu_t \in \mathcal{M}^1_+(Z)$. Then, for each $t \in [0,T]$, the mapping $\nu \mapsto u_{x,\nu}(t)$ is continuous on $\mathcal{R}$, where $\mathcal{R}$ is endowed with the stable topology.

Proof. (a) Let $\nu \in \mathcal{R}$ and let $\mathrm{bar}(\nu) : t \mapsto \mathrm{bar}(\nu_t)$, $t \in [0,T]$. It is easy to check that $\nu \mapsto \mathrm{bar}(\nu)$ from $\mathcal{R}$ into $L^1_E([0,T])$ is continuous with respect to the stable topology and the $\sigma(L^1_E, L^\infty_E)$, respectively. Note that $\mathcal{R}$ is compact metrizable for the stable topology. Now let (ν^n) be a sequence in $\mathcal{R}$ which stably converges to $\nu \in \mathcal{R}$. Then $\mathrm{bar}(\nu^n)$ $\sigma(L^1_E, L^\infty_E)$ converges to $\mathrm{bar}(\nu)$. By Theorem 2.8 we see that u_{x,ν^n} pointwise converges to $u_{x,\nu}$. □

Remark Taking into account the remark of Theorem 2.8, it is not difficult to show that, for each $t \in [0,T]$, the mapping $(x,\nu) \mapsto u_{x,\nu}(t)$ is continuous on $C(0) \times \mathcal{R}$.

We are now able to relate the Bolza type problems associated with sweeping processes, as follows:

Theorem 2.11. *With the hypotheses and notations of Theorem 2.10, assume that $J : [0,T] \times E \times Z \to \mathbf{R}$ is a Carathéodory integrand, that is, $J(t,.,.)$ is continuous on $E \times Z$ for every $t \in [0,T]$ and $J(.,x,z)$ is Lebesgue-measurable on $[0,T]$ for every $(x,z) \in E \times Z$, which satisfies the condition $(\mathcal{C})$: for every sequence (ζ_n) in $\mathcal{Z}$, the sequence $(J(.,u_{x,\zeta^n}(.),\zeta^n(.))$ is uniformly integrable in $L^1_{\mathbf{R}}([0,T],dt)$; here u_{x,ζ^n} denotes the unique BVC solution to $(\mathcal{PSW})(C;\zeta^n;x)$*

$$\begin{cases} -\dfrac{dDu_{x,\zeta^n}}{d|Du_{x,\zeta^n}|}(t) - \zeta^n(t) \in N_{C(t)}(u_{x,\zeta^n}(t)), \zeta^n \in \mathcal{Z} \\ u_{x,\zeta^n}(0) = x \in C(0) \end{cases}$$

Let us consider the control problems

$$\inf(P_{\mathcal{Z}}) := \inf_{\zeta \in \mathcal{Z}} \int_0^T J(t, u_{x,\zeta}(t), \zeta(t))\,dt$$

and

$$\inf(P_{\mathcal{R}}) := \inf_{\lambda \in \mathcal{R}} \int_0^T [\int_Z J(t, u_{x,\lambda}(t), z)\,\lambda_t(dz)]\,dt$$

where $u_{x,\zeta}$ (resp. $u_{x,\lambda}$) is the unique BVC solution to $(\mathcal{PSW})(C;\zeta;x)$ and $(\mathcal{PSW})(C;\lambda;x)$, respectively. Then one has

$$\inf(P_{\mathcal{Z}}) = \inf(P_{\mathcal{R}}).$$

Proof. Take a control $\lambda \in \mathcal{R}$. By virtue of Proposition 2.5, there is a sequence $(\zeta^n)_{n\in\mathbf{N}}$ in $\mathcal{Z}$ such that the sequence $(\delta_{\zeta^n})_{n\in\mathbf{N}}$ of Young measures associated with $(\zeta^n)_{n\in\mathbf{N}}$ stably converges to λ. By Theorem 2.8, the sequence (u_{x,ζ^n}) where u_{x,ζ^n} is the unique BVC solution to $(\mathcal{PSW})(C;\zeta^n;x)$ pointwise converges to the unique BVC solution $u_{x,\lambda}$ to $(\mathcal{PSW})(C;\lambda;x)$. As $(J(t,u_{x,\zeta^n}(t),\zeta^n(t)))$ is uniformly integrable by assumption $(\mathcal{C})$, using Theorem 2.2 (or [10, Theorem 6.3.5]), we get

$$\lim_{n\to\infty} \int_0^T J(t, u_{x,\zeta^n}(t), \zeta^n(t))\,dt = \int_0^T [\int_Z J(t, u_{x,\lambda}, z) d\lambda_t(z)]\,dt.$$

As

$$\int_0^T J(t, u_{x,\zeta^n}(t), \zeta^n(t))\,dt \geq \inf(P_{\mathcal{Z}})$$

for all $n \in \mathbf{N}$, so is

$$\int_0^T [\int_Z J(t, u_{x,\lambda}, z) d\lambda_t(z)]\,dt \geq \inf(P_{\mathcal{Z}});$$

by taking the infimum on $\mathcal{R}$ in this inequality we get

$$\inf(P_{\mathcal{R}}) \geq \inf(P_{\mathcal{O}})$$

As $\inf(P_{\mathcal{O}}) \geq \inf(P_{\mathcal{R}})$, the proof is complete. □

In the framework of Optimal Control, the above considerations lead to the study of the value function associated with the $(\mathcal{PSW})$. The following shows that the value function satisfies the dynamic programming principle (DPP).

Theorem 2.12 (of dynamic programming principle)**.** *Assume the hypothesis and notations of Theorem 2.10, and let $x \in E$, $\tau < T$ and $\sigma > 0$ such that $\tau + \sigma < T$. Assume that $J : [0,T] \times E \times Z \to \mathbf{R}$ is bounded and continuous. Let us consider the value function*

$$V_J(\tau, x) = \sup_{\nu \in \mathcal{R}} \int_\tau^T [\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz)]\, dt, \quad (\tau, x) \in [0,T] \times E$$

where $u_{\tau,x,\nu}$ is the BVC solution to the $(\mathcal{PSW})(C;\nu;x)$ defined on $[\tau,T]$ associated the control $\nu \in \mathcal{R}$ starting from x at time τ

$$\begin{cases} -\dfrac{dDu_{\tau,x,\nu}}{d|Du_{\tau,x,\nu}|}(t) - \mathrm{bar}(\nu_t) \in N_{C(t)}(u_{\tau,x,\nu}(t)) \\ u_{\tau,x,\nu}(\tau) = x \in C(\tau) \end{cases}$$

Then the following holds:

$$V_J(\tau, x) = \sup_{\nu \in \mathcal{R}} \{ \int_\tau^{\tau+\sigma} [\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz)]dt + V_J(\tau + \sigma, u_{\tau,x,\nu}(\tau + \sigma))\}$$

with

$$V_J(\tau + \sigma, u_{\tau,x,\nu}(\tau + \sigma)) = \sup_{\mu \in \mathcal{R}} \int_{\tau+\sigma}^T [\int_Z J(t, v_{\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma),\mu}(t), z)\mu_t(dz)]\, dt$$

where $v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\mu}$[1] *is the BVC solution to $(\mathcal{PSW})(C;\mu;u_{\tau,x,\nu}(\tau+\sigma))$ defined on $[\tau+\sigma,T]$ associated with the control $\mu \in \mathcal{R}$ starting from $u_{\tau,x,\nu}(\tau+\sigma)$ at time $\tau+\sigma$*

$$\begin{cases} -\dfrac{dDv_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\mu}}{d|Dv_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\mu}|}(t) - \mathrm{bar}(\mu_t) \in N_{C(t)}(v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\mu}(t)) \\ v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\mu}(\tau+\sigma) = u_{\tau,x,\nu}(\tau+\sigma) \in C(\tau+\sigma) \end{cases} \tag{2.3}$$

Proof. Let

$$W_J(\tau, x) := \sup_{\nu \in \mathcal{R}} \{ \int_\tau^{\tau+\sigma} [\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz)]dt + V_J(\tau + \sigma, u_{\tau,x,\nu}(\tau + \sigma))\}.$$

For any $\nu \in \mathcal{R}$, we have

$$\begin{aligned} \int_\tau^T [\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz)]dt = & \int_\tau^{\tau+\sigma} [\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz)]dt \\ & + \int_{\tau+\sigma}^T [\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz)]dt. \end{aligned}$$

[1]It is necessary to write completely the expression of the trajectory $v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\mu}$ that depends on $(\nu,\mu) \in \mathcal{R} \times \mathcal{R}$ in order to get the continuous dependence with respect to $\nu \in \mathcal{R}$ of $V_J(\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma))$.

By the definition of $V_J(\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma))$ we have

$$V_J(\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma)) \geq \int_{\tau+\sigma}^{T} [\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz)]dt.$$

It follows that

$$\int_{\tau}^{T} [\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz)]dt \leq \int_{\tau}^{\tau+\sigma} [\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz)]dt + V_J(\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma)).$$

By taking the supremum on $\nu \in \mathcal{R}$ in this inequality, we get

$$\begin{aligned} V_J(\tau, x) &\leq \sup_{\nu\in\mathcal{R}} \{\int_{\tau}^{\tau+\sigma} [\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz)]dt + V_J(\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma))\} \\ &= W_J(\tau, x). \end{aligned}$$

Let us prove the converse inequality.

Main fact : $\nu \mapsto V_J(\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma))$ is continuous on $\mathcal{R}$.
Let us focus on the expression of $V_J(\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma))$:

$$V_J(\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma)) = \sup_{\mu\in\mathcal{R}} \int_{\tau+\sigma}^{T} [\int_Z J(t, v_{\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma),\mu}(t), z)\mu_t(dz)]\, dt$$

where $v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\mu}$ denotes the trajectory solution on $[\tau+\sigma, T]$ associated with the control $\mu \in \mathcal{R}$ starting from $u_{\tau,x,\nu}(\tau+\sigma)$ at time $\tau+\sigma$ in (2.3). Using the remark of Theorem 2.10 concerning the continuous dependence of the solution with respect to the state and the control, it is readily seen that the mapping $(\nu, \mu) \mapsto v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\mu}(t)$ is continuous on $\mathcal{R}\times\mathcal{R}$ for each $t \in [\tau, T]$, namely if ν^n stably converges to $\nu \in \mathcal{R}$ and μ^n stably converges to $\mu \in \mathcal{R}$, then $v_{\tau+\sigma,u_{\tau,x,\nu^n}(\tau+\sigma),\mu^n}$ pointwise converges to $v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\mu}$. By using the fiber product of Young measure (see Theorem 2.1 or [10, Theorem 3.3.1]), we deduce that $(\nu,\mu) \mapsto \int_{\tau+\sigma}^{T}[\int_Z J(t, v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\mu}(t), z)\mu_t(dz)]\, dt$ is continuous on $\mathcal{R}\times\mathcal{R}$. Consequently $\nu \mapsto V_J(\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma))$ is continuous on $\mathcal{R}$. Hence the mapping $\nu \mapsto \int_{\tau}^{\tau+\sigma}[\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz)]dt + V_J(\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma))$ is continuous on $\mathcal{R}$. By compactness of $\mathcal{R}$, there is a maximum point $\nu^1 \in \mathcal{R}$ such that

$$W_J(\tau, x) = \int_{\tau}^{\tau+\sigma} [\int_Z J(t, u_{\tau,x,\nu^1}(t), z)\nu^1_t(dz)]dt + V_J(\tau+\sigma, u_{\tau,x,\nu^1}(\tau+\sigma)).$$

Similarly there is $\mu^2 \in \mathcal{R}$ such that

$$V_J(\tau+\sigma, u_{\tau,x,\nu^1}(\tau+\sigma)) = \int_{\tau+\sigma}^{T} [\int_Z J(t, v_{\tau+\sigma,u_{\tau,x,\nu^1}(\tau+\sigma),\mu^2}(t), z)\mu^2_t(dz)]dt$$

where

$$v_{\tau+\sigma,u_{\tau,x,\nu^1}(\tau+\sigma),\mu^2}(t)$$

denotes the trajectory solution associated with the control $\mu^2 \in \mathcal{R}$ starting from $u_{\tau,x,\nu^1}(\tau+\sigma)$ at time $\tau+\sigma$ to the $(\mathcal{PSW})(C;\mu^2;u_{\tau,x,\nu^1}(\tau+\sigma))$ defined on $[\tau+\sigma,T]$

$$-\frac{dDv_{\tau+\sigma,u_{\tau,x,\nu^1}(\tau+\sigma),\mu^2}}{d|Dv_{\tau+\sigma,u_{\tau,x,\nu^1}(\tau+\sigma),\mu^2}|}(t) - \mathrm{bar}(\mu^2_t) \in N_{C(t)}(v_{\tau+\sigma,u_{\tau,x,\nu^1}(\tau+\sigma),\mu^2}(t)).$$

Let us set

$$\overline{\nu} := 1_{[\tau,\tau+\sigma]}\nu^1 + 1_{[\tau+\sigma,T]}\mu^2.$$

Then $\overline{\nu} \in \mathcal{R}$. Let $w_{\tau,x,\overline{\nu}}$ be the trajectory solution on $[\tau,T]$ associated with $\overline{\nu} \in \mathcal{R}$, that is,

$$\begin{cases} -\dfrac{dDw_{\tau,x,\overline{\nu}}}{d|Dw_{\tau,x,\overline{\nu}}|}(t) - \mathrm{bar}(\overline{\nu}_t) \in N_{C(t)}(w_{\tau,x,\overline{\nu}}(t)),\ t \in [\tau,T] \\ w_{\tau,x,\overline{\nu}}(\tau) = x \in C(\tau). \end{cases}$$

By uniqueness of the solution, we have

$$w_{\tau,x,\overline{\nu}}(t) = u_{\tau,x,\nu^1}(t),\ \forall t \in [\tau,\tau+\sigma],$$
$$w_{\tau,x,\overline{\nu}}(t) = v_{\tau+\sigma,u_{\tau,x,\nu^1}(\tau+\sigma),\mu^2}(t),\ \forall t \in [\tau+\sigma,T].$$

Coming back to the expression of V_J and W_J we have

$$\begin{aligned} W_J(\tau,x) = \int_\tau^{\tau+\sigma} &[\int_Z J(t,u_{\tau,x,\nu^1}(t),z)\nu^1_t(dz)]\,dt \\ &+ \int_{\tau+\sigma}^T [\int_Z J(t,v_{\tau+\sigma,u_{\tau,x,\nu^1}(\tau+\sigma),\mu^2}(t),z)\mu^2_t(dz)]\,dt \\ &= \int_\tau^1 [\int_Z J(t,w_{\tau,x,\overline{\nu}}(t),z)\overline{\nu}_t(dz)]\,dt \\ &\le \sup_{\nu\in\mathcal{R}}\{\int_\tau^1 [\int_Z J(t,u_{\tau,x,\nu}(t),z)\nu_t(dz)]\,dt = V_J(\tau,x). \end{aligned}$$

□

In the above results we have considered the sweeping process with control Young measure $(\mathcal{PSW})(C;\nu;x)$

$$\begin{cases} -\dfrac{dDu_{\tau,x,\nu}}{d|Du_{\tau,x,\nu}|}(t) - \mathrm{bar}(\nu_t) \in N_{C(t)}(u_{\tau,x,\nu}(t)),\ t \in [\tau,T] \\ u_{\tau,x,\nu}(\tau) = x \in C(\tau) \end{cases}$$

with $\nu \in \mathcal{R}$, using the continuity of the solution with respect to the control $\nu \in \mathcal{R}$. In this case the perturbation $\mathrm{bar}(\nu_t)$ is of simple nature. Now we will treat a more general case. Namely we consider a mapping $f : [0,T] \times E \to E$ satisfying:
(i) for every fixed $t \in [0,T]$, $f(t,.)$ is continuous on E,
(ii) for every $x \in E$, $f(.,x)$ is Lebesgue-measurable on $[0,T]$,
(iii) there is a nonnegative Lebesgue integrable function g such that $f(t,x) \in g(t)(1+||x||)\bar{B}_E(0,1)$ for all (t,x) in $[0,T] \times E$,
(iv) there exists $\alpha \in L^1_{\mathbf{R}^+}[0,T])$ such that

$$||f(t,x_1) - f(t,x_2)|| \le \alpha(t)\,||x_1 - x_2||$$

for all $(t,x_1),(t,x_2) \in [0,T] \times E$.

We consider the sweeping process $(\mathcal{PSW})$ on $[0,T]$

$$\begin{cases} -\dfrac{dDu_{x,\lambda}}{d|Du_{x,\lambda}|}(t) - \displaystyle\int_Z f(t, u_{\tau,x,\lambda}(t), z)\lambda_t(dz) \in N_{C(t)}(u_{x,\lambda}(t)) \\ u_{x,\lambda}(0) = x \in C(0) \end{cases}$$

where $C : [0,T] \to ck(E)$ is a convex compact valued mapping satisfying (*) and (**), Z is a compact subset of E and $\mathcal{R}$ is the space of relaxed controls associated with Z. By virtue of Proposition 2.3 or [5, Theorem 4.4], for each $\lambda \in \mathcal{R}$ there is a unique BVC solution $u_{x,\lambda}$ for this inclusion. Further, arguing as in the proof of [7, Theorems 4.2–4.3], it is not difficult to see that $(u_{x,\lambda})_{\lambda\in\mathcal{R}}$ is uniformly bounded and uniformly bounded in variation, that is, $\sup_{t\in[0,T]} \sup_{\lambda\in\mathcal{R}} ||u_{x,\lambda}(t)|| \leq R < \infty$ and $\sup_{\lambda\in\mathcal{R}} ||Du_{x,\lambda}|| := \sup_{\lambda\in\mathcal{R}} \int_\tau^T |Du_{x,\lambda}| \leq K < \infty$.

Here is a main result in this section dealing with the continuous dependence of the solution with respect to the control Young measure in the above $(\mathcal{PSW})$.

Theorem 2.13. *Let $C : [0,T] \to ck(E)$ be a convex compact valued continuous mapping satisfying (*) and (**). Let $u_{x,\lambda}$ be the BVC solution to the following $(\mathcal{PSW})$ on $[0,T]$*

$$\begin{cases} -\dfrac{dDu_{x,\lambda}}{d|Du_{x,\lambda}|}(t) - \displaystyle\int_Z f(t, u_{x,\lambda}(t), z)\lambda_t(dz) \in N_{C(t)}(u_{x,\lambda}(t)) \\ u_{x,\lambda}(0) = x \in C(0) \end{cases}$$

Then the following hold:

(a) *For each $t \in [0,T]$, the mapping $\lambda \mapsto u_{x,\lambda}(t)$ is continuous on $\mathcal{R}$, here $\mathcal{R}$ is endowed with the stable topology.*

(b) *Let $J : [0,T]\times E\times Z \to E$ be bounded continuous mapping, then the mapping*

$$\lambda \mapsto \int_0^T [\int_Z J(t, u_{x,\lambda}(t), z)\lambda_t(dz)]dt$$

is continuous on $\mathcal{R}$ with respect to the stable topology. Consequently

$$\inf_{\zeta\in\mathcal{Z}} \int_0^T J(t, u_{x,\zeta}(t), \zeta(t))\, dt = \min_{\lambda\in\mathcal{R}} \int_0^T [\int_Z J(t, u_{x,\lambda}(t), z)\lambda_t(dz)]dt.$$

Proof. (a) Let $\lambda^n \in \mathcal{R}$ stably converge to $\lambda^\infty \in \mathcal{R}$. We need to show that (u_{x,λ^n}) pointwise converges to u_{x,λ^∞}. Since (u_{x,λ^n}) is uniformly bounded and uniformly bounded in variation, by Banach-Helly Theorem [21, Theorem 2.1], the sequence (u_{x,λ^n}) is relatively sequentially compact for the topology of pointwise convergence. Hence we may assume, by extracting a subsequence that (u_{x,λ^n}) pointwise converges to a BV function u^∞ with $u^\infty(0) = x$. Let us set

$$h_\lambda(t,x) = \int_Z f(t,x,z)\lambda_t(dz) \quad \forall (t,x,\lambda) \in [0,T] \times E \times \mathcal{R}.$$

Then we have

$$\begin{cases} -\dfrac{dDu_{x,\lambda^n}}{d|Du_{x,\lambda^n}|}(t) - h_{\lambda^n}(t, u_{x,\lambda^n}(t)) \in N(C(t); u_{x,\lambda^n}(t)) \\ u_{x,\lambda^n}(0) = x \in C(0). \end{cases}$$

Let us consider a positive Radon measure $d\nu$ such that $|Du_{\lambda^n}|+|Du_{\lambda^\infty}|+dt << d\nu$ for all $n \in \mathbf{N}$. Then $|Du_{\lambda^n}| << d\nu, |Du_{\lambda^\infty}| << d\nu$ and $dt << d\nu$. In view of the characterization of the solution to $(\mathcal{PSW})$ in [7, Theorem 4.1], we have

$$-\frac{dDu_{x,\lambda^n}}{d\nu} - h_{\lambda^n}(t, u_{x,\lambda^n}(t))\frac{dt}{d\nu}(t) \in N_{C(t)}(u_{x,\lambda^n}(t)),\ \nu\text{-a.e.} \tag{2.4}$$

Since $x \mapsto N_{C(t)}(x)$ is monotone, (2.4) implies

$$\Big\langle \Big(-\frac{dDu_{x,\lambda^n}}{d\nu} - h_{\lambda^n}(t, u_{x,\lambda^n}(t))\frac{dt}{d\nu}(t)\Big) - \Big(-\frac{dDu_{x,\lambda^\infty}}{d\nu} - h_{\lambda^\infty}(t, u_{x,\lambda^\infty}(t))\frac{dt}{d\nu}(t)\Big), u_{x,\lambda^n}(t) - u_{x,\lambda^\infty}(t)\Big\rangle \geq 0,\ \nu\text{-a.e.}$$

Equivalently,

$$\Big\langle \frac{dD(u_{x,\lambda^n} - u_{x,\lambda^\infty})}{d\nu}, u_{x,\lambda^n}(t) - u_{x,\lambda^\infty}(t)\Big\rangle \leq -\Big\langle h_{\lambda^n}(t, u_{x,\lambda^n}(t)) - h_{\lambda^\infty}(t, u_{x,\lambda^\infty}(t)), (u_{x,\lambda^n}(t) - u_{x,\lambda^\infty}(t))\frac{dt}{d\nu}\Big\rangle,\ \nu\text{-a.e.} \tag{2.5}$$

Integrating (2.5) over $[0, t]$ ($t \in [0, T]$) with respect to the measure $d\nu$ yields

$$\int_0^t \langle u_{x,\lambda^n} - u_{x,\lambda^\infty}, D(u_{x,\lambda^n} - u_{x,\lambda^\infty})\rangle \leq -\int_0^t \langle h_{\lambda^n}(s, u_{x,\lambda^n}(s)) - h_{\lambda^\infty}(s, u_{x,\lambda^\infty}(s)), u_{x,\lambda^n}(s) - u_{x,\lambda^\infty}(s)\rangle\, ds. \tag{2.6}$$

By (2.6) and [23], it follows that

$$\frac{1}{2}||u_{x,\lambda^n}(t) - u_{x,\lambda^\infty}(t)||^2 = \int_0^t \langle u_{x,\lambda^n} - u_{x,\lambda^\infty}, D(u_{x,\lambda^n} - u_{x,\lambda^\infty})\rangle \leq -\int_0^t \langle h_{\lambda^n}(s, u_{x,\lambda^n}(s)) - h_{\lambda^\infty}(s, u_{x,\lambda^\infty}(s)), u_{x,\lambda^n}(s) - u_{x,\lambda^\infty}(s)\rangle\, ds. \tag{2.7}$$

Let us set

$$L_n(t) = \int_0^t \langle u_{x,\lambda^n}(s) - u_{x,\lambda^\infty}(s), -h_{\lambda^n}(s, u_{x,\lambda^n}(s)) + h_{\lambda^\infty}(s, u_{x,\lambda^\infty}(s))\rangle\, ds.$$

Then $L_n(t) = L_n^1(t) + L_n^2(t) + L_n^3(t)$ where

$$L_n^1(t) = \int_0^t \langle u_{x,\lambda^n}(s) - u_{x,\lambda^\infty}(s), -h_{\lambda^n}(s, u_{x,\lambda^n}(s)) + h_{\lambda^n}(s, u_{x,\lambda^\infty}(s))\rangle\, ds,$$

$$L_n^2(t) = \int_0^t \langle u_{x,\lambda^n}(s) - u^\infty(s), -h_{\lambda^n}(s, u_{x,\lambda^\infty}(s)) + h_{\lambda^\infty}(s, u_{x,\lambda^\infty}(s))\rangle\, ds,$$

$$L_n^3(t) = \int_0^t \langle u^\infty(s) - u_{x,\lambda^\infty}(s), -h_{\lambda^n}(s, u_{x,\lambda^\infty}(s)) + h_{\lambda^\infty}(s, u_{x,\lambda^\infty}(s))\rangle\, ds.$$

As $u_{x,\lambda^n}(s) \to u^\infty(s)$ for every fixed $s \in [0,T]$, and, by (iii),

$$||h_{\lambda^n}(s, u_{x,\lambda^n}(s)|| \le g(s)(1 + ||u_{x,\lambda^n}(s))||) \le g(s)(1 + R)$$

for all $n \in \mathbf{N} \cup \{\infty\}$, we see that $L^2_n(t) \to 0$ when $n \to \infty$ for each $t \in [[0,T]$. By (iii), the integrand

$$h(s,z) := \langle u^\infty(s) - u_{x,\lambda^\infty}(s), f(s, u_{x,\lambda^\infty}(s), z)\rangle$$

is Carathéodory integrable and is estimated by

$$||h(s,z)|| \le g(s)(1+R)||u^\infty - u_{x,\lambda^\infty}||$$

for all $(s,z) \in [0,T] \times Z$. Hence $h \in L^1_{C(Z)}([0,T])$. As λ^n stably converges to λ^∞, for every $t \in [0,T]$, we have

$$\lim_{n\to\infty} \int_0^t [\int_Z h(s,z)\lambda^n_s(dz)]\, dt \to \int_0^t [\int_Z h(s,z)\lambda^\infty_s(dz)]\, dt.$$

So $\lim_{n\to\infty} L^3_n(t) = 0$, for every $t \in [0,T]$. By (iv) we have that $|L^1_n(t)| \le \int_0^t \alpha(s)||u_{x,\lambda^n}(s) - u_{x,\lambda^\infty}(s)||^2\, ds$. Finally by (2.7) we get

$$\frac{1}{2}||u_{x,\lambda^n}(t) - u_{x,\lambda^\infty}(t)||^2 \le L^2_n(t) + L^3_n(t) + \int_0^t \alpha(s)||u_{x,\lambda^n}(s) - u_{x,\lambda^\infty}(s)||^2\, ds.$$

As $L^2_n(t) \to 0$ and $L^3_n(t) \to 0$, for all $t \in [0,1]$, by Gronwall's lemma we have that $u_{x,\lambda^n}(t) \to u_{x,\lambda^\infty}(t)$, for all $t \in [0,1]$, and hence $u_{x,\lambda^\infty} = u^\infty$.
(b) Let λ^n stably converge to λ in $\mathcal{R}$. By (a), u_{x,λ^n} pointwise converges to u_{x,λ^∞} so that

$$\lim_{n\to\infty} \int_0^T [\int_Z J(t, u_{x,\lambda^n}(t), z)\lambda^n_t(dz)]\, dt = \int_0^T [\int_Z J(t, u_{x,\lambda^\infty})(t), z)\lambda^\infty_t(dz)]\, dt$$

using the fiber product for Young measures in Theorem 2.1 or [10, Theorem 3.3.1]. Again by compactness of $\mathcal{R}$ and by continuity of the solutions with respect to the controls in (a), we conclude that

$$\inf_{\zeta\in\mathcal{Z}} \int_0^T J(t, u_{x,\zeta}(t), \zeta(t))\, dt = \min_{\lambda\in\mathcal{R}} \int_0^T [\int_Z J(t, u_{x,\lambda}(t), z)\lambda_t(dz)]\, dt.$$

□

Remarks. (1) It is worth to mention that the condition

$$(^{**}) \qquad \forall t \in [0,T], \quad 2r\overline{B}(0,1) \subset C(t)$$

can be replaced by the following more general condition

$$(^{***}) \qquad \forall t \in [0,T], \quad \operatorname{Int} C(t) \neq \emptyset.$$

We would like to mention that, in the above dynamic models, we need some new techniques, because the closed convex moving set $C(t)$ is only assumed to be *continuous with* $\operatorname{Int} C(t) \neq \emptyset$, by contrast to earlier problems considered in the above mentioned works.

(2) The stability for the sweeping process [2] with continuous moving sets is of importance. This allows to obtain a general Skohorod problem and is decisive in the solvability of the Skorohod differential equation.

3. Viscosity problem in the sweeping process: The Lipschitzean case

In this section we present several viscosity problems for the $(\mathcal{PSW})$ when $E = \mathbf{R}^d$ and $C : [0, T] \Rightarrow E$ is a convex compact valued L-Lipschitzean mapping:

$$|d(x, C(t)) - d(y, C(\tau))| \leq L|t - \tau| + ||x - y||, \forall x, y \in E \times E, \forall t, \tau \in [0, T] \times [0, T].$$

Given a closed convex valued L-Lipschitzean mapping, then, from a classical result in the sweeping process [22], given $x \in C(0)$, there is a unique L-Lipschitzean function $u : [0, T] \to E$ with $||\dot{u}(t)|| \leq L$ a.e. such that

$$\begin{cases} -\dot{u}(t) \in N_{C(t)}(u(t)),\ t \in [0, T] \\ u(0) = x \in C(0). \end{cases}$$

Now we state and summarize a stability result in $(\mathcal{PSW})$ for the Lipschitzean case.

Theorem 3.1. *Let $(C_n), n \in \mathbf{N} \cup \{\infty\}$ be a sequence of $ck(E)$-valued L-Lipschitzean mappings. Assume that the following assumptions are fulfilled:*

(i) *for any sequence (t_n) in $[0, T]$ converging to t, $C_\infty(t) \subset Li(C_n(t_n))$*
(ii) *for any $t \in [0, T]$, $Ls(C_n(t)) \subset C_\infty(t)$.*

Let $(u_k)_{k \in \mathbf{N}}$ be the L-Lipschitzean solutions of the sweeping processes

$$-\dot{u}_k(t) \in N_{C_k(t)}(t)(u_k(t)),\ u_k(0) = a_k \in C_k(0).$$

If $(a_k)_{k \in \mathbf{N}}$ converges to $a_\infty \in E$, then $(u_k)_{k \in \mathbf{N}}$ converges uniformly to the unique L-Lipschitzean solution u_∞ of the sweeping process

$$-\dot{u}_\infty(t) \in N_{C_\infty(t)}(t)(u_\infty(t)),\ u_\infty(0) = a_\infty \in C_\infty(0).$$

Proof. Here one can adapt the proof of the stability result for the sweeping process in the BVC case [2](Theorem 4.4). For convenience we provide the proof in the Lipschitzean case that is based on an epi-lower convergence result of integral convex functional on the space of vector measures and the definition of normal cone for closed convex set in the sense of Convex Analysis. Since $(u_k)_{k \in \mathbf{N}}$ is equi-lipschitzean, it is relatively compact in $C_E([0, T])$, therefore $(u_k)_{k \in \mathbf{N}}$ converges uniformly to an L-Lipschitzean function $u_\infty \in C_E([0, T])$ with $\dot{u}_k \to \dot{u}_\infty$ with respect to the $\sigma(L^1_E, L^\infty_E)$ topology and $u_\infty(0) = a_\infty \in C_\infty(0)$. It remains to check that

$$-\dot{u}_\infty(t) \in N_{C_\infty(t)}(u_\infty(t)) \text{ a.e.}$$

Main fact: $\liminf_k \int_0^T \delta^*(-\dot{u}_k(t), C_k(t))\, dt \geq \int_0^T \delta^*(-\dot{u}_\infty(t), C_\infty(t))\, dt$.
Let $\delta^*(x, C_k(t))$ denote the support function of C_k. From (i) and (ii), it is not difficult to check that $\delta^*(., C_k(.))$ lower epiconverges to $\delta^*(., C_\infty(.))$, that is, for any sequence (t_k, x_k) in $[0, T] \times E$ converging to $(t, x) \in [0, T] \times E$ we have that $\liminf_k \delta^*(x_k, C_k(t_k)) \geq \delta^*(x, C_\infty(t))$. Let φ_∞ be a continuous selection of C_∞. By [2, Proposition 3.9], there are continuous selections φ_k of C_k which converge uniformly to φ_∞. Let us set $\Gamma_k = C_k - \varphi_k$, $k \in \mathbf{N} \cup \{\infty\}$. Then $\delta^*(., \Gamma_k(.))$ is positive and lower epiconverges to $\delta^*(., \Gamma_\infty(.))$. Now let $m_k = -\dot{u}_k\, dt$ be the vector measure of density $-\dot{u}_k$ $(k \in \mathbf{N})$. Then m_k converges weakly to $m_\infty = -\dot{u}_\infty\, dt$

in the space of bounded vector measure $\mathcal{M}^b([0,T],E)$. Hence by invoking a lower epi-convergence version of Reshetnyak Theorem [2, Theorem 3.4], we have

$$\liminf_k \int_0^T \delta^*(\frac{dm_k}{d|m_k|}(t), \Gamma_k(t)) d|m_k|(t) \geq \int_0^T \delta^*(\frac{dm_\infty}{d|m_\infty|}, \Gamma_\infty(t)), d|m_\infty|(t).$$

This gives

$$\text{(3.1)} \quad \liminf_k \int_0^T \delta^*(\frac{dm_k}{d|m_k|}(t), C_k(t)) d|m_k|(t) \geq \int_0^T \delta^*(\frac{dm_\infty}{d|m_\infty|}(t), C_\infty(t)) d|m_\infty|(t).$$

Since $m_k = -\dot{u}_k\, dt$, we have, for a.e. $t \in [0,T]$ and for all $k \in \mathbf{N}$,

$$\text{(3.2)} \qquad dm_k = \frac{dm_k}{d|m_k|} \frac{d|m_k|}{dt}\, dt = \frac{dm_k}{d|m_k|} ||\dot{u}_k||\, dt.$$

Hence (3.2) yields

$$\begin{aligned} \delta^*(\frac{dm_k}{dt}(t), C_k(t)) &= \delta^*(\frac{dm_k}{d|m_k|}(t)||\dot{u}_k(t)||, C_k(t)) \\ &= ||\dot{u}_k(t)||\delta^*(\frac{dm_k}{d|m_k|}(t), C_k(t)). \end{aligned}$$

By integrating we get

$$\begin{aligned} \int_0^T \delta^*(\frac{dm_k}{dt}(t), C_k(t))\, dt &= \int_0^T \delta^*(\frac{dm_k}{d|m_k|}(t), C_k(t)) ||\dot{u}_k(t)||\, dt \\ \text{(3.3)} \qquad &= \int_0^T \delta^*(\frac{dm_k}{d|m_k|}(t), C_k(t)) d|m_k|(t). \end{aligned}$$

Using (3.3) (also for $k = \infty$) and coming back to (3.1.1) we finally obtain

$$\liminf_k \int_0^T \delta^*(-\dot{u}_k(t), C_k(t))\, dt \geq \int_0^T \delta^*(-\dot{u}_\infty(t), C_\infty(t))\, dt.$$

From $-\dot{u}_k(t) \in N_{C_k(t)}(t)(u_k(t))$ we have

$$\delta^*(-\dot{u}_k(t), C_k(t)) + \langle u_k(t), \dot{u}_k(t) \rangle \leq 0.$$

Integrating on $[0,T]$ yields

$$\text{(3.4)} \qquad \int_0^T \delta^*(-\dot{u}_k(t), C_k(t))\, dt + \int_0^T \langle u_k(t), \dot{u}_k(t) \rangle\, dt \leq 0.$$

It is clear that

$$\lim_{k \to \infty} \int_0^T \langle u_k(t), \dot{u}_k(t) \rangle\, dt = \int_0^T \langle u_\infty(t), \dot{u}_\infty(t) \rangle\, dt.$$

Taking the $\liminf$ in (3.4) and using the main fact gives

$$\int_0^T \delta^*(-\dot{u}_\infty(t), C_\infty(t))\, dt + \int_0^T \langle u_\infty(t), \dot{u}_\infty(t) \rangle\, dt \leq 0.$$

Whence

$$\delta^*(-\dot{u}_\infty(t), C_\infty(t)) + \langle u_\infty(t), \dot{u}_\infty(t) \rangle \leq 0 \text{ a.e.}$$

thus proving the desired inclusion

$$-\dot{u}_\infty(t) \in N_{C_\infty(t)}(u_\infty(t)) \text{ a.e.}$$

□

Theorem 3.1 allows to get a stability result in the $(\mathcal{PSW})$ when C is Lipschitzean that is useful for further applications.

Theorem 3.2. *Let $C : [0,T] \to ck(E)$ be a convex compact valued L-Lipschitzean mapping. Let $h_n, h_\infty \in L^\infty_E([0,T]$ with $||h_n(t)|| \leq M$ for all $n \in \mathbf{N} \cup \{\infty\}$ and for all $t \in [0,T]$ for some positive constant M and let us consider the $(\mathcal{PSW})(C; h_n; x_n)$*

$$\begin{cases} -\dot{u}_{x_n,h_n} - h_n(t) \in N_{C(t)}(u_{x_n,h_n}(t)) \\ u_{x_n,h_n}(0) = x_n \in C(0) \end{cases}$$

where u_{x_n,h_n} denotes the Lipschitzean solution for this sweeping process. If $x_n \to x_\infty$ and if $h_n \to h_\infty$ with respect to the $\sigma(L^1_E, L^\infty_E)$ topology, then u_{x_n,h_n} converges uniformly to the Lipschitzean solution u_{x_∞,h_∞} of the $(\mathcal{PSW})(C; h_\infty; x_\infty)$

$$\begin{cases} -\dot{u}_{x_\infty,h_\infty} - h_\infty(t) \in N_{C(t)}(u_{x_\infty,h_\infty}(t)) \\ \qquad\qquad u_{x_\infty,h_\infty}(0) = x_\infty \in C(0) \end{cases}$$

Proof. The proof follows the same lines as in Theorem 2.8 by applying the stability result for Lipschitzean process in Theorem 3.1. □

Similarly we have a useful variant:

Theorem 3.3. *Let $C : [0,T] \to ck(E)$ be a convex compact valued L-Lipschitzean mapping. Let Z be a compact subset in E an $\mathcal{R}$ the space of relaxed controls associated with Z. Let $\nu^n, \nu^\infty \in \mathcal{R}$ ($n \in \mathbf{N}$ and let us consider the $(\mathcal{PSW})(C; \nu^n; x_n)$ process*

$$\begin{cases} -\dot{u}_{x_n,\nu^n} - \mathrm{bar}(\nu^n_t) \in N_{C(t)}(u_{x_n,\nu_n}(t)) \\ u_{x_n,\nu^n}(0) = x_n \in C(0) \end{cases}$$

where u_{x_n,ν^n} is the Lipschitzean solution for this sweeping process. If $x_n \to x_\infty$ and if $\nu_n \to \nu_\infty$ stably, then u_{x_n,ν^n} converges uniformly to the Lipschitzean solution u_{x_∞,ν^∞} of the $(\mathcal{PSW})(C; \nu^\infty; x_\infty)$ process

$$\begin{cases} -\dot{u}_{x_\infty,\nu^\infty} - \mathrm{bar}(\nu^\infty_t) \in N_{C(t)}(u_{x_\infty,\nu^\infty}(t)) \\ u_{x_\infty,\nu^\infty}(0) = x_\infty \in C(0) \end{cases}$$

From Theorem 3.2, it is now possible to show that the value function associated with $(\mathcal{PSW})$ when C is Lipschitzean satisfies the dynamic programming principle (DPP), namely

Theorem 3.4 (of dynamic programming principle)**.** *Let $C : [0,T] \to ck(E)$ be a convex compact valued L-Lipschitzean mapping. Let Z be a convex compact subset in E an $\mathcal{R}$ the space of relaxed controls associated with Z. Let $x \in E$, $\tau < T$ and $\sigma > 0$ such that $\tau + \sigma < T$. Assume that $J : [0,T] \times E \times Z \to \mathbf{R}$ is bounded and continuous. Let us consider the value function*

$$V_J(\tau, x) = \sup_{\nu \in \mathcal{R}} \int_\tau^T [\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz)]\, dt, \quad (\tau, x) \in [0,T[\times E$$

where $u_{\tau,x,\nu}$ is the Lipschitzean solution to the $(\mathcal{PSW})(C;\nu;x)$ defined on $[\tau,T]$ associated with the control $\nu \in \mathcal{R}$ starting from x at time τ

$$\begin{cases} -\dot{u}_{\tau,x,\nu} - \mathrm{bar}(\nu_t) \in N_{C(t)}(u_{\tau,x,\nu}(t)) \\ u_{\tau,x,\nu}(\tau) = x \in C(\tau) \end{cases}$$

Then the following holds:

$$V_J(\tau,x) = \sup_{\nu\in\mathcal{R}} \left(\int_\tau^{\tau+\sigma} [\int_Z J(t,u_{\tau,x,\nu}(t),z)\nu_t(dz)]dt + V_J(\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma)) \right)$$

with

$$V_J(\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma)) = \sup_{\mu\in\mathcal{R}} \int_{\tau+\sigma}^T [\int_Z J(t, v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\mu}(t),z)\mu_t(dz)]\,dt$$

where $v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\mu}$ is the Lipschitzean solution to $(\mathcal{PSW})(C;\mu;u_{\tau,x,\nu}(\tau+\sigma))$ defined on $[\tau+\sigma,T]$ associated with the control $\mu\in\mathcal{R}$ starting from $u_{\tau,x,\nu}(\tau+\sigma)$ at time $\tau+\sigma$.

Proof. We omit the proof since it follows the lines of the proof of Theorem 2.12 using the dependence of the solution with respect to the state $x \in E$ and the controls $\nu \in \mathcal{R}$. □

The viscosity property of the value function associated to the $(\mathcal{PSW})$ in the Lipschitz case is now within reach. Let us mention a useful lemma that is borrowed from [3, Lemma 4.1]. See also [5, 9, 10, 11] for related results.

Lemma 3.5. *Let Z be a convex compact subset in E, $\mathcal{M}^1_+(Z)$ is endowed with the vague topology and $\mathcal{R}$ the space of relaxed controls associated with Z. Let $\Lambda : [0,T]\times E\times\mathcal{M}^1_+(Z) \to \mathbf{R}$ be an upper semicontinuous function such that the restriction of Λ to $[0,T]\times B\times\mathcal{M}^1_+(Z)$ is bounded on any bounded subset B of E. Let $C : [0,T] \to ck(E)$ be a convex compact valued L-Lipschitzean mapping. Let $(t_0,x_0)\in[0,T]\times E$. If $\max_{\mu\in\mathcal{M}^1_+(Z)} \Lambda(t_0,x_0,\mu) < -\eta < 0$ for some $\eta > 0$, then there exist $\sigma > 0$ such that*

$$\sup_{\nu\in\mathcal{R}} \int_{t_0}^{t_0+\sigma} \Lambda(t, u_{t_0,x_0,\nu}(t),\nu_t)\,dt < -\frac{\sigma\eta}{2}$$

where $u_{t_0,x_0,\nu}$ is the trajectory solution of the sweeping process $(\mathcal{PSW})(C;\nu;x_0)$, associated with the control $\nu\in\mathcal{R}$ and starting from x_0 at time t_0

$$\begin{cases} -\dot{u}_{t_0,x_0,\nu}(t) - \mathrm{bar}(\nu_t) \in N_{C(t)}(u_{t_0,x_0,\nu}(t)),\ t\in[t_0,T] \\ u_{t_0,x_0,\nu}(t_0) = x_0. \end{cases}$$

Proof. By our assumption $\max_{\mu\in\mathcal{M}^1_+(Z)} \Lambda(t_0,x_0,\mu) < -\eta < 0$ for some $\eta > 0$. As the function $(t,x,\mu)\mapsto\Lambda(t,x,\mu)$ is upper semicontinuous, so is the function

$$(t,x) \mapsto \max_{\mu\in\mathcal{M}^1_+(Z)} \Lambda(t,x,\mu).$$

Hence there exists $\zeta > 0$ such that

$$\max_{\mu\in\mathcal{M}^1_+(Z)} \Lambda(t,x,\mu) < -\frac{\eta}{2}$$

for $0 < t - t_0 \leq \zeta$ and $||x - x_0|| \leq \zeta$. Thus, for small values of σ, we have

$$||u_{t_0,x_0,\nu}(t) - u_{t_0,x_0,\nu}(t_0)|| \leq \zeta$$

for all $t \in [t_0, t_0 + \sigma]$ and for all $\nu \in \mathcal{R}$ because $||\dot{u}_{t_0,x_0,\nu}(t)|| \leq L + 2|Z|$ for all $\nu \in \mathcal{R}$ and for all $t \in [0, T]$ ([1], Theorem 4.1' yields a sharper estimate). Hence $t \mapsto \Lambda(t, u_{t_0,x_0,\nu}(t), \nu_t)$ is bounded and Lebesgue-measurable on $[t_0, t_0 + \sigma]$. Then by integrating

$$\int_{t_0}^{t_0+\sigma} \Lambda(t, u_{t_0,x_0,\nu}(t), \nu_t)\, dt \leq \int_{t_0}^{t_0+\sigma} [\max_{\mu \in \mathcal{M}^1_+(Z)} \Lambda(t, u_{t_0,x_0,\nu}(t), \mu)]\, dt < -\frac{\sigma\eta}{2}.$$

□

Theorem 3.6 (of viscosity solution)**.** *Let $C : [0, T] \to ck(E)$ be a convex compact valued L-Lipschitzean mapping. Let Z be a compact subset in E and $\mathcal{R}$ be the space of relaxed control associated with Z. Assume that $J : [0, T] \times E \times Z \to \mathbf{R}$ is bounded and continuous. Let us consider the value function*

$$V_J(\tau, x) = \sup_{\nu \in \mathcal{R}} \int_\tau^T [\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz)]\, dt,\ (\tau, x) \in [0, T] \times E$$

where $u_{\tau,x,\nu}$ is the trajectory solution on $[\tau, T]$ of the sweeping process $(\mathcal{PSW})(C; \nu; x)$ associated with the control $\nu \in \mathcal{R}$ and starting from $x \in E$ at time τ

$$\begin{cases} -\dot{u}_{\tau,x,\nu}(t) - \mathrm{bar}(\nu_t) \in N_{C(t)}(u_{\tau,x,\nu}(t)),\ t \in [\tau, T] \\ u_{\tau,x,\nu}(\tau) = x \end{cases}$$

and the Hamiltonian

$$H(t, x, \rho) = \sup_{\mu \in \mathcal{M}^1_+(Z)} \Big(-\langle \rho, \mathrm{bar}(\mu) \rangle + \int_Z J(t, x, z)\mu(dz) \Big) + \delta^*(\rho, -(L + 3|Z|)\partial[d_{C(t)}](x)),$$

where $(t, x, \rho) \in [0, T] \times E \times E$ and $\partial[d_{C(t)}](x)$ denotes the subdifferential of the distance functions $x \mapsto d_{C(t)}x$. Then, V_J is a viscosity subsolution of the HJB equation

$$\frac{\partial U}{\partial t}(t, x) + H(t, x, \nabla U(t, x)) = 0,$$ [2]

that is, for any $\varphi \in C^1([0, T]) \times E)$ for which $V_J - \varphi$ reaches a local maximum at $(t_0, x_0) \in [0, T] \times E$, we have

$$H(t_0, x_0, \nabla\varphi(t_0, x_0)) + \frac{\partial \varphi}{\partial t}(t_0, x_0) \geq 0.$$

Proof. Assume by contradiction that there exists a $\varphi \in C^1([0, T] \times E)$ and a point $(t_0, x_0) \in [0, T] \times E$ for which

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + H(t_0, x_0, \nabla\varphi(t_0, x_0)) \leq -\eta < 0 \quad \text{for} \quad \eta > 0.$$

Applying Lemma 3.5, by taking

[2]here ∇U is the gradient of U with respect to the second variable

$$\Lambda(t,x,\mu) = -\langle \nabla\varphi(t,x), \mathrm{bar}(\mu)\rangle + \int_Z J(t,x,z)\mu(dz) + \delta^*(\nabla\varphi(t,x), -(L+3|Z|)\,\partial[d_{C(t)}](x)) + \frac{\partial\varphi}{\partial t}(t,x)$$

yields some $\sigma > 0$ such that

$$\sup_{\nu\in\mathcal{R}}\Bigg(\int_{t_0}^{t_0+\sigma}[\int_Z J(t,u_{t_0,x_0,\nu}(t),z)\nu_t(dz)]\,dt - \int_{t_0}^{t_0+\sigma}\langle\nabla\varphi(t,u_{t_0,x_0,\nu}(t),\mathrm{bar}(\nu_t)\rangle\,dt + \int_{t_0}^{t_0+\sigma}\delta^*(\nabla\varphi(t,u_{t_0,x_0,\nu}(t)),-(L+3|Z|)\,\partial[d_{C(t)}](u_{t_0,x_0,\nu}(t)))\,dt + \int_{t_0}^{t_0+\sigma}\frac{\partial\varphi}{\partial t}(t,u_{t_0,x_0,\nu}(t))\,dt\Bigg) \le -\frac{\sigma\eta}{2} \tag{3.5}$$

where $u_{t_0,x_0,\nu}$ is the trajectory solution of the sweeping process $(\mathcal{PSW})(C;\nu;x_0)$ associated with the control $\nu\in\mathcal{R}$ starting from x_0 at time t_0

$$\begin{cases} -\dot u_{t_0,x_0,\nu}(t) - \mathrm{bar}(\nu_t) \in N_{C(t)}(u_{t_0,x_0,\nu}(t))\; t\in[t_0,T] \\ u_{t_0,x_0,\nu}(t_0) = x_0 \end{cases}$$

Applying the dynamic programming principle (Theorem 3.4) gives

$$V_J(t_0,x_0) = \sup_{\nu\in\mathcal{R}}\Bigg(\int_{t_0}^{t_0+\sigma}[\int_Z J(t,u_{t_0,x_0,\nu}(t),z)\nu_t(dz)]dt + V_J(t_0+\sigma, u_{t_0,x_0,\nu}(t_0+\sigma))\Bigg). \tag{3.6}$$

Since $V_J-\varphi$ has a local maximum at (t_0,x_0), for small enough σ

$$V_J(t_0,x_0)-\varphi(t_0,x_0) \ge V_J(t_0+\sigma,u_{t_0,x_0,\nu}(t_0+\sigma)) - \varphi(t_0+\sigma,u_{t_0,x_0,\nu}(t_0+\sigma)) \tag{3.7}$$

for all $\nu\in\mathcal{R}$. By (3.6) for each $n\in\mathbf{N}$, there exists $\nu^n\in\mathcal{R}$ such that

$$V_J(t_0,x_0) \le \int_{t_0}^{t_0+\sigma}[\int_Z J(t,u_{t_0,x_0,\nu^n}(t)),z)\nu_t^n(dz)]\,dt + V_J(t_0+\sigma,u_{t_0,x_0,\nu^n}(t_0+\sigma)) + \frac{1}{n}. \tag{3.8}$$

From (3.7) and (3.8) we deduce that

$$V_J(t_0+\sigma,u_{t_0,x_0,\nu^n}(t_0+\sigma)) - \varphi(t_0+\sigma,u_{t_0,x_0,\nu^n}(t_0+\sigma)) \le \int_{t_0}^{t_0+\sigma}[\int_Z J(t,u_{t_0,x_0,\nu^n}(t)),z)\nu_t^n(dz)]dt + \frac{1}{n} - \varphi(t_0,x_0) + V_J(t_0+\sigma,u_{t_0,x_0,\nu^n}(t_0+\sigma)).$$

Therefore we have

$$0 \le \int_{t_0}^{t_0+\sigma}[\int_Z J(t,u_{t_0,x_0,\nu^n}(t)),z)\nu_t^n(dz)]\,dt \tag{3.9}$$

$$+\varphi(t_0+\sigma, u_{t_0,x_0,\nu^n}(t_0+\sigma)) - \varphi(t_0,x_0) + \frac{1}{n}.$$

As $\varphi \in C^1([0,T]\times E)$ we have

$$\begin{aligned}(3.10)\quad & \varphi(t_0+\sigma, u_{t_0,x_0,\nu^n}(t_0+\sigma)) - \varphi(t_0,x_0)\\ &= \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,\nu^n}(t)), \dot u_{t_0,x_0,\nu^n}(t)\rangle\, dt + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\nu^n}(t))\, dt.\end{aligned}$$

Since u_{t_0,x_0,ν^n} is the trajectory solution starting from x_0 at time t_0 to the sweeping process $(\mathcal{PSW})(C;\nu^n;x_0)$

$$\begin{cases} -\dot u_{t_0,x_0,\nu^n}(t) - \mathrm{bar}(\nu^n_t) \in N_{C(t)}(u_{t_0,x_0,\nu^n}(t)), \quad t\in[t_0,T]\\ u_{t_0,x_0,\nu^n}(t_0) = x_0\end{cases}$$

and since $\dot u_{t_0,x_0,\nu^n}(t) + \mathrm{bar}(\nu^n_t) \in \overline{B}_E(0, L+3|Z|)$, by the classical property of the normal convex cone we get

$$-\dot u_{t_0,x_0,\nu^n}(t) - \mathrm{bar}(\nu^n_t) \in (L+3|Z)|)\,\partial[d_{C(t)}](u_{t_0,x_0,\nu^n}(t))$$

so that (3.10) yields the estimate

$$\begin{aligned}& \varphi(t_0+\sigma, u_{t_0,x_0,\nu^n}(t_0+\sigma)) - \varphi(t_0,x_0)\\ (3.11)\quad &= \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,\nu^n}(t)), \dot u_{t_0,x_0,\nu^n}(t)\rangle\, dt + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\nu^n}(t))\, dt\\ &\le -\int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,\nu^n}(t)), \mathrm{bar}(\nu^n_t)\rangle\, dt\\ &+ \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t, u_{t_0,x_0,\nu^n}(t)), -(L+3|Z|)\,\partial[d_{C(t)}](u_{t_0,x_0,\nu^n}(t)))\, dt\\ &\qquad + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\nu^n}(t))\, dt.\end{aligned}$$

Inserting the estimate (3.11) into (3.9) we get

$$\begin{aligned}(3.12)\quad 0 \le & \int_{t_0}^{t_0+\sigma} [\int_Z J(t, u_{t_0,x_0,\nu^n}(t)), z)\nu^n_t(dz)]dt\\ & - \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,\nu^n}(t)), \mathrm{bar}(\nu^n_t)\rangle\, dt\\ & + \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t, u_{t_0,x_0,\nu^n}(t)), -(L+3|Z|)\,\partial[d_{C(t)}](u_{t_0,x_0,\nu^n}(t)))\, dt\\ & + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\nu^n}(t))\, dt + \frac{1}{n}.\end{aligned}$$

Then (3.5) and (3.12) yield $0 \le -\frac{\sigma\eta}{2} + \frac{1}{n}$ for all $n \in \mathbf{N}$. By passing to the limit when n goes to ∞ in this inequality, we get a contradiction: $0 \le -\frac{\sigma\eta}{2}$. The proof is therefore complete. □

4. Further extensions and applications

This study leads to several variants and the techniques developed here can be applied in other situations in differential games and also to the stochastic perturbation of the sweeping process. At this point, compare with earlier results dealing with ordinary differential equations (ODE) and evolution inclusions [3, 4, 5, 8, 9, 13, 14]. See also [11] dealing with the viscosity property of value function in second order (ODE). In this context, we assume along this section that $E = \mathbf{R}^d$ and that $C : [0,T] \rightrightarrows E$ is a convex compact valued mapping satisfying the Lipschitz condition

$$|d(x, C(t)) - d(y, C(\tau))| \leq L|t-\tau| + ||x-y||$$

for all $x, y \in E$ and for all $t, \tau \in [0,T]$ where $L > 0$ is the Lipschitz constant. Now we will study the DPP property of the value function

$$V_J(\tau, x) = \sup_{\nu \in \mathcal{R}} \int_\tau^T [\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz)]\, dt, \quad (\tau, x) \in [0, T[\times E$$

where $u_{\tau,x,\nu}$ is the Lipschitz solution to the $(\mathcal{PSW})(C;\nu;x)$ defined on $[\tau, T]$ associated to the control $\nu \in \mathcal{R}$ starting from x at time τ

$$\begin{cases} -\dot{u}_{\tau,x,\nu}(t) - \displaystyle\int_Z f(t, u_{\tau,x,\nu}(t), z)\nu_t(dz) \in N_{C(t)}(u_{\tau,x,\nu}(t)) \\ u_{\tau,x,\nu}(\tau) = x \in C(\tau) \end{cases}$$

where $f : [0,T] \times E \times Z \to E$ satisfies

(i) for every fixed $t \in [0,T], f(t,.,.)$ is continuous on $E \times Z$,
(ii) for every $(x,z) \in E \times Z, f(.,x,z)$ is Lebesgue-measurable on $[0,T]$,
(iii) there is a constant $R > 0$ such that $||f(t,x,z)|| \leq R$ for all (t,x,z) in $[0,T] \times E \times Z$,
(iv) there exists $\alpha \in L^1_{\mathbf{R}^+}[0,T])$ such that

$$||f(t,x_1,z) - f(t,x_2,z)|| \leq \alpha(t)||x_1 - x_2||$$

for all $(t,x_1,z), (t,x_2,z) \in [0,T] \times E$.

Note that, for $\nu \in \mathcal{R}$, the mapping

$$h_\nu : (t,x) \mapsto \int_Z f(t,x,z)\nu_t(dz)$$

inherits the properties

(1) for every fixed $t \in [0,T], h_\nu(t,.)$ is continuous on E,
(2) for every $x \in E, h_\nu(.,x)$ is Lebesgue-measurable on $[0,T]$,
(3) there is a constant $R > 0$ such that $||h_\nu(t,x)|| \leq R$ for all (t,x) in $[0,T] \times E$,
(4) there exists $\alpha \in L^1_{\mathbf{R}^+}[0,T])$ such that

$$||h_\nu(t,x_1) - h_\nu(t,x_2)|| \leq \alpha(t)||x_1 - x_2||$$

for all $(t,x_1), (t,x_2) \in [0,T] \times E$.

Consequently, by Theorem 4.1' in [1] the sweeping process

$$\begin{cases} -\dot{u}_{\tau,x,\nu}(t) - h_\nu(t, u_{\tau,x,\nu}(t)) \in N_{C(t)}(u_{\tau,x,\nu}(t)) \\ u_{\tau,x,\nu}(\tau) = x \in C(\tau) \end{cases}$$

admits a unique Lipschitz solution $u_{\tau,x,\nu}$ with $||\dot{u}_{\tau,x,\nu}(t)|| \le L + R$ a.e. Now using the tools developed here we will provide the following viscosity solution for the value function associated with the sweeping process. A series of lemmas will be needed.

Lemma 4.1. *Let u_{τ,x^n,ν^n} be the trajectory solution on $[\tau, T]$ associated the control $\nu^n \in \mathcal{R}$ starting from $x^n \in E$ at time τ to the sweeping process $(\mathcal{PSW})(C;\nu^n;x^n)$*

$$\begin{cases} -\dot{u}_{\tau,x^n,\nu^n}(t) - \int_Z f(t, u_{\tau,x^n,\nu^n}(t), z)\nu_t^n(dz) \in N_{C(t)}(u_{\tau,x^n,\nu^n}(t)) \\ u_{\tau,x^n,\nu^n}(\tau) = x^n \in C(\tau) \end{cases}$$

(a) *If (x^n) converges to x^∞ and ν^n stably converges to ν^∞, then u_{τ,x^n,ν^n} converges uniformly to $u_{\tau,x^\infty,\nu^\infty}$, which is the Lipschitzean solution of the sweeping process $(\mathcal{PSW})(C;\nu^\infty;x^\infty)$*

$$\begin{cases} -\dot{u}_{\tau,x^\infty,\nu^\infty}(t) - \int_Z f(t, u_{\tau,x^\infty,\nu^\infty}(t), z)\nu_t^\infty(dz) \in N_{C(t)}(u_{\tau,x^\infty,\nu^\infty}(t)) \\ u_{\tau,x^\infty,\nu^\infty}(\tau) = x^\infty \in C(\tau) \end{cases}$$

(b) *Let $J : [0,1] \times (E \times E) \to]-\infty, +\infty]$ be a normal integrand such that $J(t, x, .)$ is convex on E for all $(t, x) \in [0, T] \times E$ and that*

$$J(t, u_{\tau,x^n,\nu^n}(t), \dot{u}_{\tau,x^n,\nu^n}(t)) \ge \beta_n(t)$$

for all $n \in \mathbf{N}$ and for all $t \in [0, T]$ for some uniformly integrable sequence $(\beta_n)_{n\in\mathbf{N}}$ in $L^1_{\mathbf{R}}([0, T])$, then we have

$$\liminf_n \int_\tau^T J(t, u_{\tau,x^n,\nu^n}(t), \dot{u}_{\tau,x^n,\nu^n}(t))\, dt \ge \int_\tau^T J(t, u_{\tau,x^\infty,\nu^\infty}(t), \dot{u}_{\tau,x^\infty,\nu\infty}(t))\, dt.$$

Proof. (a) As $||\dot{u}_{\tau,x^n,\nu^n}(t)|| \le L+R$ a.e. we may extract a subsequence still denoted by $(\dot{u}_{\tau,x^n,\nu^n})$ which converges weakly in $L^1_E[\tau, T]$. For every t and for every n, $u_{\tau,x^n,\nu^n}(t) \in C(t)$. Then, by Ascoli's theorem, there is a subsequence still denoted by (u_{τ,x^n,ν^n}) which converges uniformly to an absolutely continuous function u^∞ with $||\dot{u}^\infty(t)|| \le L + R$ a.e. Clearly

$$u^\infty(\tau) = \lim_{n\to\infty} u_{\tau,x^n,\nu^n}(\tau) = \lim_{n\to\infty} x^n = x^\infty$$

and

$$u^\infty(t) = \lim_{n\to\infty} u_{\tau,x^n,\nu^n}(t) \in C(t), \forall t \in [\tau, T].$$

From the inclusions

$$-\dot{u}_{\tau,x^n,\nu^n}(t) - \int_Z f(t, u_{\tau,x^n,\nu^n}(t), z)\nu_t^n(dz) \in N_{C(t)}(u_{\tau,x^n,\nu^n}(t))$$

$$-\dot{u}_{\tau,x^\infty,\nu^\infty}(t) - \int_Z f(t, u_{\tau,x^\infty,\nu^\infty}(t), z)\nu_t^\infty(dz) \in N_{C(t)}(u_{\tau,x^\infty,\nu^\infty}(t))$$

and the monotonicity of $x \mapsto N_{C(t)}(x)$ we deduce that

$$\langle \dot{u}_{\tau,x^n,\nu^n}(t) - \dot{u}_{\tau,x^\infty,\nu^\infty}(t), u_{\tau,x^n,\nu^n}(t) - u_{\tau,x^\infty,\nu^\infty}(t)\rangle$$
$$\le -\langle h_{\nu^n}(t, u_{\tau,x^n,\nu^n}(t)) - h_{\nu^\infty}(t, u_{\tau,x^\infty,\nu^\infty}(t), u_{\tau,x^n,\nu^n}(t)) - u_{\tau,x^\infty,\nu^\infty}(t)\rangle \text{ a.e.}$$

where

$$h_{\nu^n}(t, u_{\tau,x^n,\nu^n}(t)) := \int_Z f(t, u_{\tau,x^n,\nu^n}(t), z)\nu_t^n(dz),$$
$$h_{\nu^\infty}(t, u_{\tau,x^\infty,\nu^\infty}(t)) := \int_Z f(t, u_{\tau,x^\infty,\nu^\infty}(t), z)\nu_t^\infty(dz).$$

Integrating over $[\tau, t]$ ($t \in [0, T]$) with respect to the measure ds yields

$$\begin{aligned}
&\frac{1}{2}||u_{\tau,x^n,\nu^n}(t) - u_{\tau,x^\infty,\nu^\infty}(t)||^2 - \frac{1}{2}||x^n - x^\infty||^2 \\
&= \int_\tau^t \langle u_{\tau,x^n,\nu^n} - u_{\tau,x^\infty,\nu^\infty}, \dot{u}_{\tau,x^n,\nu^n} - \dot{u}_{\tau,x^\infty,\nu^\infty}\rangle\, ds \\
&\leq -\int_\tau^t \langle h_{\nu^n}(s, u_{\tau,x^n,\nu^n}(s)) - h_{\nu^\infty}(s, u_{\tau,x^\infty,\nu^\infty}(s)), u_{\tau,x^n,\nu^n}(s) - u_{\tau,x^\infty,\nu^\infty}(s)\rangle\, ds.
\end{aligned}$$

Let us set

$$L_n(t) = \int_\tau^t \langle u_{\tau,x^n,\nu^n}(s) - u_{\tau,x^\infty,\nu^\infty}(s), -h_{\nu^n}(s, u_{\tau,x^n,\nu^n}(s)) + h_{\nu^\infty}(s, u_{\tau,x,\nu^\infty}(s))\rangle\, ds.$$

Then we have the following estimate

$$\frac{1}{2}||u_{\tau,x^n,\nu^n}(t) - u_{\tau,x^\infty,\nu^\infty}(t)||^2 \leq \frac{1}{2}||x^n - x^\infty||^2 + L_n(t).$$

Now we repeat the calculations and arguments in the proof of in Theorem 2.13. We have $L_n(t) = L_n^1(t) + L_n^2(t) + L_n^3(t)$ where

$$L_n^1(t) = \int_\tau^t \langle u_{\tau,x^n,\nu^n}(s) - u_{\tau,x^\infty,\nu^\infty}(s), -h_{\nu^n}(s, u_{\tau,x^n,\nu^n}(s)) + h_{\nu^n}(s, u_{\tau,x^\infty,\nu^\infty}(s))\rangle\, ds,$$
$$L_n^2(t) = \int_\tau^t \langle u_{\tau,x^n,\nu^n}(s) - u^\infty(s), -h_{\nu^n}(s, u_{\tau,x^\infty,\nu^\infty}(s)) + h_{\nu^\infty}(s, u_{\tau,x^\infty,\nu^\infty}(s))\rangle\, ds,$$
$$L_n^3(t) = \int_\tau^t \langle u^\infty(s) - u_{\tau,x^\infty,\nu^\infty}(s), -h_{\nu^n}(s, u_{\tau,x^\infty,\nu^\infty}(s)) + h_{\nu^\infty}(s, u_{\tau,x^\infty,\nu^\infty}(s))\rangle\, ds.$$

As by (iii) $||h_{\nu^n}(s, u_{\tau,x^n,\nu^n}(s)|| \leq R$ for all $n \in \mathbf{N} \cup \{\infty\}$ and for all $s \in [0, T]$, and $u_{\tau,x^n,\nu^n}(s) \to u^\infty(s)$ for every $s \in [0, T]$, we see that $L_n^2(t) \to 0$ when $n \to \infty$ for each $t \in [0, T]$. By (iii), the integrand

$$h(s, z) := \langle u^\infty(s) - u_{\tau,x^\infty,\nu^\infty}(s), f(s, u_{\tau,x^\infty,\nu^\infty}(s), z)\rangle$$

is Carathéodory integrable and is bounded by

$$|h(s, z)| \leq R\, ||u^\infty(s) - u_{\tau,x^\infty,\nu^\infty}(s)||$$

for all $(s, z) \in [0, T] \times Z$. Hence $h \in L^1_{C(Z)}([0, T])$. As ν^n stably converges to ν^∞, for every $t \in [0, T]$, we have

$$\lim_{n\to\infty} \int_\tau^t [\int_Z h(s, z)\nu_s^n(dz)]\, dt = \int_\tau^t [\int_Z h(s, z)\nu_s^\infty(dz)]\, dt.$$

So $\lim_{n\to\infty} L_n^3(t) = 0$, for every $t \in [0,T]$. By (iv) and (4) we have that $|L_n^1(t)| \leq \int_0^t \alpha(s)||u_{\tau,x,\nu^n}(s) - u_{\tau,x,\nu^\infty}(s)||^2\, ds$. Finally we get

$$\frac{1}{2}||u_{\tau,x,\nu^n}(t) - u_{\tau,x,\nu^\infty}(t)||^2 \leq \frac{1}{2}||x^n - x^\infty||^2 + L_n^2(t) + L_n^3(t) + \int_0^t \alpha(s)||u_{\tau,x,\nu^n}(s) - u_{\tau,x,\nu^\infty}(s)||^2\, ds.$$

As $\frac{1}{2}||x^n - x^\infty||^2 \to 0$, $L_n^2(t) \to 0$ and $L_n^3(t) \to 0$, for all $t \in [0,T]$, by Gronwall's lemma we have that $u_{\tau,x^n,\nu^n}(t) \to u_{\tau,x^\infty,\nu^\infty}(t)$, for all $t \in [0,T]$, and hence $u_{\tau,x,\nu^\infty} = u^\infty$.

(b) Follows by using a general lower semicontinuity of integral functionals, see [10, Theorem 8.1.6]. □

Lemma 4.2. *The value function*

$$V_J(\tau,x) = \sup_{\nu\in\mathcal{R}} \int_\tau^T [\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz)]\, dt, \quad (\tau,x) \in [0,T[\times E$$

where $u_{\tau,x,\nu}$ is the Lipschitzean solution of the $(\mathcal{PSW})(C;\nu;x)$ defined on $[\tau,T]$ associated the control $\nu \in \mathcal{R}$ starting from x at time τ

$$\begin{cases} -\dot{u}_{\tau,x,\nu} - \displaystyle\int_Z f(t, u_{\tau,x,\nu}(t), z)\nu_t(dz) \in N_{C(t)}(u_{\tau,x,\nu}(t)) \\ u_{\tau,x,\nu}(\tau) = x \in C(\tau) \end{cases}$$

satisfies the dynamic programming principle.

Proof. Follows from the scheme of the proof of Theorem 2.12, using the continuous dependence of the solution with respect to the state and the control obtained in Lemma 4.1. □

Lemma 4.3. *Let Z be a convex compact subset in E, $\mathcal{M}_+^1(Z)$ is endowed with the vague topology and $\mathcal{R}$ the space of relaxed controls associated with Z. Let $\Lambda : [0,T] \times E \times \mathcal{M}_+^1(Z) \to \mathbf{R}$ be an upper semicontinuous function such that the restriction of Λ to $[0,T] \times B \times \mathcal{M}_+^1(Z)$ is bounded on any bounded subset B of E. Let $C : [0,T] \to ck(E)$ be a convex compact valued L-Lipschitzean mapping. Let $(t_0,x_0) \in [0,T] \times E$. If $\max_{\mu\in\mathcal{M}_+^1(Z)} \Lambda(t_0,x_0,\mu) < -\eta < 0$ for some $\eta > 0$, then there exists $\sigma > 0$ such that*

$$\sup_{\nu\in\mathcal{R}} \int_{t_0}^{t_0+\sigma} \Lambda(t, u_{t_0,x_0,\nu}(t), \nu_t)\, dt < -\frac{\sigma\eta}{2}$$

where $u_{t_0,x_0,\nu}$ is the trajectory solution of the $(\mathcal{PSW})(C;\nu;x_0)$ associated with the control $\nu \in \mathcal{R}$ starting from x_0 at time t_0

$$\begin{cases} -\dot{u}_{t_0,x_0,\nu}(t) - \displaystyle\int_Z f(t, u_{\tau,x,\nu}(t), z)\nu_t(dz) \in N_{C(t)}(u_{t_0,x_0,\nu}(t)),\ t \in [t_0,T] \\ u_{t_0,x_0,\nu}(t_0) = x_0. \end{cases}$$

Proof. It is similar to Lemma 3.5, using the estimate $||\dot{u}_{t_0,x_0,\nu}(t)|| \leq L + R$ a.e. for all $\nu \in \mathcal{R}$ □

From these lemmas it is not difficult to get the following viscosity solution of the value function using the techniques of Theorem 3.6.

Theorem 4.4 (of viscosity solution)**.** *Let $C : [0, T] \to ck(E)$ be a convex compact valued L-Lipschitzean mapping. Let Z be a compact subset in E and $\mathcal{R}$ the space of relaxed controls associated with Z. Assume that $J : [0, T] \times E \times Z \to \mathbf{R}$ is bounded and continuous and $f : [0, T] \times E \times Z \to E$ is continuous satisfying*

(i) *there is a constant $R > 0$ such that $||f(t, x, z)|| \leq R$ for all (t, x, z) in $[0, T] \times E \times Z$,*

(ii) *there exists $\alpha \in L^1_{\mathbf{R}^+}[0, T])$ such that*

$$||f(t, x_1, z) - f(t, x_2, z)|| \leq \alpha(t)||x_1 - x_2||$$

for all $(t, x_1, z), (t, x_2, z) \in [0, T] \times E \times Z$.

Let us consider the value function

$$V_J(\tau, x) = \sup_{\nu\in\mathcal{R}} \int_\tau^T [\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz)]\, dt,\ (\tau, x) \in [0, T] \times E$$

where $u_{\tau,x,\nu}$ is the trajectory solution of the sweeping process $(\mathcal{PSW})(C; \nu; x)$ associated with the control $\nu \in \mathcal{R}$ starting from $x \in E$ at time τ

$$\begin{cases} -\dot{u}_{\tau,x,\nu}(t) - \displaystyle\int_Z f(t, u_{\tau,x,\nu}(t), z)\nu_t(dz) \in N_{C(t)}(u_{\tau,x,\nu}(t)), & t \in [\tau, T] \\ u_{\tau,x,\nu}(\tau) = x \in C(\tau) \end{cases}$$

and the Hamiltonian

$$H(t, x, \rho) = \sup_{\mu\in\mathcal{M}^1_+(Z)} \Big(-\langle\rho, \int_Z f(t, x, z)\mu(dz)\rangle + \int_Z J(t, x, z)\mu(dz)\Big) + \delta^*(\rho, -(L + 2R)\partial[d_{C(t)}](x))$$

where $(t, x, \rho) \in [0, T] \times E \times E$ and $\partial[d_{C(t)}](x)$ denotes the subdifferential of the distance functions $x \mapsto d_{C(t)}x$. Then, V_J is a viscosity subsolution of the HJB equation

$$\frac{\partial U}{\partial t}(t, x) + H(t, x, \nabla U(t, x)) = 0$$

that is, for any $\varphi \in C^1([0, T]) \times E)$ for which $V_J - \varphi$ reaches a local maximum at $(t_0, x_0) \in [0, T] \times E$, we have

$$H(t_0, x_0, \nabla\varphi(t_0, x_0)) + \frac{\partial\varphi}{\partial t}(t_0, x_0) \geq 0.$$

Proof. Follows the scheme of the proof of Theorem 3.6, using Lemmas 4.1–4.2–4.3. □

It is interesting to study the existence and properties of BVC solutions for an evolution inclusion governed by the upper semicontinuous perturbation F of a maximal monotone operator $A(t)$ depending on time $t \in [0, T]$ of the form $-\frac{dDu}{d|Du|}(t) - F(t, u(t)) \in A(t)u(t)$, $u(0) \in \operatorname{dom} A(0)$ extending the pioneering work in [19]. Existence of BVC solutions for $\mathcal{PSW}(C, F, x)$ with upper semicontinuous perturbation F

$$-\frac{dDu}{d|Du|}(t) - F(t, u(t)) \in N_{C(t)}(u(t)), \quad u(0) = x$$

is available in [7, Theorem 4.3]. When C is Lipschitz, this inclusion modelizes some applications in Mathematical Economics. See [17]. Dealing with BVC solutions for the aforementioned inclusions, the assumption

$$(^{***}) \qquad \forall t \in [0, T], \quad \operatorname{Int} C(t) \neq \emptyset$$

is crucial and so is the stability of sweeping process. Along the paper, the convexity of the moving set C is needed. One may consult [20] dealing with the sweeping process without convexity where several references and related results can be found. See also [15, 16] for related results on relaxation and sweeping process.

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