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BREGMAN DISTANCE AND RELATED RESULTS ON BANACH SPACES

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Abstract. In this paper, we first study existence theorems of solution for optimization problems which is related to Bregman distance. From these results, we study fixed point problems for nonlinear mappings, contractive type mappings, Caritsti type mappings, graph contractive type mappings with the Bregman distance on Banach spaces. We also study some properties of Bregman projection. Our results on the properties of Bregman projection improve recent results of Honda and Takahashi. We combine the techniques of optimization theory and fixed point theory to study these problems in this paper. Our results are different from many existence theorems for optimization problem and fixed point theorems of nonlinear mappings, contractive type mappings, and graph contractive mappings.

1. Introduction

In 1967, Bregman [9] introduced the concept of the Bregman distance, and he has discovered an elegant and effective technique for the use of the Bregman distance in the process of designing and analyzing feasibility and optimization algorithms. More recently researchers in geometric algorithms have shown that many important algorithms can be generalized from Euclidean metrics to distances defined by Bregman distance. Some researchers consider nonlinear mappings with Bregman distances. For example, one can see [2, 3, 4, 5, 8, 11, 21, 24, 32, 33, 35, 36, 37, 38, 40, 41] and related literatures.

Note that the Bregman distance is similar to a metric, but does not satisfy the triangle inequality nor symmetry. So, we know that the Bregman distance are different from the metric, w-distance [20], τ-distance [42], τ-function [27], and weak τ-function [22]. Indeed, these generalized distance functions satisfy the triangle inequality, but the Bregman distance does not satisfy the triangle inequality.

Since the Bregman distance does not satisfy the triangle inequality nor symmetry, it is important to consider the fixed point theory and related problems with the Bregman distance.

However, for many problems, like optimization problems and fixed point problems, we do not see many existence theorems of optimization theory with the Bregman distance, and fixed point theory with the Bregman distance. In fact, many researchers considered various iteration processes with Bregman distance for optimization problem, variational inequality, and fixed point. But, they do not consider the existence theorems for optimization problem and fixed point theorems. (For details, one refers to [2, 3, 4, 5, 8, 11, 21, 24, 31, 35, 36, 37, 38, 40, 41].)

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Few researches consider the existence theorems for optimization problems and related problems. For example, Li, Song, and Yao [25] studied the convexity problem of Chebyshev sets with the Bregman distance in a Banach space.

In this paper, we first study the properties of Bregman distance and an existence and uniqueness theorem of solution for an optimization problem which is related to Bregman distance. From these results, we study fixed point problems of nonlinear mappings, contractive type mappings, Caristi type mappings, graph contractive type mappings with the Bregman distance on Banach spaces. We also study some properties of Bregman projection. Our results on the properties of Bregman projection improve recent results of Honda and Takahashi [16]. It is worth noting that we combine the techniques of optimization theory and fixed point theory to study these problems. Our results are different from any results on fixed point theorems of nonlinear mappings, contractive type mappings, and graph contractive mappings.

2. Preliminaries

Now, we recall some definitions. Let E be a Banach space with dual space E^*. Then the duality mapping $J : E \multimap E^*$ is defined by

$$Jx := \{x^* \in E^* : \langle x, x^*\rangle = ||x||^2 = ||x^*||^2\},\ \forall x \in E.$$

Let $S(E)$ be the unit sphere centered at the origin of E. Then the space E is said to be smooth if the limit

$$\lim_{t\to 0} \frac{||x + ty|| - ||x||}{t}$$

exists for all $x, y \in S(E)$. It is also said to be uniformly smooth if the limit exists uniformly in $x, y \in S(E)$. A Banach space E is said to be strictly convex if $||\frac{x+y}{2}|| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $||\frac{x+y}{2}|| < 1 - \delta$ whenever $x, y \in S(E)$ and $||x - y|| \geq \varepsilon$. Further, we have: (i) if E is smooth, then J is single-valued; (ii) if E is reflexive, then J is onto; (iii) if E is strictly convex, then J is one-to-one; (iv) if E is strictly convex, then J is strictly monotone; (v) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E [43].

Now, let E be a smooth Banach space, and let C be a nonempty closed convex subset of E. A function $\phi : C \times C \to \mathbb{R}$ is defined by

$$\phi(x, y) := ||x||^2 - 2\langle x, Jy\rangle + ||y||^2$$

for all $x, y \in C$. We know that this function plays an important role for convex analysis and nonlinear analysis. (For example, see [23, 45] and related results). Besides, we know that

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2 \tag{2.1}$$

for all $x, y \in C$.

Let E be a Banach space. Let $f : E \to (-\infty, \infty]$ be a convex function. Let $\mathcal{D}$ (or $D(f)$) denote the domain of f, that is,

$$\mathcal{D} := \{x \in E : f(x) < \infty\}.$$

Let $\mathcal{D}^\circ$ denote the algebraic interior of $\mathcal{D}$, i.e., the subset of $\mathcal{D}$ consisting of all those points $x \in \mathcal{D}$ such that, for any $y \in E \setminus \{x\}$, there is z in the open segment (x, y)

with $[x, z] \subseteq \mathcal{D}$. The topological interior of $\mathcal{D}$, denoted by $\mathrm{int}(\mathcal{D})$, is contained in D.

A function $f : E \to (-\infty, \infty]$ is said to be proper provided that $\mathcal{D} \neq \emptyset$. It is also called lower semicontinuous if $\{x \in E : f(x) \leq r\}$ is closed for each $r \in \mathbb{R}$. A function f is upper semicontinuous if $\{x \in E : f(x) \geq r\}$ is closed for each $r \in \mathbb{R}$. A function f is convex if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for all $x, y \in E$ and $t \in [0, 1]$. A function f is strictly convex if $f(tx + (1-t)y) < tf(x) + (1-t)f(y)$ for all $x, y \in D(f)$ with $x \neq y$ and $t \in (0, 1)$. A function $f : E \to (-\infty, \infty]$ is said to be Gâteaux differentiable at $x \in E$ if there is $\nabla f(x) \in E^*$ such that

$$\lim_{t\to 0} \frac{f(x + ty) - f(x)}{t} = \langle y, \nabla f(x)\rangle$$

for each $y \in E$.

The Bregman distance D_f associated with a proper convex function f is the function $D_f : \mathcal{D} \times \mathcal{D} \to [0, \infty]$ defined by

$$D_f(y, x) := f(y) - f(x) + f^\circ(x, x - y),$$

where

$$f^\circ(x, x - y) := \lim_{t\to 0^+} \frac{f(x + t(x - y)) - f(x)}{t}.$$

For the Bregman distance, we need the following result and it plays an important and essential role in this paper.

Lemma 2.1 ([10, Proposition 1.3.9]). *Let E be a Banach space. Let f be a lower semicontinuous convex function on $\mathcal{D}$ with int$(\mathcal{D}) \neq \emptyset$. Suppose that f is Gâteaux differentiable on $\mathcal{D}$. Then we have*

(i) $D_f(y, x) = f(y) - f(x) - \langle y - x, \nabla f(x)\rangle$ *for all* $x, y \in$ *int*$(\mathcal{D})$.

(ii) $D_f(u, x) + \langle u - x, \nabla f(x) - \nabla f(y)\rangle + D_f(x, y) = D_f(u, y)$ *for all* $u, x, y \in$ *int*$(\mathcal{D})$.

Note that if $f(x) = ||x||^2$ in a smooth Banach space E, then it is known that f is Gâteaux differentiable and $\nabla f(x) = 2J(x)$ for each $x \in E$, where $J : E \to E^*$ is the duality mapping [12, 15, 33]. This implies that

$$D_f(y, x) = ||x||^2 + ||y||^2 - 2\langle y, J(x)\rangle = \phi(y, x)$$

for each $x, y \in \mathrm{int}(\mathcal{D})$ [15, 33]. Further, if E is a real Hilbert space, then $D_f(x, y) = ||x - y||^2$ for all $x, y \in E$.

For a proper lower semicontinuous convex function $f : E \to (-\infty, \infty]$, the subdifferential ∂f of f and the conjugate function f^* of f are defined by

$$\partial f(x) := \{x^* \in E^* : f(x) + \langle y - x, x^*\rangle \leq f(y) \text{ for each } y \in E\}$$

for each $x \in E$, and

$$f^*(x^*) := \sup_{x\in E}\{\langle x, x^*\rangle - f(x)\}$$

for each $x^* \in E^*$. It is well-known that $f(x) + f^*(x^*) \geq \langle x, x^*\rangle$ for each $(x, x^*) \in E \times E^*$. It is also known that

$$x^* \in \partial f(x) \Leftrightarrow f(x) + f^*(x^*) = \langle x, x^*\rangle.$$

We also know that if $f : E \to (-\infty, \infty]$ is a proper convex lower semicontinuous function, then $f^* : E^* \to (-\infty, \infty]$ is a proper, convex, and weak* lower semicontinuous. Besides, f is convex and lower semicontinuous if and only if $f^{**} = f$. (For details, one can refer to [12].)

We need the following results as tools to consider optimization theory, fixed point theory, and related results in the sequel.

Lemma 2.2 ([10, Proposition 1.1.5]). *Let E be a Banach space. If $f : E \to (-\infty, \infty]$ is a proper lower semicontinuous and convex function on $int(\mathcal{D})$, then f is continuous on $int(\mathcal{D})$.*

Lemma 2.3 ([10, Proposition 1.1.9]). *Let E be a Banach space. If $f : E \to (-\infty, \infty]$ is a proper convex function and Gâteaux differentiable on $int(D)$, then the following are equivalent:*

(i) *f is strictly convex on $int(\mathcal{D})$;*
(ii) *For each $x, y \in int(\mathcal{D})$ with $x \neq y$, one has $D_f(y, x) > 0$;*
(iii) *For each $x, y \in int(\mathcal{D})$ with $x \neq y$, one has $\langle x - y, \nabla f(x) - \nabla f(y)\rangle > 0$.*

Lemma 2.4 ([10, Proposition 1.1.10]). *Let E be a Banach space. Let $f : E \to (-\infty, \infty]$ be a lower semicontinuous convex function with $int(\mathcal{D}) \neq \emptyset$.*

(i) *f is Gâteaux differentiable at $x \in int(\mathcal{D})$ if and only if $\partial f(x)$ consists of a single element. In this case, $\partial f(x) = \{\nabla f(x)\}$.*
(ii) *If f is Gâteaux differentiable on $int(\mathcal{D})$, then ∇f is norm-to weak* continuous on $int(\mathcal{D})$;*
(iii) *If f is Fréchet differentiable, then D_f is continuous on $int(\mathcal{D})\times int(\mathcal{D})$.*

Lemma 2.5 ([10, Proposition 1.1.11]). *Let E be a Banach space. If $f : E \to \mathbb{R}$ is a continuous convex function, then the multivalued mapping $\partial f : E \to E^*$ is bounded on bounded sets if and only if f is bounded on bounded sets.*

Let E be a Banach space, and let $f : E \to (-\infty, +\infty]$ be a convex function. Define $\delta_f : (0, +\infty) \to [0, +\infty]$ as

$$\delta_f(t) := \inf\Big\{\frac{1}{2}f(x) + \frac{1}{2}f(y) - f\Big(\frac{x+y}{2}\Big) : ||x - y|| = t, x, y \in D(f)\Big\}$$

for each $t > 0$, where the infimum over the empty set is $+\infty$. We say that f is uniformly convex when $\delta_f(t) > 0$ for all $t > 0$ [7, 46].

Define $\rho_f : (0, +\infty) \to [0, +\infty]$ as

$$\rho_f(t) := \sup\Big\{\frac{1}{2}f(x) + \frac{1}{2}f(y) - f\Big(\frac{x+y}{2}\Big) : ||x - y|| = t, x, y \in D(f)\Big\}$$

for each $t > 0$. We say f is uniformly smooth if $\lim_{t\to 0^+} \frac{\rho_f(t)}{t} = 0$ [7, 46].

Lemma 2.6 ([24]). *Let E be a Banach space, and let $f : E \to \mathbb{R}$ be a convex Gâteaux differentiable function which is uniform convex on bounded sets. If $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E and $\lim_{n\to\infty} D_f(x_n, y_n) = 0$, then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.*

Definition 2.7 ([10]). Let E be a Banach space. Then a function $f : E \to \mathbb{R}$ is said to be a Bregman function if the following conditions hold:

(i) f is continuous, strictly convex, and Gâteaux differentiable;
(ii) the set $\{y \in E : D_f(x, y) \leq r\}$ is bounded for each $x \in E$ and each $r > 0$.

Let E be a Banach space. A function $f : E \to \mathbb{R}$ is said to be coercive if $||x_n|| \to \infty$ implies that $f(x_n) \to \infty$. A function $f : E \to \mathbb{R}$ is said to be strongly coercive if $||x_n|| \to \infty$ implies that $f(x_n)/||x_n|| \to \infty$ [33].

Lemma 2.8 ([33]). *Let E be a reflexive Banach space, and let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then*
(i) *$\nabla f : E \to E^*$ is one-to-one, onto, and norm-to-weak continuous;*
(ii) *$\{x \in E : D_f(x, y) \leq r\}$ is bounded for each $y \in E$ and each $r > 0$.*
(iii) *$D(f^*) = E^*$, f^* is Gâteaux differentiable, and $\nabla f^* = (\nabla f)^{-1}$.*

Let ℓ^∞ be the Banach space of bounded sequences with the supremum norm. A linear functional μ on ℓ^∞ is called a mean if $\mu(e) = ||\mu|| = 1$, where $e = (1, 1, 1, \dots.)$. For $x = (x_1, x_2, x_3, \dots.)$, the value $\mu(x)$ is also denoted by $\mu_n(x_n)$. A mean μ on ℓ^∞ is called a Banach limit if it satisfies $\mu_n(x_n) = \mu_n(x_{n+1})$. If μ is a Banach limit on ℓ^∞, then for $x = (x_1, x_2, x_3, \dots) \in \ell^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $x = (x_1, x_2, x_3, \dots) \in \ell^\infty$ and $\lim_{n\to\infty} x_n = a \in \mathbb{R}$, then $\mu(x) = \mu_n(x_n) = a$. For details, we can refer to [43].

The following result comes from the proof of Theorem 3.2 in [28].

Lemma 2.9 ([28]). *Let E be a reflexive Banach space with dual space E^*. Let $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous, and strictly convex function. Suppose that f is Gâteaux differentiable on int($\mathcal{D}$) and is bounded on bounded subsets of int($\mathcal{D}$). Let $C \subseteq$ int($\mathcal{D}$) be a nonempty closed convex set. Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in C and let μ be a mean on ℓ^∞. Let $g : C \to (-\infty, \infty]$ be defined by $g(z) := \mu_n D_f(x_n, z)$ for each $z \in C$. Then there is a unique element $\bar{x} \in C$ such that $\mu_n\langle x_n, y\rangle = \langle \bar{x}, y\rangle$ for all $y \in E^*$, and $g(\bar{x}) = \min_{z\in C} g(z)$. Note that $\bar{x} \in \overline{co}\{x_n : n \in \mathbb{N}\}$.*

Proposition 2.10. *Let E be a smooth, strictly convex, and reflexive Banach space. Let C be nonempty closed convex subset of E. Let $f : E \to \mathbb{R}$ be a proper, lower semicontinuous, strictly convex, and Gâteaux differentiable function. Let $T : C \to C$ be a mapping, μ be a Banach limit on ℓ^∞, and $\{x_n\}$ be a sequence in C with $x_n \rightharpoonup x$ for some $x \in C$ with $\mu_n D_f(x_n, Tx) \leq \mu_n D_f(x_n, x)$. Then x is a fixed point of T.*

Proof. By Lemma 2.1, we have

$$D_f(x_n, y) - D_f(x_n, x) = \langle x_n - x, \nabla f(x) - \nabla f(y)\rangle + D_f(x, y)$$

for each $y \in C$. Since $x_n \rightharpoonup x$, we have

$$\mu_n D_f(x_n, y) - \mu_n D_f(x_n, x) = D_f(x, y) \geq 0.$$

By Lemma 2.3, for each $y \in C$ with $y \neq x$, we know that $D_f(x, y) > 0$ and

$$\mu_n D_f(x_n, x) < \mu_n D_f(x_n, y).$$

By assumption, $Tx = x$. □

Proposition 2.11. *Let E be a smooth, strictly convex, and reflexive Banach space. Let C be nonempty closed convex subset of E. Let $f : E \to \mathbb{R}$ be a proper, lower semicontinuous, strictly convex, and Gâteaux differentiable function. Let $T : C \to C$ be a mapping, $\{x_n\}$ be a sequence in C with $x_n \rightharpoonup x$ for some $x \in C$ with $\limsup_{n\to\infty} D_f(x_n, Tx) \le \limsup_{n\to\infty} D_f(x_n, x)$. Then x is a fixed point of T.*

Proof. Following the similar argument as the proof of Proposition 2.10, we get the conclusion of Proposition 2.11. □

The following results are special cases of Propositions 2.10 and 2.11, respectively.

Corollary 2.12. *Let E be a smooth, strictly convex, and reflexive Banach space. Let C be nonempty closed convex subset of E. Let $T : C \to C$ be a mapping. Let μ be a Banach limit on ℓ^∞. Let $\{x_n\}$ be a sequence in C with $x_n \rightharpoonup x$ for some $x \in C$ with $\mu_n\phi(x_n, Tx) \le \mu_n\phi(x_n, x)$, where $\phi(x, y) := ||x||^2 - 2\langle x, Jy\rangle + ||y||^2$. Then x is a fixed point of T.*

Proof. Let $f : E \to \mathbb{R}$ be defined by $f(x) := ||x||^2$ for each $x \in E$. Then f is a proper, lower semicontinuous, strictly convex, Gâteaux differentiable function, and $\nabla f(x) = 2J(x)$ for each $x \in E$. So, we get Corollary 2.12 by Proposition 2.10. □

Corollary 2.13. *Let E be a smooth, strictly convex, and reflexive Banach space. Let C be nonempty closed convex subset of E. Let $T : C \to C$ be a mapping. Let $\{x_n\}$ be a sequence in C with $x_n \rightharpoonup x$ for some $x \in C$ with $\limsup_{n\to\infty} \phi(x_n, Tx) \le \limsup_{n\to\infty} \phi(x_n, x)$, where $\phi(x, y) := ||x||^2 - 2\langle x, Jy\rangle + ||y||^2$. Then x is a fixed point of T.*

3. Existence theorem for optimization problem and fixed point problem with the Bregman distance

Theorem 3.1. *Let E be a reflexive Banach space, and let $f : E \to \mathbb{R}$ be a lower semicontinuous, strictly convex, and Gâteaux differentiable function. Suppose that f is bounded on bounded sets, and uniform convex on bounded sets. Let C be a nonempty closed convex subset of E, let $\{x_n\}$ be a bounded sequence in C, $\psi : C \to (-\infty, \infty]$ be a proper, bounded below, and lower semicontinuous function, and let $T : C \to C$ be a mapping. Suppose that there exists $m \in \mathbb{N} \cup \{0\}$ such that*

$$\mu_n D_f(x_n, T^m y) + \psi(Ty) \le \psi(y) \text{ for each } y \in C. \tag{3.1}$$

Then there exists $\bar{x} \in C$ such that

(a) $\lim_{k\to\infty} \mu_n D_f(x_n, T^k y) = \mu_n D_f(x_n, \bar{x}) = 0$ *for each $y \in C$ with $\psi(y) < \infty$;*

(b) $\lim_{k\to\infty} D_f(\bar{x}, T^k y) = 0$ *for each $y \in C$ with $\psi(y) < \infty$.*

Further, if C is bounded or f is a Bregman function, then we know that

(c) $\lim_{k\to\infty} T^k y = \bar{x}$ *for each $y \in C$ with $\psi(y) < \infty$;*

(d) $\psi(\bar{x}) = \inf_{y\in X} \psi(y)$*;*

(e) *$\bar{x}$ is the unique fixed point of T.*

Proof. (a) and (b): Let $g : E \to [0, \infty)$ be defined by $g(z) := \mu_n D_f(x_n, z)$ for each $z \in E$. By Lemma 2.9, there is a unique element $\bar{x} \in C$ such that

$$g(\bar{x}) = \min_{z \in C} g(z), \text{ and } \mu_n \langle x_n, y \rangle = \langle \bar{x}, y \rangle \text{ for each } y \in E^*. \tag{3.2}$$

Take any $y \in C$ with $\psi(y) < \infty$. Then it follows from (3.1) that

$$\mu_n D_f(x_n, T^m T^k y) \le \psi(T^k y) - \psi(T^{k+1} y)$$

for each $k \in \mathbb{N} \cup \{0\}$. Hence, $\{\psi(T^k y)\}_{k=0}^{\infty}$ is a decreasing sequence which is bounded below. Then $\lim_{k\to\infty} \psi(T^k y)$ exists. Put $s = \lim_{k\to\infty} \psi(T^k y)$. Since

$$\begin{aligned}\sum_{k=0}^{N} g(T^{m+k} y) &= \sum_{k=0}^{N} \mu_n d(x_n, T^{m+k} y) \\ &\le \sum_{k=0}^{N} \psi(T^k y) - \psi(T^{k+1} y) \\ &= \psi(y) - \psi(T^{N+1} y)\end{aligned}$$

for any $N \in \mathbb{N}$, we have that

$$\sum_{k=0}^{\infty} g(T^{m+k} y) = \sum_{k=0}^{\infty} \mu_n d(x_n, T^{m+k} y) \le \psi(y) - s < \infty.$$

Thus,

$$\lim_{k\to\infty} \mu_n D_f(x_n, T^{k+m} y) = \lim_{k\to\infty} g(T^k T^m y) = 0. \tag{3.3}$$

So, it follows from (3.2) and (3.3) that

$$g(\bar{x}) = \min_{z \in C} g(z) \le \lim_{k\to\infty} g(T^k T^m y) = 0.$$

Therefore, $g(\bar{x}) = 0$. Since y is any point of C with $\psi(y) < \infty$, we know that

$$\lim_{k\to\infty} \mu_n D_f(x_n, T^k y) = g(\bar{x}) = 0 \tag{3.4}$$

for each $y \in C$ with $\psi(y) < \infty$. By (3.4), for each $y \in C$ with $\psi(y) < \infty$, we have

$$\begin{aligned}&\lim_{k\to\infty} \mu_n f(x_n) - f(T^k y) - \mu_n \langle x_n - T^k y, \nabla f(T^k y) \rangle \\ =&\mu_n f(x_n) - f(\bar{x}) - \mu_n \langle x_n - \bar{x}, \nabla f(\bar{x}) \rangle.\end{aligned} \tag{3.5}$$

By (3.2) and (3.5),

$$\lim_{k\to\infty} \mu_n f(x_n) - f(T^k y) - \langle \bar{x} - T^k y, \nabla f(T^k y) \rangle = \mu_n f(x_n) - f(\bar{x}).$$

This implies that

$$\lim_{k\to\infty} D_f(\bar{x}, T^k y) = \lim_{k\to\infty} f(\bar{x}) - f(T^k y) - \langle \bar{x} - T^k y, \nabla f(T^k y) \rangle = 0. \tag{3.6}$$

Further, suppose that C is bounded or f is a Bregman function. Then $\{T^k y\}$ is a bounded sequence for each $y \in C$ with $\psi(y) < \infty$. (c): By (3.6) and Lemma 2.6, we know that $\lim_{k\to\infty} T^k y = \bar{x}$ for each $y \in C$ with $\psi(y) < \infty$.

(d): By assumption, we have that

$$\psi(T\bar{x}) \le \mu_n D_f(x_n, T^m\bar{x}) + \psi(T\bar{x}) \le \psi(\bar{x}). \tag{3.7}$$

Since ψ is lower semicontinuous and $\lim_{k\to\infty} T^kT^m u = \bar{x}$ for each $u \in C$ with $\psi(u) < \infty$, we have that

$$\psi(\bar{x}) \le \liminf_{k\to\infty} \psi(T^kT^m u) = \lim_{k\to\infty} \psi(T^kT^m u) = \inf_{m\in\mathbb{N}\cup\{0\}} \psi(T^kT^m u) \le \psi(u) \tag{3.8}$$

for each $u \in C$ with $\psi(u) < \infty$. By (3.7) and (3.8),

$$\psi(\bar{x}) \le \inf_{y\in X} \psi(y) \le \psi(T\bar{x}) \le \psi(\bar{x}).$$

Hence, $\psi(\bar{x}) = \inf_{y\in X} \psi(y)$.

(e): Next, we show that $\bar{x}$ is the unique fixed point of T. Since

$$0 \le g(T^m\bar{x}) = \mu_n D_f(x_n, T^m\bar{x}) \le \psi(\bar{x}) - \psi(T\bar{x}) \le 0,$$

we have that $g(T^m\bar{x}) = g(\bar{x}) = 0$. By (3.2), $T^m\bar{x} = \bar{x}$. Hence,

$$0 \le \mu_n D_f(x_n, T\bar{x}) = \mu_n D_f(x_n, T^{m+1}\bar{x}) \le \psi(T\bar{x}) - \psi(T^2\bar{x}) \le \psi(\bar{x}) - \psi(T^2\bar{x}) \le 0.$$

Thus, $g(\bar{x}) = g(T\bar{x}) = 0$. By (3.2) again, $T\bar{x} = \bar{x}$. We show that $\bar{x}$ is a unique fixed point of T. Indeed, if v is a fixed point of T, then

$$0 \le g(v) = g(T^m v) \le \psi(v) - \psi(Tv) = 0.$$

Hence, $v = \bar{x}$. Therefore, $\bar{x}$ is the unique fixed point of T. □

Remark 3.2. The conclusion of Theorem 3.1 is still true if (3.1) is replaced by one of the following conditions:

(a) if $\frac{1}{2}D_f(x, Tx) \le D_f(x, y)$, then $\mu_n D_f(x_n, y) + \psi(y) \le \psi(x)$;
(b) $D_f(y, Tx) + \psi(Tx) \le \psi(x)$ for all $x, y \in C$.

Further, we know that following result is a special case of Theorem 3.1. Note that we do not assume that C is bounded. Indeed, by (2.1) and following the same argument as the proof of Theorem 3.1, we get the following result.

Corollary 3.3. *Let E be a reflexive, smooth, and strictly convex Banach space. Let C be a nonempty closed convex subset of E, and let $\{x_n\}$ be a bounded sequence in C. Let $\psi : C \to (-\infty, \infty]$ be a proper, bounded below, and lower semicontinuous function. Let $T : C \to C$ be a mapping. Suppose that there exists $m \in \mathbb{N} \cup \{0\}$ such that*

$$\mu_n \phi(x_n, T^m y) + \psi(Ty) \le \psi(y) \text{ for each } y \in C, \tag{3.9}$$

where $\phi(x, y) := ||x||^2 - 2\langle x, Jy\rangle + ||y||^2$ and J is the duality mapping. Then there exists $\bar{x} \in C$ such that (a) $\lim_{k\to\infty} \mu_n \phi(x_n, T^k y) = \mu_n \phi(x_n, \bar{x}) = 0$ for each $y \in C$ with $\psi(y) < \infty$; (b) $\lim_{k\to\infty} \phi(\bar{x}, T^k y) = 0$ for each $y \in C$ with $\psi(y) < \infty$; (c) $\lim_{k\to\infty} T^k y = \bar{x}$ for each $y \in C$ with $\psi(y) < \infty$; (d) $\psi(\bar{x}) = \inf_{y\in X} \psi(y)$; and (e) $\bar{x}$ is the unique fixed point of T.

Following similar argument as the proof of Theorem 3.1, we have the following result. For completeness, we give the proof of the following result.

Theorem 3.4. *Let E be a reflexive Banach space, and let $f : E \to \mathbb{R}$ be a lower semicontinuous, strictly convex, and Gâteaux differentiable function. Suppose that f is bounded on bounded sets, and uniform convex on bounded sets. Let C be a nonempty closed convex subset of E, let $\{x_n\}$ be a bounded sequence in C, and let $T : C \to C$ be a mapping. Suppose that $r \in [0, 1)$ and*

$$\mu_n D_f(x_n, Ty) \le r \cdot \mu_n D_f(x_n, y) \text{ for each } y \in C. \tag{3.10}$$

Then there exists $\bar{x} \in C$ such that

(a) $\lim_{k\to\infty} \mu_n D_f(x_n, T^k y) = \mu_n D_f(x_n, \bar{x}) = 0$ *for each* $y \in C$;
(b) $\lim_{k\to\infty} D_f(\bar{x}, T^k y) = 0$ *for each* $y \in C$;
(c) $\mu_n f(x_n) = f(\bar{x})$;
(d) $\bar{x}$ *is the unique fixed point of* T.

Further, if C is bounded or f is a Bregman function, then $\lim_{k\to\infty} T^k y = \bar{x}$ for each $y \in C$.

Proof. Let $g : E \to [0, \infty)$ be defined by $g(z) := \mu_n D_f(x_n, z)$ for each $z \in E$. By Lemma 2.9, there is a unique element $\bar{x} \in C$ such that

$$g(\bar{x}) = \min_{z\in C} g(z), \text{ and } \mu_n\langle x_n, y\rangle = \langle \bar{x}, y\rangle \text{ for each } y \in E^*. \tag{3.11}$$

By (3.10) and (3.11), we know that $\bar{x}$ is a fixed point of T. Besides, it follows from (3.10) that

$$\lim_{k\to\infty} \mu_n D_f(x_n, T^k y) = 0 \text{ for each } y \in C. \tag{3.12}$$

So, by (3.11) and (3.12), we know that

$$\mu_n D_f(x_n, \bar{x}) = 0. \tag{3.13}$$

This implies that

$$\mu_n f(x_n) - f(\bar{x}) - \mu_n\langle x_n - \bar{x}, \nabla f(\bar{x})\rangle = 0. \tag{3.14}$$

By (3.11) and (3.14), we know that $\mu_n f(x_n) = f(\bar{x})$. Besides, from the proof of Theorem 3.1, we know that $\lim_{k\to\infty} D_f(\bar{x}, T^k y) = 0$ for each $y \in C$.

If $\bar{y}$ is a fixed point of T, then it follows from (3.12) that $\mu_n D_f(x_n, \bar{y}) = 0$. So, by (3.11) and (3.13), we have $\bar{x} = \bar{y}$. Hence, $\bar{x}$ is the unique fixed point of T.

Further, suppose that C is bounded or f is a Bregman function. Then $\{T^k y\}$ is a bounded sequence for each $y \in C$. By (3.12) and Lemma 2.6, we know that $\lim_{k\to\infty} T^k y = \bar{x}$ for each $y \in C$. □

Remark 3.5. We follow the same method as in [13, Theorem 3.4], we can get a result which is similar to Theorem 3.4. Indeed, if we let $\psi : C \to [0.\infty)$ as follows:

$$\psi(y) = \frac{1}{1-r}\mu_n D_f(x_n, y).$$

Then we know from Lemmas 2.2 and 2.4 that $\psi : C \to [0.\infty)$ is a proper bounded below and continuous function. Further, we know that the following (3.15) and (3.16) are equivalent:

$$\mu_n D_f(x_n, y) + \psi(Ty) \le \psi(y) \text{ for each } y \in C. \tag{3.15}$$

$$\mu_n D_f(x_n, Ty) \leq r \cdot \mu_n D_f(x_n, y) \text{ for each } y \in C. \tag{3.16}$$

Then, by Theorem 3.1, we can get a result which is similar to Theorem 3.4.

The following results are special cases of Theorem 3.4.

Corollary 3.6. *Let E be a reflexive, smooth, and strictly convex Banach space. Let C be a nonempty closed convex subset of E, $\{x_n\}$ be a bounded sequence in C, and $T : C \to C$ be a mapping. Suppose that $r \in [0, 1)$ and*

$$\mu_n \phi(x_n, Ty) \leq r \cdot \mu_n \phi(x_n, y) \text{ for each } y \in C,$$

where $\phi(x, y) := ||x||^2 - 2\langle x, Jy\rangle + ||y||^2$ and J is the duality mapping. Then there exists $\bar{x} \in C$ such that $\bar{x}$ is the unique fixed point of T, and $\lim_{k\to\infty} T^k y = \bar{x}$ for each $y \in C$.

Corollary 3.7. *Let E be a reflexive Banach space, and let $f : E \to \mathbb{R}$ be a lower semicontinuous, strictly convex, and Gâteaux differentiable function. Suppose that f is bounded on bounded sets, and uniform convex on bounded sets. Let C be a nonempty closed convex subset of E, let $T : C \to C$ be a mapping with $\{T^n x\}$ is a bounded sequence in C for some $x \in C$. Suppose that $r \in [0, 1)$ and*

$$\mu_n D_f(T^n x, Ty) \leq r \cdot \mu_n D_f(T^n x, y) \text{ for each } y \in C.$$

Then there exists $\bar{x} \in C$ such that

(a) $\lim_{k\to\infty} \mu_n D_f(T^n x, T^k y) = \mu_n D_f(T^n x, \bar{x}) = 0$ *for each* $y \in C$;
(b) $\lim_{k\to\infty} D_f(\bar{x}, T^k y) = 0$ *for each* $y \in C$;
(c) $\mu_n f(T^n x) = f(\bar{x})$;
(d) $\bar{x}$ *is the unique fixed point of* T.

Further, if C is bounded or f is a Bregman function, then $\lim_{k\to\infty} T^k y = \bar{x}$ for each $y \in C$.

Corollary 3.8. *Let E be a Hilbert space. Let C be a nonempty closed convex subset of E, let $\{x_n\}$ be a bounded sequence in C, and let $T : C \to C$ be a mapping. Suppose that $r \in [0, 1)$ and*

$$\mu_n ||x_n - Ty|| \leq r \cdot \mu_n ||x_n - y|| \text{ for each } y \in C.$$

Then there exists $\bar{x} \in C$ such that

(a) $\lim_{k\to\infty} \mu_n ||x_n - T^k y|| = \mu_n ||x_n - \bar{x}|| = 0$ *for each* $y \in C$;
(b) $\lim_{k\to\infty} T^k y = \bar{x}$ *for each* $y \in C$;
(c) $\mu_n ||x_n|| = ||\bar{x}||$;
(d) $\bar{x}$ *is the unique fixed point of* T.

Remark 3.9. Although Corollary 3.8 is a special case of Theorem 3.4, it is also different from [14, Theorem 3.1]. Indeed, we have the conclusion (c) of Corollary 3.8.

4. Fixed point theorems for Caristi type and graph contraction type mappings with the Bregman distance

In this section, we first consider the Caristi's type fixed point theorem with the Bregman distance on Banach spaces.

Theorem 4.1. *Let E be a reflexive Banach space, and let $f : E \to \mathbb{R}$ be a lower semicontinuous, strictly convex, and Gâteaux differentiable function. Suppose that f is bounded on bounded sets, and uniform convex on bounded sets. Let C be a nonempty bounded subset of E. Let $\psi : C \to [0, \infty)$ be a function, and let $T : C \to C$ be a map. Assume that:*

$$D_f(x, Tx) \leq \psi(x) - \psi(Tx) \text{ for each } x \in C. \tag{4.1}$$

Then we have:

- (A1) *If C is also a closed convex subset of E and T has demiclosed property, then there exists $\bar{x} \in C$ such that $T\bar{x} = \bar{x}$;*
- (A2) *If C is a compact set and T is continuous, then there exists $\bar{x} \in C$ such that $T\bar{x} = \bar{x}$;*
- (A3) *If C is a compact set and $x \to D_f(x, Tx)$ is lower semicontinuous, then there exists $\bar{x} \in C$ such that $T\bar{x} = \bar{x}$.*

Proof. Take any $x \in C$ and let x be fixed. For each $n \in \mathbb{N} \cup \{0\}$, let $x_n = T^n x$ and $T^0 x = x$. Then, for each $n \in \mathbb{N} \cup \{0\}$, it follows from (4.1) that

$$D_f(x_n, Tx_n) \leq \psi(x_n) - \psi(Tx_n).$$

That is,

$$D_f(x_n, x_{n+1}) \leq \psi(x_n) - \psi(x_{n+1}). \tag{4.2}$$

By (4.2), we know that $\{\psi(x_n)\}$ is a nonincreasing sequence, $\lim_{n\to\infty} \psi(x_n)$ exists, and $\lim_{n\to\infty} D_f(x_n, x_{n+1}) = 0$. By Lemma 2.6, $\lim_{n\to\infty} ||x_n - x_{n+1}|| = 0$. That is, $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$.

(A1) Since C is a bounded closed convex subset of a reflexive Banach space E, we may assume that $x_n \rightharpoonup \bar{x}$ for some $\bar{x} \in C$. Since T has demiclosed property, then $T\bar{x} = \bar{x}$.

(A2) If C is a compact set, then we may assume that $x_n \to \bar{x}$ for some $\bar{x} \in C$. Since T is continuous and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, we have $T\bar{x} = \bar{x}$.

(A3) If C is a compact set, then we may assume that $x_n \to \bar{x}$ for some $\bar{x} \in C$. Since $x \to D_f(x, Tx)$ is lower semicontinuous, then

$$D_f(\bar{x}, T\bar{x}) \leq \liminf_{n\to\infty} D_f(x_n, Tx_n) = \lim_{n\to\infty} D_f(x_n, Tx_n) = 0.$$

Hence, it follows from Lemma 2.3 that $T\bar{x} = \bar{x}$. □

Remark 4.2. In Theorem 4.1, if $f : E \to \mathbb{R}$ is Fréchet differentiable and $T : C \to C$ is continuous, then it follows from Lemma 2.4 that $x \to D_f(x, Tx)$ is lower semicontinuous for each $x \in C$.

Remark 4.3. Theorem 4.1 is not a special case of the classical Caristi's fixed point theorem or generalized Caristi's fixed point theorem with generalized distance.

Following the same argument as in Remark 3.5, we get the following result by Theorem 4.1.

Theorem 4.4. *Let E be a reflexive Banach space, and let $f : E \to \mathbb{R}$ be a lower semicontinuous, strictly convex, and Gâteaux differentiable function. Suppose that f is bounded on bounded sets, and uniform convex on bounded sets. Let C be a nonempty bounded subset of E. Let $\psi : C \to [0, \infty)$ be a function, and let $T : C \to C$ be a map. Assume that $r \in [0, 1)$ and*

$$D_f(Tx, T^2x) \leq r \cdot D_f(x, Tx) \text{ for each } x \in C.$$

Then we have:

- (A1) *If C is also a closed convex subset of E and T has demiclosed property, then there exists $\bar{x} \in C$ such that $T\bar{x} = \bar{x}$;*
- (A2) *If C is a compact set and T is continuous, then there exists $\bar{x} \in C$ such that $T\bar{x} = \bar{x}$;*
- (A3) *If C is a compact set and $x \to D_f(x, Tx)$ is lower semicontinuous, then there exists $\bar{x} \in C$ such that $T\bar{x} = \bar{x}$.*

The following results are special cases of Theorem 4.6.

Corollary 4.5. Let E be a reflexive, smooth, and strictly convex Banach space. Let C be a nonempty closed convex subset of E, and $T : C \to C$ be a mapping. Suppose that $r \in [0, 1)$ and

$$\phi(Tx, T^2x) \leq r \cdot \phi(x, Tx) \text{ for each } y \in C,$$

where $\phi(x, y) := ||x||^2 - 2\langle x, Jy\rangle + ||y||^2$ and J is the duality mapping. Then we have:

- (A1) If C is also a closed convex subset of E and T has demiclosed property, then there exists $\bar{x} \in C$ such that $T\bar{x} = \bar{x}$;
- (A2) If C is a compact set and T is continuous, then there exists $\bar{x} \in C$ such that $T\bar{x} = \bar{x}$;
- (A3) If C is a compact set and $x \to \phi(x, Tx)$ is lower semicontinuous, then there exists $\bar{x} \in C$ such that $T\bar{x} = \bar{x}$.

Corollary 4.6. *Let E be a Hilbert space. Let C be a nonempty closed convex subset of E, and let $T : C \to C$ be a mapping. Suppose that $r \in [0, 1)$ and*

$$||T^2x - Tx|| \leq r \cdot ||x - Tx|| \text{ for each } x \in C.$$

Then there exists $\bar{x} \in C$ such that

- (A1) *If C is also a closed convex subset of E and T has demiclosed property, then there exists $\bar{x} \in C$ such that $T\bar{x} = \bar{x}$;*
- (A2) *If C is a compact set and T is continuous, then there exists $\bar{x} \in C$ such that $T\bar{x} = \bar{x}$;*
- (A3) *If C is a compact set and $x \to ||x - Tx||$ is lower semicontinuous, then there exists $\bar{x} \in C$ such that $T\bar{x} = \bar{x}$.*

Let (X, d) be a metric space. Then $f : X \to \mathbb{R}$ is said to be a graph contraction map [39] if it has closed graph and there exists $r \in [0, 1)$ such that

$$d(fx, f^2x) \leq r \cdot d(x, fx) \text{ for each } x \in C.$$

Note that our results in this section are different from graph contraction mapping existing in the literatures.

For Theorem 4.1, if C is not assumed to be a bounded set, then we modify Theorem 4.1 to the following result.

Theorem 4.7. *Let E be a reflexive Banach space, and let $f : E \to \mathbb{R}$ be a lower semicontinuous, strictly convex, and Gâteaux differentiable function. Suppose that f is bounded on bounded sets, and uniform convex on bounded sets. Let C be a nonempty subset of E. Let $\psi : C \to [0, \infty)$ be a coercive function, and let $T : C \to C$ be a continuous map or has demiclosed property. Assume that:*

$$D_f(x, Tx) \leq \psi(x) - \psi(Tx) \text{ for each } x \in C. \tag{4.3}$$

Then there exists $\bar{x} \in C$ such that $T\bar{x} = \bar{x}$.

Proof. Take any $x \in C$ and let x be fixed. For each $n \in \mathbb{N} \cup \{0\}$, let $x_n = T^n x$ and $T^0 x = x$. In the proof of Theorem 4.1, we know that $\{\psi(x_n)\}$ is a nonincreasing sequence, $\lim_{n\to\infty} \psi(x_n)$ exists, and $\lim_{n\to\infty} D_f(x_n, x_{n+1}) = 0$. Clearly, $\{\psi(x_n)\}_{n\in\mathbb{N}}$ is a bounded sequence.

Let $B := \{x_n : n \in \mathbb{N}\}$. Suppose that B is an unbounded set. Then for each $k \in \mathbb{N}$, there exists an element $y_k \in B$ such that $||y_k|| \geq k$. Hence, we get a sequence $\{y_k\}_{k\in\mathbb{N}}$ in C such that $||y_k|| \to \infty$. Since ψ is coercive, we know that $\psi(y_k) \to \infty$. So, $\{\psi(y_k)\}$ is an unbounded set. Since $\{\psi(y_k)\} \subseteq \psi(B) = \{\psi(x_n)\}_{n\in\mathbb{N}}$, we know that $\{\psi(x_n)\}_{n\in\mathbb{N}}$ is an unbounded set. This leads to contraction. Hence, $\{x_n\}_{n\in\mathbb{N}}$ is a bounded set. Next, following similar argument as the proof of Theorem 4.1, we get the conclusion of Theorem 4.7. □

Next, we consider a multivalued fixed point theorem with the Bregman distance on Banach spaces.

Theorem 4.8. *Let E be a reflexive Banach space, and let $f : E \to \mathbb{R}$ be a lower semicontinuous, strictly convex, and Gâteaux differentiable function. Suppose that f is bounded on bounded sets, and uniform convex on bounded sets. Let C be a nonempty compact subset of E. Let $T : C \multimap C$ be a multivalued map with nonempty closed values. Let $D_f(x, T(x)) := \inf_{y\in T(x)} D_f(x, y)$. Let $0 < a < 1$, and let $\varphi : [0, \infty) \to [0, 1)$ and $\psi : [0, \infty] \to [a, 1]$ be two functions with the properties:*

$$\limsup_{r\to t^+} \frac{\varphi(r)}{\psi(r)} < 1 \text{ for each } t \in [0, \infty). \tag{4.4}$$

For each $x \in X$, there exists $y \in T(x)$ such that

$$\psi(D_f(x, T(x))) \cdot D_f(x, y) \leq D_f(x, T(x)), \tag{4.5}$$

and

$$D_f(y, T(y)) \leq \varphi(D_f(x, T(x))) \cdot D_f(x, y). \tag{4.6}$$

Assume that one of the following conditions holds:

(A1) *T is closed (i.e., $Gr(T) := \{(x, y) \in C \times C : y \in T(x)\}$ is a closed set);*
(A2) *$x \to D_f(x, T(x))$ is lower semicontinuous;*
(A3) *$\inf\{D_f(x, z) + D_f(x, T(x)) : x \in C\} > 0$ for each $z \in C$ with $z \notin T(z)$.*

Then there exists $\bar{x} \in C$ such that $\bar{x} \in T(\bar{x})$.

Proof. Take any point $x_0 \in E$ and let x_0 be fixed. By (4.5) and (4.6), there exists $x_1 \in T(x_0)$ such that

$$\psi((D_f(x_0, T(x_0)))) \cdot D_f(x_0, x_1) \leq D_f(x_0, T(x_0)), \tag{4.7}$$

and

$$D_f(x_1, T(x_1)) \leq \varphi(D_f(x_0, T(x_0))) \cdot D_f(x_0, x_1). \tag{4.8}$$

Continuing this process, we can choose a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_{n+1} \in T(x_n)$ such that

$$\psi((D_f(x_n, T(x_n)))) \cdot D_f(x_n, x_{n+1}) \leq D_f(x_n, T(x_n)), \tag{4.9}$$

and

$$D_f(x_{n+1}, T(x_{n+1})) \leq \varphi(D_f(x_n, T(x_n))) \cdot D_f(x_n, x_{n+1}) \tag{4.10}$$

for each $n \in \mathbb{N} \cup \{0\}$. By (4.9) and (4.10), for each $n \in \mathbb{N} \cup \{0\}$, we have:

$$D_f(x_{n+1}, T(x_{n+1})) < \frac{\varphi(D_f(x_n, T(x_n))) \cdot D_f(x_n, T(x_n))}{\psi(D_f(x_n, T(x_n)))}. \tag{4.11}$$

Clearly, $\{D_f(x_n, T(x_n))\}_{n=0}^{\infty}$ is a nonincreasing sequence in $[0, \infty)$. Then there exists $\delta \geq 0$ such that

$$\delta = \lim_{n\to\infty} D_f(x_n, T(x_n)) = \inf\{D_f(x_n, T(x_n)) : n \in \mathbb{N} \cup \{0\}\}. \tag{4.12}$$

Suppose that $\delta > 0$. By (4.4), (4.11), and (4.12),

$$\delta \leq \lim_{D_f(x_n, T(x_n)) \to \delta^+} \frac{\varphi(D_f(x_n, T(x_n))) \cdot D_f(x_n, T(x_n))}{\psi(D_f(x_n, T(x_n)))} < \delta.$$

And this is a contradiction. Thus $\delta = 0$. Further, $\lim_{n\to\infty} D_f(x_n, x_{n+1}) = 0$. By Lemma 2.6, $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$. Since C is a compact set, we may assume that $\lim_{n\to\infty} x_n = \bar{x}$ for some $\bar{x} \in C$.

If (A1) holds, then T is closed. Since $x_{n+1} \in T(x_n)$ for each $n \in \mathbb{N} \cup \{0\}$ and $x_n \to \bar{x}$ as $n \to \infty$, $\bar{x} \in T(\bar{x})$.

If (A2) holds, then we have:

$$0 \leq D_f(\bar{x}, T(\bar{x})) \leq \liminf_{n\to\infty} D_f(x_n, T(x_n)) = 0.$$

Then $D_f(\bar{x}, T(\bar{x})) = 0$. So, for each $k \in \mathbb{N}$, there exists $y_k \in T(\bar{x})$ such that $D_f(\bar{x}, y_k) < 1/k$. By Lemma 2.6 again, $y_k \to \bar{x}$ as $k \to \infty$. Since $T(\bar{x})$ is closed, $\bar{x} \in T(\bar{x})$.

If (A3) holds, suppose that $\bar{x} \notin T(\bar{x})$, then we have:

$$\begin{aligned} 0 &< \inf\{D_f(x, \bar{x}) + D_f(x, T(x)) : x \in C\} \\ &\leq \inf\{D_f(x_n, \bar{x}) + D_f(x_n, T(x_n)) : n \in \mathbb{N}, \text{ and } n > n_0\} \\ &\leq \inf\{D_f(x_n, \bar{x}) + D_f(x_n, x_{n+1}) : n \in \mathbb{N} \text{ and } n > n_0\} = 0. \end{aligned}$$

And this is a contradiction. Hence, $\bar{x} \in T(\bar{x})$. □

Remark 4.9. Although Theorem 4.8 is similar to [26, Theorem 3.1], [34, Theorem 2.2], and [30, Theorem 2.1], we know that these results are different.

Remark 4.10. (a) If T is an upper semicontinuous multivalued map with nonempty closed values, then the condition (A1) of Theorem 4.8 holds [1].
(b) If f is Fréchet differentiable and T is a lower semicontinuous multivalued map, then it follows from Lemma 2.4 and [29, Theorem 1] that the condition (A2) of Theorem 4.8 holds.

Remark 4.11. Following the same argument as the proof of Theorem 4.8, we can get similar results from Theorems 3.2 and 3.3 in [26].

In Theorem 4.8, if T is a map, and $\psi(t) = 1$ for each $t \in [0, \infty)$, then we have the following result.

Theorem 4.12. *Let E be a reflexive Banach space, and let $f : E \to \mathbb{R}$ be a lower semicontinuous, strictly convex, and Gâteaux differentiable function. Suppose that f is bounded on bounded sets, and uniform convex on bounded sets. Let C be a nonempty compact subset of E. Let $T : C \to C$ be a map. Let $\varphi : [0, \infty) \to [0, 1)$ and*

$$\limsup_{r \to t^+} \varphi(r) < 1 \text{ for each } t \in [0, \infty). \tag{4.13}$$

For each $x \in X$,

$$D_f(T(x), T^2(x)) \leq \varphi(D_f(x, T(x))) \cdot D_f(x, T(x)). \tag{4.14}$$

Assume that one of the following conditions holds:
(A1) *T is closed (i.e., $Gr(T) := \{(x, y) \in C \times C : y = T(x)\}$ is a closed set);*
(A2) *$x \to D_f(x, T(x))$ is lower semicontinuous;*
(A3) *$\inf\{D_f(x, z) + D_f(x, T(x)) : x \in C\} > 0$ for each $z \in C$ with $z \neq T(z)$.*
Then there exists $\bar{x} \in C$ such that $T(\bar{x}) = \bar{x}$.

5. Properties of Bregman projection

Lemma 5.1. [10, Proposition 2.1.5] *Let E be a reflexive Banach space, let C be a nonempty closed convex set of $D(f)$. Let $f : E \to (-\infty, \infty]$ be a strongly coercive Bregman function on $int(\mathcal{D})$. Then for each $x \in int(\mathcal{D})$, there exists a unique $z \in C$ such that $D_f(z, x) = \min_{y \in C} D_f(y, x)$. Here, let $P_C x = z$, and P_C is called the Bregman projection from E onto C. Further, we have*
(i) *$z = P_C x$ if and only if $z \in C$ and $\langle y - z, \nabla f(x) - \nabla f(z)\rangle \leq 0$ for all $y \in C$;*
(ii) *$D_f(y, P_C x) + D_f(P_C x, x) \leq D_f(y, x)$ for each $y \in C$ and each $x \in E$.*

Lemma 5.2 ([46])**.** *Let E be a Banach space, and let $f : E \to (-\infty, +\infty]$ be a convex function. Then we get:*
(i) *If f is uniformly convex on bounded sets, then*

$$0 < \inf_{||x|| \leq r, ||y|| \leq r, ||x-y|| = t, \alpha \in (0,1)} \frac{\alpha f(x) + (1-\alpha) f(y) - f(\alpha x + (1-\alpha) y)}{\alpha(1-\alpha)}$$

for all $r > 0$ and $t > 0$;
(ii) *If f is uniformly smooth on bounded sets, then*

$$\lim_{t \downarrow 0} \sup_{||x|| \leq r, ||y|| = 1, \alpha \in (0,1)} \frac{\alpha f(x + (1-\alpha) t y) + (1-\alpha) f(x - \alpha t y) - f(x)}{t \alpha (1-\alpha))} = 0$$

for all $r > 0$.

Lemma 5.3. [10, 33, 46] *Let* E *be a reflexive Banach space, and let* $f : E \to \mathbb{R}$ *be a continuous convex and strongly coercive function. Then the following assertions are equivalent.*

(i) f *is bounded on bounded sets, and uniformly smooth on bounded sets;*
(ii) f^* *is Fréchet differentiable and* ∇f^* *is uniformly norm-to-norm continuous on bounded sets;*
(iii) $D(f^*) = E^*$, f^* *is strongly coercive and uniformly convex on bounded sets.*

By Lemmas 5.2 and 5.3, it is easy to get the following result.

Lemma 5.4. *Let* E *be a reflexive Banach space, and let* $f : E \to \mathbb{R}$ *be a continuous convex and strongly coercive function. If* f *is bounded on bounded sets, and uniformly smooth on bounded sets, then* f^* *is strictly convex.*

Lemma 5.5 ([43])**.** *Let* E *be a reflexive Banach space, and let* C *be a nonempty closed convex subset of* E*. Let* $f : C \to (\infty, \infty]$ *be a proper convex lower semicontinuous function, and suppose that* $f(x_n) \to \infty$ *as* $||x_n|| \to \infty$*. Then there exists* $\bar{x} \in C$ *such that* $f(\bar{x}) = \inf\{f(x) : x \in C\}$*.*

By Lemma 5.5, it is easy to get the following result.

Lemma 5.6. *Let* E *be a reflexive Banach space, and let* C *be a nonempty closed convex subset of* E*. Let* $f : E \to \mathbb{R}$ *be a proper, continuous, strictly convex, and strongly coercive function. Then, for each* $x \in E$*, there exists a unique* $\bar{x} \in C$ *such that* $f(x - \bar{x}) = \min_{y \in C} f(x - y)$*. Here, we use the notation* $P_C^f(x) = \bar{x}$*.*

For P_C^f, we have the following result by Lemma 2.1. For detail, we give the proof of the following result. Note that it is very important and essential for Theorem 5.21. In fact, if we do not have this result, then we could not get Theorem 5.21. Besides, it is a generalization of Problem 5.3.3 in [43, page154].

Lemma 5.7. *Let* C *be a nonempty closed convex subset of a reflexive Banach space* E*. Let* $f : E \to \mathbb{R}$ *be a proper, continuous, strictly convex, Gateaux differentiable, and strongly coercive function. Let* $(x, z) \in E \times C$*. Then* $z = P_C^f(x)$ *if and only if* $\langle y - z, \nabla f(x - z)\rangle \leq 0$ *for all* $y \in C$*.*

Proof. Suppose that $\langle y - z, \nabla f(x - z)\rangle \leq 0$ for all $y \in C$. Then for each $y \in C$,

$$D_f(x - y, x - z) := f(x - y) - f(x - z) - \langle (x - y) - (x - z), \nabla f(x - z)\rangle \geq 0.$$

Hence,

$$f(x - y) - f(x - z) \geq \langle z - y, \nabla f(x - z)\rangle \geq 0 \text{ for each } y \in C.$$

So, $f(x - z) = \min_{y \in C} f(x - y)$ and $z = P_C^f(x)$.

Conversely, suppose that $z = P_C^f(x)$. Take any $y \in C$, and let y be fixed. Since f is Gateaux differentiable,

$$\begin{aligned}\langle z - y, \nabla f(x - z)\rangle &= \lim_{t \to 0} \frac{f(x - z + t(z - y)) - f(x - z)}{t} \\ &= \lim_{t \downarrow 0} \frac{f(x - (1 - t)z - ty) - f(x - z)}{t}.\end{aligned}$$

Clearly, $(1-t)z+ty \in C$ for $t \in [0,1]$. This implies that $\langle y-z, \nabla f(x-z)\rangle \le 0$ for each $y \in C$. Therefore, the proof is completed. □

The following result is another version of Lemma 5.1.

Lemma 5.8. *Let E be a reflexive Banach space, $f: E \to \mathbb{R}$ be a proper strongly coercive Bregman function, and let D_f be the Bregman distance associated with f. Suppose that f is bounded on bounded sets, uniformly convex on bounded sets, and uniformly smooth on bounded sets. Let C^* be a closed convex subset of E^*. Let f^* be the conjugate function of f. Define $D_{f^*}: E^* \times E^* \to \mathbb{R}$ as*

$$D_{f^*}(x^*, y^*) := f^*(x^*) - f^*(y^*) - \langle x^* - y^*, \nabla f^*(y^*)\rangle$$

for each $(x^*, y^*) \in E^* \times E^*$. Then, for each $x^* \in E^*$, there is a unique element $z^* \in C^*$ such that $D_{f^*}(z^*, x^*) = \min_{y^* \in C^*} D_{f^*}(y^*, x^*)$.

Proof. By definition, we know that

$$D_{f^*}(\nabla f(x), \nabla f(y)) = D_f(y, x)$$

for each $(x, y) \in E \times E$ [33]. f^* is proper, convex, and Gâteaux differentiable. By Lemma 2.2, f^* is w^*-continuous. By Lemma 5.3, f^* is strongly coercive and uniformly convex on bounded sets. By Lemma 5.4, f^* is strictly convex.

Next, for each $x^* \in E^*$ and each $r > 0$, it follows from Lemma 2.8 that there exists a unique $x \in E$ such that $x^* = \nabla f(x)$. Hence,

$$\{y^* \in E^* : \ D_{f^*}(x^*, y^*) \le r\} = \nabla f(\{y \in E : \ D_f(y, x) \le r\}).$$

By Lemma 2.8, $\{y \in E : \ D_f(y, x) \le r\}$ is a bounded set. By Lemmas 2.4 and 2.5, $\{y^* \in E^* : \ D_{f^*}(x^*, y^*) \le r\}$ is a bounded set.

So, all conditions of Lemma 5.1 are satisfied. Therefore, for each $x^* \in E^*$, there is a unique element $z^* \in C^*$ such that $D_{f^*}(z^*, x^*) = \min_{y^* \in C^*} D_{f^*}(y^*, x^*)$. Therefore, the proof is completed. □

Let E be a normed linear space and let $x, y \in E$. We say that x is orthogonal to y if $||x|| \le ||x + \lambda y||$ for each $\lambda \in \mathbb{R}$. Here, we use $x \perp y$ to denote x is orthogonal to y [6, 17, 18, 19]. We know that for $x, y \in E$, $x \perp y$ if and only if there exists $g \in Jx$ such that $\langle y, g\rangle = 0$. (For example, one can refer to [43].)

Definition 5.9. Let E be a normed linear space and let $x, y \in E$, and let $f: E \to \mathbb{R}$ be a function. We say that x is f-orthogonal to y if $f(x) \le f(x+\lambda y)$ for each $\lambda \in \mathbb{R}$. Here, we use $x \perp_f y$ to denote x is f-orthogonal to y.

Remark 5.10. In Definition 5.9, if $f(x) = ||x||^2$, then $x \perp_f y$ is reduced to $x \perp y$.

By the definition of Bregman distance and related results, it is easy to get the following result. For detail, we give the proof of the following result.

Proposition 5.11. *Let E be a Banach space and let $x, y \in E$, let $f: E \to \mathbb{R}$ be a proper, lower semicontinuous, convex, and Gâteaux differentiable on $D(f)$. Then the following conditions are equivalent:*

(i) *$f(x) \le f(x + \lambda y)$ for each $\lambda > 0$;*
(ii) *$\langle y, \nabla f(x)\rangle \ge 0$.*

Proof. (i) ⇒ (ii): Suppose that $f(x) \leq f(x+\lambda y)$ for each $\lambda > 0$. Since f is Gâteaux differentiable on $D(f)$,

$$\langle y, \nabla f(x)\rangle = \lim_{t\to 0} \frac{f(x+ty)-f(x)}{t} = \lim_{t\downarrow 0} \frac{f(x+ty)-f(x)}{t} \geq 0.$$

(ii) ⇒ (i): Suppose that $\langle y, \nabla f(x)\rangle \geq 0$. Then, for each $\lambda > 0$, we get

$$D_f(x+\lambda y, x) = f(x+\lambda y) - f(x) - \langle x+\lambda y - x, \nabla f(x)\rangle \geq 0.$$

This implies that $f(x+\lambda y) - f(x) \geq \lambda\langle y, \nabla f(x)\rangle \geq 0$. □

Further, we get the following proposition by following the similar argument as in the proof of Proposition 5.11.

Proposition 5.12. *Let E be a Banach space, and let $f : E \to \mathbb{R}$ be a proper, lower semicontinuous, convex, and Gâteaux differentiable on $D(f)$. Then $x \perp_f y$ if and only if $\langle y, \nabla f(x)\rangle = 0$.*

Before we consider a sunny generalized nonexpansive retraction mapping, we show the following lemmas. First, it is easy to get this result and it is a generalized version of [16, Lemma 2.2]. Further, it is an important tool for Lemma 5.14 and related results.

Lemma 5.13. *Let E be a reflexive Banach space, and let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function. Let M be a closed linear subspace of E, and let $(x, z) \in E \times M$. Then $z = P_M x$ if and only if $\langle y, \nabla f(x) - \nabla f(z)\rangle = 0$ for each $y \in M$.*

Next, we get the following result and it is a generalization of [16, Lemma 3.1].

Lemma 5.14. *Let E be a reflexive Banach space. Let $f : E \to \mathbb{R}$ be a proper, strongly coercive, and Bregman function. Let Y be a closed linear subspace of E, and let P_Y be the Bregman projection from E onto Y. Then the following hold.*

(i) $\langle P_Y x, \nabla f(x)\rangle = \langle P_Y x, \nabla f(P_Y x)\rangle$ *for each* $x \in E$.
(ii) $f^*(\nabla f(P_Y x)) \leq f^*(\nabla f(x))$ *for each* $x \in E$.
(iii) *If X is also a closed linear subspace of E with $X \subseteq Y$, then $P_X P_Y = P_X$.*
(iv) $P_Y \alpha x = \alpha P_Y x$ *for each $x \in E$ and each $\alpha \geq 0$.*

Proof. (i) Since P_Y is the Bregman projection from E onto Y, it follows Lemma 5.1 that $\langle y - P_Y x, \nabla f(x) - \nabla f(P_Y x)\rangle \leq 0$. for each $x \in E$ and each $y \in Y$. Let $y_1 := 0$ and $y_2 := 2P_Y x$. Clearly, $y_1, y_2 \in Y$. Then, for each $x \in E$,

$$\langle P_Y x, \nabla f(x) - \nabla f(P_Y x)\rangle \geq 0 \text{ and } \langle P_Y x, \nabla f(x) - \nabla f(P_Y x)\rangle \leq 0.$$

Hence, $\langle P_Y x, \nabla f(x)\rangle = \langle P_Y x, \nabla f(P_Y x)\rangle$ for each $x \in E$.

(ii) Since P_Y is the Bregman projection from E onto Y, for each $x \in E$ and each $y \in Y$, it follows from Lemmas 2.1 ,5.13 and 5.1 and the properties of the conjugate function that

$$
\begin{aligned}
0 &\le D_f(y,x) - D_f(y,P_Y x)\\
&= -\langle y-x, \nabla f(x) - \nabla f(P_Y x)\rangle - D_f(x,P_Y x)\\
&= -\langle y-x, \nabla f(x) - \nabla f(P_Y x)\rangle - f(x) + f(P_Y x) + \langle x - P_Y x, \nabla f(P_Y x)\rangle\\
&= \langle x, \nabla f(x) - \nabla f(P_Y x)\rangle - f(x) + f(P_Y x) + \langle x - P_Y x, \nabla f(P_Y x)\rangle\\
&= \langle x, \nabla f(x)\rangle - \langle x, \nabla f(P_Y x)\rangle - f(x) + f(P_Y x) + \langle x, \nabla f(P_Y x)\rangle - \langle P_Y x, \nabla f(P_Y x)\rangle\\
&= \langle x, \nabla f(x)\rangle - f(x) + f(P_Y x) - \langle P_Y x, \nabla f(P_Y x)\rangle\\
&= f^*(\nabla f(x)) - f^*(\nabla f(P_Y x)).
\end{aligned}
$$

Therefore, $f^*(\nabla f(P_Y x)) \le f^*(\nabla f(x))$ for each $x \in E$. □

By Lemmas 5.8, 5.13, and 5.14, we give another versions of Lemmas 5.13 and 5.14, respectively.

Lemma 5.15. *Let E be a reflexive Banach space. Let $f : E \to \mathbb{R}$ be a proper, strongly coercive, and Bregman function. Suppose that f is bounded on bounded sets, uniformly convex on bounded sets, and uniformly smooth on bounded sets. Let M^* be a closed linear subspace of E^*, and let $(x^*, z^*) \in E^* \times M^*$. Then $z^* = P_{M^*}x^*$ if and only if $\langle y^*, \nabla f^*(x^*) - \nabla f^*(z^*)\rangle = 0$ for each $y^* \in M^*$.*

Lemma 5.16. *Let E be a reflexive Banach space. Let $f : E \to \mathbb{R}$ be a proper, strongly coercive, and Bregman function. Suppose that f is bounded on bounded sets, uniformly convex on bounded sets, and uniformly smooth on bounded sets. Let Y^* be a closed linear subspace of E^*, and let P_{Y^*} be the Bregman projection from E^* onto Y^*. Then the following hold.*

(i) *$\langle P_{Y^*}x^*, \nabla f^*(x^*)\rangle = \langle P_{Y^*}x^*, \nabla f^*(P_{Y^*}x^*)\rangle$ for each $x^* \in E^*$;*
(ii) *$f(\nabla f^*(P_{Y^*}x^*)) \le f(\nabla f^*(x^*))$ for each $x^* \in E^*$;*
(iii) *If X^* is a closed linear subspace of E^* with $X^* \subseteq Y^*$, then $P_{X^*}P_{Y^*} = P_{X^*}$;*
(iv) *$P_{Y^*}\alpha x = \alpha P_{Y^*}x$ for each $x \in E^*$ and $\alpha \ge 0$.*

Following the same argument as the proof of Theorem 3.3 and Corollary 3.1 in [16] and using the above lemmas, we get the following result. In fact, it is a generalization of [16, Theorem 3.3].

Theorem 5.17. *Let E be a smooth, strictly convex, and reflexive Banach space, and let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded sets, and uniformly convex on bounded sets, and uniformly smooth on bounded sets. Let Y^* be a nonempty closed linear subspace of E^*. Then the mapping $E_{Y^*} := (\nabla f)^{-1}P_{Y^*}\nabla f$ has the following properties:*

(i) *For each $x \in E$, $\nabla f(x) \in Y^*$ if and only if $E_{Y^*}(x) = x$;*
(ii) *$f(E_{Y^*}x) \le f(x)$ for each $x \in E$;*
(iii) *For each $x \in E$, the following are equivalent:*
(a) $f(E_{Y^*}(x)) = f(x)$; (b) $E_{Y^*}(x) = x$; and (c) $\nabla f(x) \in Y^*$.
(iv) *$E_{Y^*}(x_1 + x_2) = E_{Y^*}(E_{Y^*}(x_1) + E_{Y^*}(x_2))$ for all $x_1, x_2 \in E$.*
(v) *For $n \ge 2$, and $x_1, x_2, \ldots, x_n \in E$. Then*

$$E_{Y^*}(x_1 + x_2 + \cdots + x_n) = E_{Y^*}(E_{Y^*}(x_1) + \cdots + E_{Y^*}(x_n)).$$

(vi) *$E_{Y^*}(\alpha x) = \alpha E_{Y^*}(x)$ for each $x \in E^*$ and each $\alpha > 0$.*

Following the same argument as the proof of [16, Theorem 3.4] and using the above lemmas, we get the following result. In fact, it is a generalization of [16, Theorem 3.4].

Theorem 5.18. *Let E be a smooth, strictly convex, and reflexive Banach space, and let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded sets, and uniformly convex and uniformly smooth on bounded sets. Let Y_1^* and Y_2^* be a nonempty closed linear subspace of E^* with $Y_1^* \subseteq Y_2^*$. Then we have: (i) $E_{Y_2^*}E_{Y_1^*} = E_{Y_1^*}$; (ii) $E_{Y_1^*}E_{Y_2^*} = E_{Y_1^*}$; and (iii) If $\nabla f(0) = 0$, then $E_{Y_2^*}^{-1}(0) \subseteq E_{Y_1^*}^{-1}(0)$.*

Lemma 5.19. *Let E be a normed linear space, and let $f : E \to \mathbb{R}$ be a function. Let $T : E \to E$ be a map such that $T(Tx + \beta(x - Tx)) = Tx$ for each $x \in E$ and each $\beta \in \mathbb{R}$. Then $f(Tx) \leq f(x)$ for each $x \in E$ if and only if $Tx \perp_f (x - Tx)$ for each $x \in E$.*

Proof. Suppose that $f(Tx) \leq f(x)$ for each $x \in E$. For each $x \in E$ and each $\beta \in \mathbb{R}$, since $T(Tx + \beta(x - Tx)) = Tx$, we know that

$$f(Tx) = f(T(Tx + \beta(x - Tx))) \leq f(Tx + \beta(x - Tx)).$$

Then $Tx \perp_f (x - Tx)$ for each $x \in E$.

Conversely, suppose that $Tx \perp_f (x - Tx)$ for each $x \in E$. Then $f(Tx) \leq f(Tx + \beta(x - Tx))$ for each $x \in E$ and each $\beta \in \mathbb{R}$. Let $\beta = 1$. Then $f(Tx) \leq f(x)$ for each $x \in E$. □

By Lemmas 2.3, 5.15, 5.19, and Theorem 5.17, and following the same argument as the proof of [16, Theorem 4.1], we get the following result.

Theorem 5.20. *Let E be a reflexive, strictly convex, and smooth Banach space. Let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded sets, and uniformly convex and uniformly smooth on bounded sets. Let Y^* be a nonempty closed linear subspace of E^*. Then $E_{Y^*}(x) \perp_f (x - E_{Y^*}(x))$ for each $x \in E$.*

Let Y be a nonempty subset of a Banach space E, and let Y^* be a nonempty subset of the dual space E^*. Then we define the following two sets:

$$Y_\perp^* := \{x \in E : g(x) = 0 \text{ for all } g \in Y^*\}, \tag{5.1}$$

and

$$Y^\perp := \{g \in E^* : g(x) = 0 \text{ for all } x \in Y\}. \tag{5.2}$$

For these two sets, we give the following result.

Following the same argument as the proof of [16, Theorem 4.2], and using Lemmas 5.15 and 5.7, we get the following result. In fact, Lemma 5.7 is essential and important for Theorem 5.21.

Theorem 5.21. *Let E be a reflexive, strictly convex, and smooth Banach space. Let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded sets, and uniformly convex and uniformly smooth on bounded sets. Let Y^* be a* *nonempty closed linear subspace of* E^*. *Let* I *be the identity map on* E. *Then we have:*

(i) $I - E_{Y^*} = P^f_{Y^*_\perp}$;

(ii) *If* Y *is a closed linear subspace of* E, *then* $(I - P^f_Y) = E_{Y^\perp}$.

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