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FORMALLY SELF-ADJOINT SCHERÖDINGER OPERATORS WITH REAL MEASURABLE POTENTIAL

KIYOKO FURUYA

ABSTRACT. A Schrödinger operator, (not self-adjoint but) formally self-adjoint, generates a (not unitary but) contraction semigroup. Our class of potentials U in Schrödinger equation is wide enough : the real measurable potential U should be locally essentially bounded except a closed set of measure zero.

1. INTRODUCTION

We shall construct a family of unique solutions to the Schrödinger equation in $\mathbb{R}^N$

$$h\frac{\partial}{\partial t}u(t,x) = \frac{ih^2}{2m}\triangle u(t,x) - iU(x)u(t,x), \quad u(0,x) = \varphi(x), \tag{1.1}$$

for $U \in L^\infty_{loc}(\mathbb{R}^N\backslash\mathcal{N};\mathbb{R})$ where $\mathcal{N}$ is a closed set of measure 0. For further information, see (1.2) and (2.2). Here h and m are positive constants.

In the previous paper [4] we discussed this problem under the condition for $U \in C(\mathbb{R}^N\backslash\mathcal{N};\mathbb{R}^+)$ with the subdifferential of lower semicontinuous and convex functional (see section 7).

Example 1.1. First we consider the movement of a classical particle with mass $m > 0$ in the field of a (real) potential U:

$$mu''(t) = -U'(u(t)), \quad u(0) = u_0,\ u'(0) = v_0.$$

The energy $E \equiv \frac{m}{2}v_0^2 + U(u_0)$ (which is constant) and the solution u satisfy

$$E = \frac{m}{2}(u'(t))^2 + U(u(t)), \quad u(t) = u_0 + \int_0^t \sqrt{\frac{2E - 2U(u(s))}{m}}ds.$$

If U decreases rapidly enough, the particle vanishes in a finite time : $\lim_{t\uparrow t_\infty}|u(t)| = \infty$ for some $t_\infty \in (0,\infty)$. In fact, $|u(t)| \to \infty$ as $t \to t_\infty$ for $U(x) = -|x|^n, n > 2$.

Now we consider the wave function u corresponding to the classical particle above. For such a potential U the existence probability $\|u(t)\|^2$ of the particle described by (1.1) may decrease : $\|u(t)\| < \|u(0)\|$ for $t > 0$. If a new particle comes from infinity at t_∞, the existence probability may be constant : $\|u(t)\| \equiv \|u(0)\|$. However the movement of the new particle is not unique, and hence $u(t)$ may not be unique.

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Remark 1.2. We define a sequence of functions $\{U_n\}_{n\in\mathbb{N}}$ as follows:

$$(1.2) \qquad U_n(x) = \begin{cases} n & \text{if} \quad n < U(x), \\ U(x) & \text{if} \quad -n \le U(x) \le n \\ -n & \text{if} \quad U(x) < -n \end{cases} \qquad for \quad n \in \mathbb{N}\,.$$

It is easily checked that $U_n(x) = \min\{n, \max\{-n, U(x)\}\}$. Then we shall approximate the potential U by U_n. The unique solution obtained by this approximation seems to correspond to the case no particle comes from infinity in the example above. This solution seems natural for the theory of path integrals. The physical meaning of the solution by Nelson[11] is unclear to the author.

Definition 1.3. Let B be a densely defined operator on Hilbert space $\mathcal{H}$. Then

(i) B is essentially self-adjoint if it has a unique self-adjoint extension, necessarily its closure $\bar{B}$,

(ii) B is formally self-adjoint if $\langle B\varphi, \psi\rangle = \langle \varphi, B\psi\rangle$ for all φ and ψ in $\mathcal{H}$.

We consider a closed extension of (not necessarily essentially self-adjoint but) formally self-adjoint operator $iA \equiv -(h^2/2m)\triangle + U$ on $\mathcal{C}_0^\infty(\mathbb{R}^N\backslash\mathcal{N})$. Here $\mathcal{C}_0^\infty(E)$ denote the set of all infinitely differentiable functions with compact support in E. The semigroup of our solution family, which is obtained by the approximation (1.2), is not necessarily a group of unitary operators but a semigroup of contractions. Our result improves one of the Nelson [11]'s, which says the contraction semigroup of his solution family exists (*not for all but*) for a. e. $m > 0$ and for any $U \in \mathcal{C}(\mathbb{R}^N\backslash\mathcal{N}_0;\mathbb{R})$, where $\mathcal{N}_0$ is a closed subset of capacity 0. In these half a century, no essential progress in this direction has been obtained as far as the author knows.

Remark 1.4. From a physical standpoint, we obtain the following conjecture :

The operator $-iA \equiv \frac{h^2}{2m}\triangle - U$ for $U \in \mathcal{C}(\mathbb{R}^N;\mathbb{R})$ is essentially self-adjoint if and only if $t_\infty = \infty$.

Our paper is organized in the following way: In Section 2, the fundamental assumptions are made and a full statement of our main theorem is given. In Section 3, basic facts and preliminary properties are given. Section 4 deals with the general theory for existence of weak limit of unitary group. Section 5 discusses our equation with the subdifferential of lower semicontinuous and convex functional. Section 6 is devoted to the proof of our main theorem. Previous results [4] are given in Section 7.

2. Schrödinger equation

For simplicity we consider the following normalized equation :

$$(2.1) \quad \frac{\partial}{\partial t}u(t,x) = i\triangle u(t,x) - iU(x)u(t,x),\ u(0,x) = \varphi(x) \quad \text{for} \quad \varphi \in H^{(2)}(\mathbb{R}^N;\mathbb{C}),$$

where $H^{(2)}(\mathbb{R}^N;\mathbb{C})$ denote the Sobolev space of L^2-functions with first and second distributional derivatives also in L^2 on $\mathbb{R}^N$ to $\mathbb{C}$.

If $\triangle - U$ is essentially self-adjoint, the operator family $\{T(t)\}$ defined by $T(t)\varphi = u(t)$ is uniquely extended to a group of unitary operators from $L^2(\mathbb{R}^N;\mathbb{C})$ to $L^2(\mathbb{R}^N;\mathbb{C})$. Let $\mathcal{N}$ be a fixed closed subset of $\mathbb{R}^N$ of measure 0 and $\mathcal{D} = \{D\}$ be the maximum family such that each element $D \subset \overline{D} \subset \mathbb{R}^N\backslash\mathcal{N}$ is a finite union of connected bounded open sets. The family $\mathcal{D} = \{D\}$ satisfies $\bigcup_{D\in\mathcal{D}} D = \mathbb{R}^N\backslash\mathcal{N}$. We denote the restriction of f to D by $f|_D$. We use the following notation

$$L^\infty_{loc}(\mathbb{R}^N\backslash\mathcal{N};\mathbb{R}) = \big\{f \;\big|\; f(x)\in\mathbb{R},\ x\in\mathbb{R}^N,\ f|_D\in L^\infty(D;\mathbb{R}),\ \forall D\in\mathcal{D}\big\}. \tag{2.2}$$

Let $U\in L^\infty_{loc}(\mathbb{R}^N\backslash\mathcal{N};\mathbb{R})$. We assume for any neighbourhood of any point of $\mathcal{N}$, U is not essentially bounded. By this assumption, U uniquely determines $\mathcal{N}$ in the following sense :

$$\mathcal{N} = \bigcap_\nu \{\mathcal{N}_\nu \;\big|\; U\in L^\infty_{loc}(\mathbb{R}^N\backslash\mathcal{N}_\nu;\mathbb{R})\}. \tag{2.3}$$

Let

$$B_n = \{\ x\in\mathbb{R}^N \;\big|\; -n < U(x) < n\ \} \quad \text{for} \quad n\in\mathbb{N}. \tag{2.4}$$

We have $B_m \supset B_n$ for $m > n$ and

$$\text{for any } D\in\mathcal{D},\ \text{there exists } B_n \text{ such that } D\subset\overline{D}\subset B_n. \tag{2.5}$$

(Strictly speaking, $\overline{D}\backslash B_n$ is not necessarily empty, but a null set.) We denote $U_n(x) = \min\big\{n, \max\{-n, U(x)\}\big\}$. Thus $U_n\in L^\infty(\mathbb{R}^N;\mathbb{R})$. For $U\in L^\infty_{loc}(\mathbb{R}^N\backslash\mathcal{N};\mathbb{R})$ we consider the approximative equation

$$\frac{d}{dt}u_n(t) = A_n u_n(t) \quad \text{where} \quad A_n = i(\triangle - U_n). \tag{2.6}$$

In this case the operator $-iA_n$ is essentially self-adjoint. Hence the semigroup $\{T_n(t)\}$ generated by $-iA_n$ is the family of solutions to (2.6) and is a group of unitary operators : $\|T_n(t)\varphi\| = \|\varphi\|$ for $t\in\mathbb{R}$ and $\varphi\in L^2(\mathbb{R}^N;\mathbb{C})$.

The main theorem in this paper is the following :

Theorem 2.1. *For any $U\in L^\infty_{loc}(\mathbb{R}^N\backslash\mathcal{N};\mathbb{R})$, there exists a closed extension of $(i\triangle - iU)|_{C_0^\infty(\mathbb{R}^N\backslash\mathcal{N})}$ in $L^2(\mathbb{R}^N;\mathbb{C})$ to $L^2(\mathbb{R}^N;\mathbb{C})$ which generates a unique contraction C_0-semigroup $\{T(t) \;\big|\; t\geq 0\}$ such that*

$$T(t)\varphi = w\text{-}\lim_{n\to\infty} T_n(t)\varphi \quad \textit{for} \quad \textit{all} \quad \varphi\in L^2(\mathbb{R}^N;\mathbb{C}), \tag{2.7}$$

where $T_n(t)\varphi$ is the solution to (2.6) *and w-lim means the weak convergence.*

For the proof of existence of $\{T(t)\}$, we use an abstract theory of semigroups, which will be give in the section 4. The proof of uniqueness is given in the section 6.2.

3. Preliminaries

In this section we begin by introducing some terminology and notation and present those aspects of the basic theory which are required in subsequent sections.

3.1. **Filter.**

Definition 3.1. Given a set E, a partial ordering $\subset$ can be defined on the powerset $\mathcal{P}(E)$ by subset inclusion. Define a filter $\mathcal{F}$ on E as a subset of $\mathcal{P}(E)$ with the following properties:

(i) $\emptyset \notin \mathcal{F}$.(The empty set is not in $\mathcal{F}$.)
(ii) If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \bigcap B \in \mathcal{F}$. ($\mathcal{F}$ is closed under finite meets.)
(iii) If $A \in \mathcal{F}$ and $A \subset B$, then $B \in \mathcal{F}$. (Therefore $E \in \mathcal{F}$.)

Definition 3.2. Let $\mathcal{B}$ be a subset of $\mathcal{P}(E)$. $\mathcal{B}$ is called filter base on E if

(i) The intersection of any two sets of $\mathcal{B}$ contains a set of $\mathcal{B}$,
(ii) $\mathcal{B}$ is non-empty and the empty set is not in $\mathcal{B}$.

Let X be a topological space.

Definition 3.3. $\mathcal{U}(x)$ is called the neighourhood filter at point x for X if $\mathcal{U}(x)$ is the set of all topological neighbourhoods of the point x.

Definition 3.4. To say that filter base $\mathcal{B}$ converges to x, denoted $\mathcal{B} \to x$, means that for every neighourhood U of x, there is a $B \in \mathcal{B}$ such that $B \subset U$. In this case, x is called a limit of $\mathcal{B}$ and $\mathcal{B}$ is called a convergent filter base

Lemma 3.5. *For every neighourhood base $\mathcal{U}(x)$ of x, $\mathcal{U}(x) \to x$.*

Lemma 3.6. *X is a Hausdorff space if every filter base on X has at most one limit.*

Definition 3.7. A filter $\mathcal{F}$ in a topological spacec is called ultra filter if which is maximal, in the sense that every filter containing it coincides with it.

For details concerning the filter, we refer to Bourbak[1].

3.2. **Compact open topology.**

Definition 3.8. A linear topological space X is called a locally convex linear topological space, or, in short, a locally convex space, if its open sets $\ni 0$ contains a convex, balanced and absorbing open set.

Let X and X' be two linear spaces over the complex field $\mathbb{C}$ and a scalar product $\langle x, x' \rangle \in \mathbb{C}$ for $x \in X$ and $x' \in X'$ be defined.

Definition 3.9. Let X be topological vector space. The weak topology on X, denote by $\sigma(X, X')$, is the weakest topology such that all elements of X' remains continuous.

Definition 3.10. The strong topology β of X' is the topology of uniform convergence on every $\sigma(X, X')$-bounded set in X. X'_β denotes $(X')_\beta$.

Definition 3.11. Let τ_0 be the locally convex topology on X, defined by the seminorm system $\mathcal{P} = \{p_\gamma \mid p_\gamma(f) = \sup_{g \in C_\gamma} |\langle f, g \rangle|, C_\gamma \in \mathcal{C}\}$, where $\mathcal{C} = \{C_\gamma\}$ denotes the family of the compact subsets of X'_β. Equivalently, $\mathcal{U}_{\tau_0} = \{U_p\}_{p \in \mathcal{P}}$, where $U_p = \{x \in X \mid p(x) < 1\}$, is a fundamental system of τ_0-neighbourhoods of zero. τ_0 is called the compact open topology.

In the case of Banach space J. Dieudonné has proved the following theorem.

Theorem 3.12 (Dieudonné [2])**.** *The bounded weak* topology in a Banach space is identical with the compact open topology.*

We denote by $X^{'*}$ the space of linear functionals bounded on every bounded set in X'_β.

Proposition 3.13 (Kōmura and Furuya [9, Proposition 1])**.** *Let $\overline{X}_{\tau_0}$ be the completion of the space X_{τ_0}. Then we have:*

$$(X'_\beta)' \subset \overline{X}_{\tau_0} \subset X^{'*}.$$

Lemma 3.14 (Kōmura and Furuya [9, Lemma 5])**.** *Let $x'' \in X''$. $x'' \in \overline{X}_{\tau_0}$ if and only if x'' is $\sigma(X', X)$-continuous on every τ_0-equi-continuous set $\{U_p^o | U_p \in \mathcal{U}_{\tau_0}\}$. Here U_p^o is a polar set of U_p.*

Corollary 3.15. *If X is a Banach space, we have $X'' = \overline{X}_{\tau_0}$.*

4. Existence of weak limit of unitary groups in abstract case

Let $(\mathcal{H}, \|\cdot\|)$, or simply $\mathcal{H}$, denote a Hilbert space with norm $\|\cdot\|$. Instead of the convergence of subsequences we use the convergence of filters. We consider an infinite semi-ordered index set $\mathcal{A} = \{\alpha\}$. We assume that there exists an ultra-filter Φ of infinite subsets of $\mathcal{A}$ satisfying

$$\forall \phi \in \Phi, \ \forall \alpha \in \mathcal{A}, \ \exists \alpha' \in \phi : \alpha' \succ \alpha. \tag{4.1}$$

In the following $\{\Phi\}$ denotes the family of ultra-filters whose element satisfies (4.1).

Remark 4.1. Note that we can use subsequences $\{\alpha_k\}_{k=1}^\infty$ instead of ultra-filters, if $\mathcal{H}$ is separable.

Let a family $\{T_\alpha(t) \mid -\infty < t < \infty\}_{\alpha \in \mathcal{A}}$ of groups of unitary operators in $\mathcal{H}$ be given. Let A_α denote the generator of $\{T_\alpha(t)\}$:

$$\frac{d}{dt} T_\alpha(t)\varphi = A_\alpha T_\alpha(t)\varphi, \quad \varphi \in D(A_\alpha).$$

Definition 4.2. For an ultra-filter Φ satisfying (4.1), the operators $(I - A_\Phi)^{-1}$ and $T_\Phi(t)$ are defined as follows :

$$(I - A_\Phi)^{-1} f = w\text{-}\lim_{\alpha \in \phi \in \Phi} (I - A_\alpha)^{-1} f \quad \text{for} \quad \forall f \in \mathcal{H}, \tag{4.2}$$

$$T_\Phi(t)\varphi = w\text{-}\lim_{\alpha \in \phi \in \Phi} T_\alpha(t)\varphi \quad \text{for} \quad \forall \varphi \in \mathcal{H}. \tag{4.3}$$

In this section we shall show the existence of a semigroup $\{T_\Phi(t)\}$ in (4.3). As is well known, iA_α is self-adjoint : $\langle A_\alpha \varphi, \psi \rangle = -\langle \varphi, A_\alpha \psi \rangle$ for $\varphi, \psi \in D(A_\alpha)$, and the resolvent $(I - A_\alpha)^{-1}$ is a contraction : $\|(I - A_\alpha)^{-1}\| \le 1$. Since a bounded subset of $\mathcal{H}$ is relatively $\sigma(\mathcal{H}, H)$-compact, $\|(I - A_\alpha)^{-1}\varphi\| \le \|\varphi\|$ and $\|T_\alpha(t)\varphi\| = \|\varphi\|$, there exist $w\text{-}\lim_{\alpha \in \phi \in \Phi}(I - A_\alpha)^{-1}\varphi$ and $w\text{-}\lim_{\alpha \in \phi \in \Phi} T_\alpha(t)\varphi$. Hence $(I - A_\Phi)^{-1}$ and $T_\Phi(t)$ are well defined. Note that A_Φ may be multi-valued. The following condition implies A_Φ is single-valued, which will be verified later (See.Theorem 4.7).

Condition 4.3. There exist a dense subspace $\mathcal{H}_0$ of $\mathcal{H}$ satisfying $\mathcal{H}_0 \subset \bigcap_{\alpha\in\phi} D(A_\alpha)$ and a linear operator $A_0 : \mathcal{H}_0 \longrightarrow \mathcal{H}$ such that

$$\forall\psi \in \mathcal{H}_0,\ \exists\alpha(\psi) \in \mathcal{A}\ :\ A_0\psi = A_\alpha\psi \quad \text{for} \quad \forall\alpha \succ \alpha(\psi). \tag{4.4}$$

By definition, (4.2) means

$$\forall f \in \mathcal{H}, \forall\varepsilon > 0, \forall C_\beta \in \mathcal{C}, \exists\phi \in \Phi : \sup_{\alpha\in\phi,\varphi\in C_\beta} |\langle (I - A_\Phi)^{-1}f - (I - A_\alpha)^{-1}f, \varphi\rangle| < \varepsilon.$$

Lemma 4.4. *For a fixed $f \in \mathcal{H}$, put*

$$\varphi_\alpha = (I - A_\alpha)^{-1}f \quad \textit{and} \quad \varphi_\Phi = (I - A_\Phi)^{-1}f. \tag{4.5}$$

Under the condition 4.3, we have

(a) *Let*

$$f = (I - A_0)\varphi \quad \textit{for} \quad \varphi \in \mathcal{H}_0$$

then

$$\varphi = \lim_{\alpha\in\phi\in\Phi} (I - A_\alpha)^{-1}f.$$

(b) *Let*

$$\varphi_\alpha = (I - A_\alpha)^{-1}f \quad \textit{and} \quad \varphi_\Phi = w\text{-}\lim_{\alpha\in\phi\in\Phi} \varphi_\alpha, \tag{4.6}$$

then

$$w\text{ - }\lim_{\alpha\in\phi\in\Phi} A_\alpha\varphi_\alpha = A_\Phi\varphi_\Phi. \tag{4.7}$$

Moreover if A_Φ is single-valued, it follows that

$$\mathcal{H}_0 \subset D(A_\Phi), \quad A_\Phi|_{\mathcal{H}_0} = A_0 \tag{4.8}$$

and

$$\langle A_\Phi\varphi, \psi\rangle = -\langle\varphi, A_0\psi\rangle \quad \textit{for} \quad \forall\varphi \in D(A_\Phi) \quad \textit{and} \quad \forall\psi \in \mathcal{H}_0. \tag{4.9}$$

Proof. Since $f = (I - A_\alpha)\varphi_\alpha$ for any $\alpha \in \phi \in \Phi$ such that $\alpha \succ \alpha(\varphi)$, we have $\varphi_\alpha = (I - A_\alpha)^{-1}f$. This implies **(a)**. By the definition of A_Φ, we have $\varphi_\Phi \in D(A_\Phi)$. Thus (4.8) is verified. (4.5) and (4.6) mean

$$\varphi_\Phi - A_\Phi\varphi_\Phi = f, \quad \varphi_\alpha - A_\alpha\varphi_\alpha = f \quad \text{and} \quad \varphi_\Phi = w\text{-}\lim_{\alpha\in\phi\in\Phi} \varphi_\alpha. \tag{4.10}$$

Therefore (4.7) follows from $A_\alpha\varphi_\alpha = \varphi_\alpha - f \xrightarrow{\tau_0} \varphi_\Phi - f = A_\Phi\varphi_\Phi$ by (4.10). At last we shall show (4.9). Note that $\varphi \in D(A_\Phi)$ means

$$\exists f \in H : \varphi_\alpha - A_\alpha\varphi_\alpha = f, \quad \varphi = w\text{-}\lim_{\alpha\in\phi\in\Phi} \varphi_\alpha.$$

Since $A_\alpha\psi = A_0\psi$ for $\alpha \succ \alpha(\psi)$ such that $\alpha \in \phi \in \Phi$, we have

$$\langle A_\alpha\varphi_\alpha, \psi\rangle = -\langle\varphi_\alpha, A_\alpha\psi\rangle = -\langle\varphi_\alpha, A_0\psi\rangle \longrightarrow -\langle\varphi, A_0\psi\rangle, \quad \forall\psi \in \mathcal{H}_0.$$

□

From Definition 4.2 and Lemma 4.4 we obtain the following proposition:

Proposition 4.5. *Under the condition 4.3, the range of* $(I - A_\Phi)^{-1} : \mathcal{R}((I - A_\Phi)^{-1}) = (I - A_\Phi)^{-1}\mathcal{H}$ *is dense in* $\mathcal{H}$.

We cite Theorem 9 in [9] as Theorem 4.6. Let X be a reflexive Banach space and $\{T_\alpha(t)\}_{\alpha\in\mathcal{A}}$ be a family of contraction C_0-semigroups in X.

Theorem 4.6 (Kōmura and Furuya [9, Theorem 9])**.** *Suppose for some filter* Φ

$$\forall f \in X,\ \exists \varphi_\Phi = w\text{-}\lim_{\alpha\in\phi\in\Phi}(I - A_\alpha)^{-1} f. \tag{4.11}$$

Thus the operator $(I - A_\Phi)^{-1}$ *is defined. If the range* $\mathcal{R}((I - A_\Phi)^{-1})$ *is dense in* X, A_Φ *is a densely defined closed operator and generates a semigroup* $\{T_\Phi(t)\}$:

$$w\text{-}\lim_{\alpha\in\phi\in\Phi} T_\alpha(t)x = T_\Phi(t)x, \quad \text{for} \quad \forall x \in X. \tag{4.12}$$

Moreover, we have $\{T_\Phi(t)\}$ *is a contraction* C_0*-semigroup in* X.

Theorem 4.7. *Under condition 4.3,* A_Φ *is a closed operator and generates a contraction* C_0*-semigroup* $\{T_\Phi(t)\}$.

Proof. Since the range $\mathcal{R}((I - A_\Phi)^{-1})$ is dense in $\mathcal{H}$ by Proposition 4.5, our Theorm follows from Theorem 4.6. □

5. Approximations

5.1. **Approximation by bounded domains.** Let $\mathcal{D} = \{D\}$ be the maximum family such that each element $D \subset \overline{D} \subset \mathbb{R}^N \backslash \mathcal{N}$ is a finite union of connected bounded open sets. For $D \in \mathcal{D}$, $L^2(D;\mathbb{C})$ denotes the L^2-space on D to $\mathbb{C}$ and $H^{(1)}(D;\mathbb{C})$ denotes the Sobolev space of L^2-functions with first distributional derivatives also in L^2 on D to $\mathbb{C}$. $H^{(2)}(D;\mathbb{C})$ denote the Sobolev space of L^2-functions with first and second distributional derivatives also in L^2 on D to $\mathbb{C}$ with norm $\|\cdot\|_{(2)}$. $H_0^{(1)}(D;\mathbb{C})$ is defined as the closure in $H^{(1)}(D;\mathbb{C})$ of $\mathcal{C}_0^\infty(D;\mathbb{C})$. For $U \in L^\infty(D;\mathbb{R})$, the functional $\Psi^D(\varphi) \equiv \frac{1}{2}\|(-\triangle)^{-1/2}\varphi\|^2 + \frac{1}{2}\|\sqrt{U+C}\varphi\|^2$ is lower semicontinuous and convex, where $C = \max\{0, -\text{ess inf } U\}$. The domain of Ψ^D is $H^{(1)}(D;\mathbb{C})$.

Definition 5.1. We denote by Ψ_0^D if the domain of Ψ^D is restricted to the closure of $\mathcal{C}_0^\infty(D;\mathbb{C})$: $D(\Psi_0^D) = H_0^{(1)}(D;\mathbb{C})$.

Definition 5.2. Let $\Psi : \mathcal{H} \to]-\infty, +\infty]$ be a propery convex function. The subdifferential of Ψ is the (possibly multivalued) operator $\partial\Psi : \mathcal{H} \to \mathcal{H}$ defined by

$$\partial\Psi(x) = \{w \in \mathcal{H}; \Psi(x) - \Psi(v) \le (w, x - v),\ \forall v \in \mathcal{H}\}.$$

Since

$$-\partial\Psi_0^D = \triangle - U - C \quad \text{and} \quad D(\partial\Psi_0^D) = H_0^{(1)}(D;\mathbb{C}) \bigcap H^{(2)}(D;\mathbb{C}),$$

the equation in $L^2(D;\mathbb{C})$ with Dirichlet condition is written as

$$\frac{d}{dt}u_D(t) = -i\partial\Psi_0^D(u_D(t)) \ \big(= i(\triangle - U - C)u_D(t)\big) \quad \text{and} \quad u_D(0) = \varphi. \tag{5.1}$$

If the boundary ∂D of D is smooth, the normal derivative ∂_n is defined on ∂D, and we have

$$-\partial\Psi^D = \triangle - U - C \quad \text{where} \quad D(\partial\Psi^D) = \{\varphi \in H^{(2)}(D;\mathbb{C}) \ \big|\ \partial_n\varphi|_{\partial D} = 0\}.$$

Hence the equation in $L^2(D;\mathbb{C})$ with (generalized) Neumann condition is written as

$$(5.2)\qquad \frac{d}{dt}u_D(t) = -i\partial\Psi^D(u_D(t)) \quad \text{and} \quad u_D(0) = \varphi.$$

The semigroup $\{T_D(t)\}, T_D(t)\varphi = e^{-iCt}u_D(t)$, of solution family to (5.1) or (5.2) is a group of unitary operators, respectively. We define an order in $\mathcal{D}$ as follows :

$$(5.3)\qquad D_\alpha, D_\beta \in \mathcal{D}: \quad D_\alpha \prec D_\beta \Longleftrightarrow \overline{D_\alpha} \subset D_\beta.$$

We consider an ultra-filter $\Phi = \{\phi\}$ whose element ϕ consists of infinite subsets of $\mathcal{D}$ satisfying (4.1) for $\mathcal{A} = \mathcal{D}$ in the next section :

$$\lim_{D\in\phi\in\Phi} D = \bigcup_{D\in\phi\in\Phi} D = \mathbb{R}^N\backslash\mathcal{N}.$$

Proposition 5.3. *We define an operator $T_\Phi(t)$ for $T_D(t)$ associated with* (5.1) *or* (5.2) *by*

$$(5.4)\quad T_\Phi(t)\varphi = w\text{-}\lim_{D\in\phi\in\Phi} T_D(t)\varphi\Big(= \tau_0\text{-}\lim_{D\in\phi\in\Phi} T_D(t)\varphi\Big) \quad \textit{for} \quad \forall\varphi \in L^2(\mathbb{R}^N;\mathbb{C}).$$

Then $\{T_\Phi(t)\}$ is a contraction semigroup.

The proof is given in the next section.

5.2. **More general approximation** $\{U_{m,n}\}_{m,n\in\mathbb{N}}$**.** For an approximation of U more general than (1.2) :

$$(5.5)\quad U_{m,n}(x) = \begin{cases} m & \text{if} \quad m < U(x), \\ U(x) & \text{if} \quad -n \le U(x) \le m, \\ -n & \text{if} \quad U(x) < -n, \end{cases} \qquad \text{for} \quad m, n \in \mathbb{N}.$$

the result is almost the same as the case of approximation $\{U_n\}_{n\in\mathbb{N}}$.

Theorem 5.4 (Furuya [4, Theorem 9])**.** *For any $U \in L^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{R})$, there exists a closed extension of $i(\triangle - U)|_{C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})}$ in $L^2(\mathbb{R}^N;\mathbb{C})$ to $L^2(\mathbb{R}^N;\mathbb{C})$ which generates a contraction C_0-semigroup $\{T(t)\,|\,t \ge 0\}$ such that*

$$(5.6)\quad T(t)\varphi = w\ \text{-}\lim_{n\to\infty} T_{m,n}(t)\varphi\Big(= \tau_0\text{-}\lim_{m,n\to\infty} T_{m,n}(t)\varphi\Big) \quad \textit{for} \quad \forall\varphi \in L^2(\mathbb{R}^N;\mathbb{C}),$$

where $T_{m,n}(t)\varphi$ is the solution to

$$(5.7)\qquad \frac{d}{dt}u_{m,n}(t) = A_{m,n}u_{m,n}(t),\ u_{m,n}(0) = \varphi \quad \textit{where} \quad A_{m,n} = i(\triangle - U_{m,n})$$

and w-lim means the weak convergence.

In this paper, for simplicity we discuss the case of $\{U_n\}_{n\in\mathbb{N}}$.

Example 5.5. We shall show (2.7) and (5.4) give different solutions in a certain case. We consider the one-dimensional case : $N = 1$. Let $U(x) = |x|^{-1/3}$. In this case, $\triangle - U$ is essentially self-adjoint, hence, the resolvent of A_n converges to that of the minimal closed extension $\bar{A}$ of $(i\triangle - iU)|_{C_0^\infty(\mathbb{R}\backslash\mathcal{N})}$ and $\bar{A} = i(\triangle - U)$ generates a unitary group $\{T(t)\}$. Fix an initial data φ, with supp $\varphi \subset (0,\infty)$. We have $T(t)\varphi|_{(-\infty,0]} \ne 0$ for $t > 0$ in general.

On the other hand, $\mathcal{N} = \{0\}$. We may assume Φ consists of subsets of $\mathcal{D} = \{D(r) = (-\infty, -r) \bigcup (r, \infty) \bigm| r > 0\}$. The solution $u_{D(r)}$ to (5.1) satisfies Dirichlet condition : $u_{D(r)}(t, r) = 0$ for all $t > 0$. Moreover $u_{D(r)}(t, x) = 0$ for all $x \in (-\infty, -r)$. $u_{D(r)}(t)$ converges to the solution $u_\Phi(t)$ satisfying $u_\Phi(t, 0) = 0$ for all $x \in (-\infty, 0]$. Thus we have

$$T(t)\varphi \neq \tau_0\text{-}\lim_{D\in\phi\in\Phi} T_D(t)\varphi.$$

6. Weak limit of unitary groups

6.1. **Existence in L^2 case.** In this section we shall show the existence of a semi-group $\{T_t^\Phi\}$ in (4.3).

Theorem 6.1. *For each approximation $\{A_n\}$, $\{A_D\}$ or $\{A_{m,n}\}$, the limit $T_\Phi(t) = \lim_\Phi exp(tA_n)$, $\lim_\Phi exp(tA_D)$ or $\lim_\Phi exp(tA_{m,n})$ exists and $\{T_\Phi(t)\}$ is a contraction C_0-semigroup. Here $A_D = \partial\Phi_0^D$ in (5.1) or $\partial\Phi^D$ in (5.2).*

Lemma 6.2. *Condition 1 is satisfied for $\mathcal{H}_0 = C_0^\infty(\mathbb{R}^N \backslash \mathcal{N}; \mathbb{C})$.*

Proof. We have for any $\psi \in C_0^\infty(\mathbb{R}^N \backslash \mathcal{N})$, there exists $D \in \mathcal{D}$ and B_m such that supp $\psi \subset D \subset B_m$, where B_m is defined in (2.4), and if supp $\psi \subset D \subset B_m$ then we have $U_n\psi = U_m\psi$ and $A_n\psi = A_m\psi$ for all $n > m$. Thus for the case of $\{A_n\}$ Condition 1 is satisfied for $\alpha(\psi) = m$. For the case of $\{A_D\}$ and $\{A_{m,n}\}$, the proof is the same. □

We consider an ultra-filter $\mathcal{D}_1 = \{\delta\}$ by the order (5.3) whose element ϕ consists of infinite subsets of $\mathcal{D}$. $\mathcal{D}_1$ satisfies (4.1) and

$$\lim_{D\in\delta\in\mathcal{D}_1} D = \bigcup_{D\in\delta\in\mathcal{D}_1} D = \mathbb{R}^N \backslash \mathcal{N}. \tag{6.1}$$

Lemma 6.3. *Let $\varphi_\Phi = w\text{-}\lim_{m\in\phi_1\in\Phi}(I - A_m)^{-1}f$ for $f \in L^2(\mathbb{R}^N; \mathbb{C})$. Then on any fixed $D \in \mathcal{D}$, the filter $\{\varphi_m \bigm| \varphi_m = (I - A_m)^{-1}f, m \in \phi \in \Phi\}$ strongly converges to φ_Φ on D, that is,*

$$\forall D \in \mathcal{D},\ \forall\varepsilon > 0,\ \exists\phi \in \Phi : \left\|(\varphi_\Phi - \varphi_m)\big|_D\right\| < \varepsilon \quad for\ all \quad m \in \phi, \tag{6.2}$$

and the filter $\{\varphi_\Phi|_D \bigm| D \in \mathcal{D}_1\}$ strongly converges to φ_Φ, that is,

$$\forall\varepsilon > 0,\ \exists D \in \mathcal{D} \quad such\ that \quad \left\|\varphi_\Phi - \varphi_\Phi\big|_D\right\| < \varepsilon. \tag{6.3}$$

Proof. (6.3) is trivial. We shall show (6.2). By (2.5) we have $\exists B_l : D \subset B_l$. (For the definition of B_n, see (2.4)). Note that for $l < m$ if $D \subset B_l$ and $D \subset B_m$ then

$$\big((I - A_m)\varphi\big)(x) = \big((I - A_l)\varphi\big)(x) \quad \text{for} \quad \forall x \in D. \tag{6.4}$$

In fact, for $\varphi \in D(A_l)$, the relation supp $\varphi \subset D \subset B_l \subset B_m$ implies $U_l = U_m$ on D, that is, $A_l = A_m$ on D. □

Hence for $m > l$, we have

$$\int_D |(I - A_l)\varphi_m(x)|^2 dx = \int_D |(I - A_m)\varphi_m(x)|^2 dx \leq \int_{\mathbb{R}^N} |(I - A_m)\varphi_m(x)|^2 dx = \|f\|^2.$$

That is, $\{\varphi_m|_D \mid m \in \phi\}$ is contained in a bounded subset of $H^{(2)}(\mathbb{R}^N;\mathbb{C})|_D \equiv \{\varphi|_D \mid \varphi \in H^{(2)}(\mathbb{R}^N;\mathbb{C})\}$, since two norms $\|\cdot\|_{(2)}$ and $\|\cdot\|_l = \|(I-A_l)^{-1}\cdot\|$ are equivalent on $H^{(2)}(\mathbb{R}^N;\mathbb{C})|_D$. A closed bounded subset of $H^{(2)}(\mathbb{R}^N;\mathbb{C})|_D$ is a compact subset of $L^2(D;\mathbb{C})$, since D is bounded in $\mathbb{R}^N$. Since the filter $\{\varphi_m|_D \mid m \in \phi \in \Phi\}$ is weakly convergent in $L^2(D;\mathbb{C})$, it is strongly convergent in $L^2(D;\mathbb{C})$.

Proposition 6.4. *Let $A = A_\Phi$ with domain $D(A) = H^{(2)}_{loc}(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$. Then A is a closed operator from $L^2_{loc}(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$ to $L^2_{loc}(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$.*

Proof. Proof follows from Lemma 6.3. □

Proof of Theorem 6.1. From Lemma 6.2, Proposition 6.4 and Theorem 4.7 we obtain Thorem 6.1. □

6.2. **Uniqueness of $T_\Phi(t)$.** In this subsection we shall show the uniqueness of $T_\Phi(t)$ in (4.3) for the approximative equation (2.6).

Let $\Phi = \{\phi = \{n_k\} \mid n_k \in \mathbb{N}\}$ be an ultra-filter of subsequences of natural numbers.

Theorem 6.5. *$T_\Phi(t)$, defined in (4.3) , is independ of the choice of Φ for the approximation $U_n(x) = \min\{n, \max\{-n, U(x)\}\}$.*

Proof. From Theorem 4.7 it is sufficient to show the generator A_Φ is independent of Φ. We make the following Assumption:

Assumption 6.6. $A_{\Phi_1} \neq A_{\Phi_2}$ for two ultra-filters Φ_1 and Φ_2 with $\Phi_1 \neq \Phi_2$.

In the following we shall show that Assumption 6.6 implies a contradiction. We shall begin with several Lemmas.

Lemma 6.7. *Suppose $T_{\Phi_1}(t) \neq T_{\Phi_2}(t)$. There exists $\varphi_0 \in C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$ which satisfies*

$$\exists t_1 > 0, \exists c_0 > 0 \quad \textit{such that} \quad \frac{d}{dt}\|T_{\Phi_1}(t)\varphi_0 - T_{\Phi_2}(t)\varphi_0\|\Big|_{t=t_1} \geq c_0. \tag{6.5}$$

Proof. If $T_{\Phi_1}(t)\varphi = T_{\Phi_2}(t)\varphi$ for $\forall\varphi \in C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$, we have $T_{\Phi_1}(t) = T_{\Phi_2}(t)$, since $C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$ is dense in $L^2(\mathbb{R}^N;\mathbb{C})$. Thus there exisits $\varphi_0 \in C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$ such that $T_{\Phi_1}(t)\varphi_0 \neq T_{\Phi_2}(t)\varphi_0$ for some $t > 0$. Since $\varphi_0 \in C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C}) \subset D(A_{\Phi_1}) \bigcap D(A_{\Phi_2})$, $\frac{d}{dt}T_{\Phi_1}(t)\varphi_0$ and $\frac{d}{dt}T_{\Phi_2}(t)\varphi_0$ are continuous in t, and hence $\frac{d}{dt}\|T_{\Phi_1}(t)\varphi_0 - T_{\Phi_2}(t)\varphi_0\|$ is continuous in t. Thus there exists $t_0 > 0$ such that

$$0 < \|T_{\Phi_1}(t_0)\varphi_0 - T_{\Phi_2}(t_0)\varphi_0\| = \int_0^{t_0} \frac{d}{dt}\|T_{\Phi_1}(t)\varphi_0 - T_{\Phi_2}(t)\varphi_0\|dt. \tag{6.6}$$

If $\frac{d}{dt}\|T_{\Phi_1}(t)\varphi_0 - T_{\Phi_2}(t)\varphi_0\| \leq 0$ for $\forall t \in [0, t_0]$, we have $\|T_{\Phi_1}(t_0)\varphi_0 - T_{\Phi_2}(t_0)\varphi_0\| \leq 0$. This is cotradiction to (6.6) and (6.5) is verified. □

Put $\varphi_1 = T_{\Phi_1}(t_1)\varphi_0$ and $\varphi_2 = T_{\Phi_2}(t_1)\varphi_0$. (6.5) means

$$\frac{d}{dt}\|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_2}(t)\varphi_2\|\big|_{t=0} \geq c_0 > 0. \tag{6.7}$$

Note that $\varphi_1 \in D(A_{\Phi_1})$ and $\varphi_2 \in D(A_{\Phi_2})$.

Case 1. In the case that $\varphi_2 \in D(A_{\Phi_1})$. This means $\frac{d^+}{dt} T_{\Phi_1}(t)\varphi_2\big|_{t=0}$ exists, where $\frac{d^+}{dt}$ denotes the right derivative:

$$\frac{d^+}{dt} f(t) = \lim_{h\downarrow 0} \frac{(f(t+h) - f(t))}{h}.$$

We have

$$\frac{d^+}{dt} \|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_1}(t)\varphi_2\|\Big|_{t=0} \le 0, \tag{6.8}$$

or equivalently,

$$\begin{aligned} Re\frac{d^+}{dt} \langle T_{\Phi_1}(t)\varphi_2 - T_{\Phi_1}(t)\varphi_1, \varphi_2 - \varphi_1\rangle\Big|_{t=0} \\ = \frac{1}{2}\frac{d^+}{dt} \|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_1}(t)\varphi_2\|^2\Big|_{t=0} \le 0, \end{aligned} \tag{6.9}$$

since $T_{\Phi_1}(t)$ is a contraction. In fact,

$$\|T_{\Phi_1}(h)\left(T_{\Phi_1}(t)\varphi_1 - T_{\Phi_1}(t)\varphi_2\right)\| \le \|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_1}(t)\varphi_2\| \quad \text{for} \quad \forall h > 0,$$

implies

$$\frac{1}{h}\left(\|T_{\Phi_1}(t+h)\varphi_1 - T_{\Phi_1}(t+h)\varphi_2\| - \|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_1}(t)\varphi_2\|\right) \le 0 \quad \text{for} \quad \forall h > 0.$$

From this the relation (6.12) follows. Note that the left hand of (6.8) exists since $\frac{d^+}{dt} T_{\Phi_1}(t)\varphi_1\big|_{t=0}$ and $\frac{d^+}{dt} T_{\Phi_1}(t)\varphi_2\big|_{t=0}$ exist for $\varphi_1, \varphi_2 \in D(A_{\Phi_1})$. We have

$$\begin{aligned} \|T_{\Phi_1}(h)\varphi_2 - T_{\Phi_2}(h)\varphi_2\| - \|\varphi_2 - \varphi_2\| + \|T_{\Phi_1}(h)\varphi_1 - T_{\Phi_1}(h)\varphi_2\| - \|\varphi_1 - \varphi_2\| \\ \ge \|T_{\Phi_1}(h)\varphi_1 - T_{\Phi_2}(h)\varphi_2\| - \|\varphi_1 - \varphi_2\|. \end{aligned}$$

Hence

$$\begin{aligned} \frac{d^+}{dt} \|T_{\Phi_1}(t)\varphi_2 - T_{\Phi_2}(t)\varphi_2\|\big|_{t=0} + \frac{d^+}{dt} \|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_1}(t)\varphi_2\|\big|_{t=0} \\ \ge \frac{d^+}{dt} \|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_2}(t)\varphi_2\|\big|_{t=0}. \end{aligned}$$

We have by (6.7) and (6.8)

$$\begin{aligned} &\frac{d^+}{dt} \|T_{\Phi_1}(t)\varphi_2 - T_{\Phi_2}(t)\varphi_2\|\big|_{t=0} \\ &\qquad \ge \frac{d^+}{dt} \|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_2}(t)\varphi_2\|\big|_{t=0} - \frac{d^+}{dt} \|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_1}(t)\varphi_2\|\big|_{t=0} > 0. \end{aligned}$$

Hence

$$\begin{aligned} Re\langle (A_{\Phi_2} - A_{\Phi_1})\varphi_2, \varphi_2\rangle &= Re\frac{d^+}{dt} \langle (T_{\Phi_2}(t) - T_{\Phi_1}(t))\varphi_2, \varphi_2\rangle\Big|_{t=0} \\ &= \frac{1}{2}\frac{d^+}{dt} \|(T_{\Phi_2}(t) - T_{\Phi_1}(t))\varphi_2\|^2\Big|_{t=0} > 0. \end{aligned} \tag{6.10}$$

Thus we have $A_{\Phi_2}\varphi_2 \neq A_{\Phi_1}\varphi_2$. Since $C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$ is dense in $L^2(\mathbb{R}^N;\mathbb{C})$, there exists $\psi \in C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$ such that

$$\langle (A_{\Phi_2} - A_{\Phi_1})\varphi_2, \psi\rangle \neq 0.$$

Nevertheless, from (6.11) of lemma 6.8 we obtain that

$$\langle (A_{\Phi_2} - A_{\Phi_1})\varphi_2, \psi\rangle = \langle \varphi_2, ({}^tA_{\Phi_2} - {}^tA_{\Phi_1})\psi\rangle = -\langle \varphi_2, (A_0 - A_0)\psi\rangle = 0.$$

This is a contradiction.

Lemma 6.8. *Let* $A_0 = A|_{C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})}$ *(see Condition 1). We have*

$$A_0 = -{}^tA_{\Phi_1}|_{C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})} = -{}^tA_{\Phi_2}|_{C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})}. \tag{6.11}$$

Proof. Proof follows from (4.9) by Lemma 6.2. □

Corollary 6.9. *Let* $f \in L^2(\mathbb{R}^N;\mathbb{C})$ *and* $\psi \in C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$*. Then* $\langle T_{\Phi_2}(t)f, \psi\rangle$ *is differentiable in* $t \geq 0$ *:*

$$\frac{d}{dt}\langle T_{\Phi_2}(t)f, \psi\rangle = \langle T_{\Phi_2}(t)A_{\Phi_2}f, \psi\rangle = \langle f, {}^t(T_{\Phi_2}(t){}^tA_{\Phi_2})\psi\rangle = -\langle f, {}^tT_{\Phi_2}(t)A_0\psi\rangle.$$

Case 2. In the case that $\varphi_2 \notin D(A_{\Phi_1})$.

This means

$$\|A_{\Phi_1}\varphi_2\|^2 = \lim_{D\in\delta\in\mathcal{D}_1}\int_D |A_{\Phi_1}\varphi_2|^2 dx = \infty, \tag{6.12}$$

since $\varphi_2 \in D(A_{\Phi_2}) \subset H_{loc}^{(2)}(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$ and $A_{\Phi_1}\varphi_2 = A_0\varphi_2 \in L_{loc}^2(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$ by Proposition 6.4.

Lemma 6.10. *There exists* $\delta > 0$ *such that*

$$0 < Re\frac{d}{dt}\langle T_{\Phi_2}(t)\varphi_2 - T_{\Phi_1}(t)\varphi_1, \psi\rangle\big|_{t=0}, \tag{6.13}$$

if $\|\varphi_2 - \varphi_1 - \psi\| < \delta$ *and* $\psi \in C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$.

Proof. Let δ satisfy

$$0 < \delta < \frac{c_0\|\varphi_2 - \varphi_1\|}{2\|A_{\Phi_2}\varphi_2 - A_{\Phi_1}\varphi_1\|}.$$

From Lemma 6.7 we have

$$\begin{aligned} c_0\|\varphi_2 - \varphi_1\| &\leq \frac{1}{2}\frac{d}{dt}\|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_2}(t)\varphi_2\|^2\big|_{t=0} \\ &= Re\frac{d}{dt}\langle T_{\Phi_2}(t)\varphi_2 - T_{\Phi_1}(t)\varphi_1, \varphi_2 - \varphi_1\rangle\big|_{t=0} \\ &\leq Re\langle A_{\Phi_2}\varphi_2 - A_{\Phi_1}\varphi_1, \psi\rangle + |Re\langle A_{\Phi_2}\varphi_2 - A_{\Phi_1}\varphi_1, \varphi_2 - \varphi_1 - \psi\rangle| \\ &\leq Re\langle A_{\Phi_2}\varphi_2 - A_{\Phi_1}\varphi_1, \psi\rangle + \|A_{\Phi_2}\varphi_2 - A_{\Phi_1}\varphi_1\| \cdot \|\varphi_2 - \varphi_1 - \psi\| \\ &< Re\langle A_{\Phi_2}\varphi_2 - A_{\Phi_1}\varphi_1, \psi\rangle + \frac{1}{2}c_0\|\varphi_2 - \varphi_1\|, \quad if \quad \|\varphi_2 - \varphi_1 - \psi\| < \delta. \end{aligned}$$

Thus we obtain that

$$0 < \frac{1}{2}c_0\|\varphi_2 - \varphi_1\| < Re\langle A_{\Phi_2}\varphi_2 - A_{\Phi_1}\varphi_1, \psi\rangle.$$

□

Lemma 6.11. *Let δ be in Lemma 6.10. Then there exists $\psi_1 \in D(A_{\Phi_1})$ with $\|\varphi_2 - \varphi_1 - \psi_1\| < \delta$, such that*

$$Re\frac{d}{dt}\langle T_{\Phi_1}(t)\varphi_2 - T_{\Phi_1}(t)\varphi_1, \psi_1\rangle\big|_{t=0} = Re\langle A_{\Phi_1}\varphi_2 - A_{\Phi_1}\varphi_1, \psi_1\rangle \leq 0, \tag{6.14}$$

where

$$\frac{d}{dt}\langle T_{\Phi_1}(t)\varphi_2 - T_{\Phi_1}(t)\varphi_1, \psi_1\rangle = \frac{d}{dt}\langle \varphi_2 - \varphi_1, {}^tT_{\Phi_1}(t)\psi_1\rangle \left(= \langle \varphi_2 - \varphi_1, \frac{d}{dt}{}^tT_{\Phi_1}(t)\psi_1\rangle \right).$$

Proof. We recall $\|A_{\Phi_1}\varphi_2\| = \infty$ (see (6.12)). That is, for any $L > 0$ and δ in Lemma 6.10, there exists $\psi_\varepsilon \in C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$ such that $|\langle A_{\Phi_1}\varphi_2 - A_{\Phi_1}\varphi_1, \psi_\varepsilon\rangle| > L$ and $\|\psi_\varepsilon\| < \delta/2$. Therfore $\langle A_{\Phi_1}\varphi_2 - A_{\Phi_1}\varphi_1, e^{i\theta}\psi_\varepsilon\rangle < -L$ for some real θ.
For $\psi_0 \in C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$ satisfying $\|\varphi_2 - \varphi_1 - \psi_0\| < \delta/2$, put

$$L := |\langle A_{\Phi_1}(\varphi_2 - \varphi_1), \psi_0\rangle|.$$

For $\psi_1 = \psi_0 + e^{i\theta}\psi_\varepsilon$ we have $\|\varphi_2 - \varphi_1 - \psi_1\| < \delta$. Hence by Lemma 6.10

$$\begin{aligned} Re\langle A_{\Phi_1}\varphi_2 - A_{\Phi_1}\varphi_1, \psi_1\rangle &= Re\langle A_{\Phi_1}\varphi_2 - A_{\Phi_1}\varphi_1, \psi_0 + e^{i\theta}\psi_\varepsilon\rangle \\ &\leq |\langle A_{\Phi_1}\varphi_2 - A_{\Phi_1}\varphi_1, \psi_0\rangle| + \langle A_{\Phi_1}\varphi_2 - A_{\Phi_1}\varphi_1, e^{i\theta}\psi_\varepsilon\rangle \\ &\leq L - L = 0. \end{aligned}$$

□

Lemma 6.12. *For $\psi_1 \in C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$ in Lemma 6.11 we have*

$$Re\frac{d}{dt}\langle (T_{\Phi_2}(t) - T_{\Phi_1}(t))\varphi_2, \psi_1\rangle\big|_{t=0} > 0, \tag{6.15}$$

where $\frac{d}{dt}\langle (T_{\Phi_2}(t) - T_{\Phi_1}(t))\,\varphi_2, \psi_1\rangle = \frac{d}{dt}\langle \varphi_2, \left({}^tT_{\Phi_2}(t) - {}^tT_{\Phi_1}(t)\right)\psi_1\rangle$.

Proof. By using (6.13) and (6.14) we get

$$\begin{aligned} 0 &< Re\frac{d}{dt}\langle T_{\Phi_2}(t)\varphi_2 - T_{\Phi_1}(t)\varphi_1, \psi_1\rangle\big|_{t=0} + Re\frac{d}{dt}\langle T_{\Phi_1}(t)\varphi_1 - T_{\Phi_1}(t)\varphi_2, \psi_1\rangle\big|_{t=0} \\ &= Re\frac{d}{dt}\langle T_{\Phi_2}(t)\varphi_2 - T_{\Phi_1}(t)\varphi_2, \psi_1\rangle\big|_{t=0}. \end{aligned}$$

□

On the other hand, from Lemma 6.8 and Corollary 6.9 it follows that

$$\begin{aligned} Re\frac{d}{dt}\langle T_{\Phi_2}(t)\varphi_2 - T_{\Phi_1}(t)\varphi_2, \psi_1\rangle\big|_{t=0} &= Re\langle \varphi_2, ({}^tA_{\Phi_2} - {}^tA_{\Phi_1})\psi_1\rangle \\ &= Re\langle \varphi_2, (A_0 - A_0)\psi_1\rangle = 0. \end{aligned}$$

This is a contradiction to (6.15). Thus in both cases we get a contradiction and the proof of Theorem 6.5 is complete. □

Proof of Theorem 2.1. Now Theorem 2.1 follows from Theorem 4.7 and Theorem 6.5, since the weak topology is equal to τ_0 on a bounded set of L^2. □

7. Previous result

Note that for $U \in C(\mathbb{R}^N \backslash \mathcal{N}, \mathbb{R}^+)$, $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$, the functional $\Psi(\varphi) \equiv \frac{1}{2}\|\nabla\varphi\|^2 + \frac{1}{2}\|\sqrt{U}\varphi\|^2$ is lower semicontinuous and convex. The subdifferential $\partial\Psi$ of Ψ is a self-adjoint positive operator. (In this case $\partial\Psi$ is single-valued.) Our previous result is that:

Theorem 7.1 (Furuya [4, Theorem 6]). *If a function $U \in C(\mathbb{R}^N \backslash \mathcal{N}; \mathbb{R}^+)$ then the Schrödinger equation*

$$\frac{d}{dt}u(t) = -i\partial\Psi(u(t)) \ \big(= i(\triangle - U)u(t)\big) \quad \textit{and} \quad u(0) = \varphi \in D(\partial\Psi)$$

has a unique solution. The operator family $\{T(t)\}$ defined by $T(t)\varphi = u(t)$ is uniquely extended to a group of unitary operators $\{T(t)\}$.

In this case $\{T_n(t)\varphi\}$ defined by (2.6) weakly converges to $\{T(t)\varphi\}$ and

$$\|\varphi\| = \lim_{n\to\infty} \|T_n(t)\varphi\| \geq \|\lim_{n\to\infty} T_n(t)\varphi\| = \|T(t)\varphi\| = \|\varphi\|,$$

hence $\lim \|T(t)\varphi\| = \|\lim_{n\to\infty} T_n(t)\varphi\|$ and $\{T_n(t)\varphi\}$ strongly converges to $\{T(t)\varphi\}$. Note that $D(\Psi) \equiv \{\varphi \in L^2(\mathbb{R}^N; \mathbb{C}) \bigm| \Psi(\varphi) < \infty\}$, effective domain of Ψ, is a Hilbert space with respect to the norm $\|\varphi\| = \sqrt{\Psi(\varphi)}$.

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Kiyoko Furuya
Graduate School of Humanities and Science, Ochanomizu University, 2-1-1 Ōtsuka, bunkyou-ku, Tokyo, Japan
E-mail address: furuya.kiyoko@ocha.ac.jp