Journal of Nonlinear and Convex Analysis
Volume 15, Number 4, 2014, 819–829

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

APPROXIMATING FIXED POINTS IN THE HILBERT BALL

EVA KOPECKÁ AND SIMEON REICH

Abstract. We establish a strong convergence theorem for an iterative algorithm that approximates fixed points of those self-mappings of the Hilbert ball which are nonexpansive with respect to the hyperbolic metric. We also prove an analogous strong convergence theorem regarding the behavior of approximating curves.

1. Introduction

Our main purpose in this note is to establish a strong convergence theorem (see Theorem 3.1 below) for an iterative procedure which approximates fixed points of those self-mappings of the Hilbert ball $\mathbb{B}$ which are nonexpansive (that is, 1-Lipschitz) with respect to the hyperbolic metric ρ. This result improves upon a theorem [7, Theorem 4.1] we established a few years ago. It provides, in particular, positive answers to two questions raised on page 366 of [7]. We also prove an analogous strong convergence theorem (see Theorem 4.1 below) regarding the behavior of approximating curves. This result improves upon [8, Theorem 3.12] and solves an open problem that was left open on page 3193 of [8]. Our theorems may be considered Hilbert ball analogues of the Hilbert space theorems in [11] and [18]. Other such analogues are presented in [10].

In the next section we recall several relevant properties of the hyperbolic metric $\rho : \mathbb{B} \times \mathbb{B} \to \mathbb{R}^+$ and of ρ-nonexpansive (in particular, holomorphic) self-mappings of $\mathbb{B}$. In the third section we state and prove our main result. In the fourth and last section we establish our strong convergence theorem for approximating curves.

2. The hyperbolic metric

In this section we collect several pertinent properties of the hyperbolic metric ρ on the Hilbert ball $\mathbb{B}$ [4]. For more recent results concerning $(\mathbb{B}, \rho)$ and ρ-nonexpansive mappings see, for example, [1, Section 9], [5], [6, Theorem 2.10], [7], [10], [9] and [15].

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $| \cdot |$, and let $\mathbb{B} := \{x \in H : |x| < 1\}$ be its open unit ball. We denote the sets of natural numbers, the real line, the interval $[0, \infty)$ and the complex plane by $\mathbb{N}$,

2010 *Mathematics Subject Classification.* 32F45, 46G20, 46T25, 47H09, 47H10, 47J25, 65J15.

Key words and phrases. Approximating curve, approximating sequence, asymptotic center, fixed point, Hilbert ball, holomorphic mapping, hyperbolic metric, iterative algorithm, Möbius transformation, nearest point projection, nonexpansive mapping.

The first author was supported by Grant FWF P23628-N18. The second author was partially supported by the Israel Science Foundation (Grant 389/12), the Fund for the Promotion of Research at the Technion (Grant 2001893), and by the Technion General Research Fund (Grant 2016723).

$\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{C}$, respectively. The *hyperbolic metric* $\rho : \mathbb{B} \times \mathbb{B} \to \mathbb{R}^+$ [4, page 98] is defined by

$$\rho(x, y) := \operatorname{argtanh}\big(1 - \sigma(x, y)\big)^{\frac{1}{2}}, \tag{2.1}$$

where

$$\sigma(x, y) := \frac{(1 - |x|^2)(1 - |y|^2)}{|1 - \langle x, y\rangle|^2}, \quad x, y \in \mathbb{B}. \tag{2.2}$$

This metric is the infinite-dimensional analogue of the Poincaré metric on the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. We let $B(a, r) := \{x \in \mathbb{B} : \rho(a, x) < r\}$ stand for the ρ-ball of center a and radius r. A subset of $\mathbb{B}$ is called *ρ-bounded* if it is contained in a ρ-ball. We say that a mapping $e : \mathbb{R} \to \mathbb{B}$ is a *metric embedding* of the real line $\mathbb{R}$ into $\mathbb{B}$ if $\rho(e(s), e(t)) = |s - t|$ for all real s and t. The image of $\mathbb{R}$ under a metric embedding is called a *metric line*. The image of a real interval $[a, b] = \{t \in \mathbb{R} : a \le t \le b\}$ under such a mapping is called a *metric segment*. It is known [4, page 102] that for any two distinct points x and y in $\mathbb{B}$, there is a unique metric line (also called a *geodesic*) which passes through x and y. This metric line determines a unique metric segment joining x and y. For each $0 \le t \le 1$, there is a unique point z on this metric segment such that $\rho(x, z) = t\rho(x, y)$ and $\rho(z, y) = (1 - t)\rho(x, y)$. This point will be denoted by $(1 - t)x \oplus ty$.

The following inequality [4, page 104] shows that the metric space $(\mathbb{B}, \rho)$ is *hyperbolic* in the sense of [14].

Lemma 2.1. *For any four points a, b, x and y in $\mathbb{B}$, and any number $t \in [0, 1]$,*

$$\rho((1 - t)a \oplus tx, (1 - t)b \oplus ty) \le (1 - t)\rho(a, b) + t\rho(x, y). \tag{2.3}$$

Next, we mention another useful property of the hyperbolic metric.

Lemma 2.2. *For any three points $u, v, w \in \mathbb{B}$ and any number $0 \le t \le 1$,*

$$\rho^2(tv \oplus (1 - t)w, u) \le t\rho^2(v, u) + (1 - t)\rho^2(w, u) - t(1 - t)\rho^2(v, w). \tag{2.4}$$

This is Lemma 2.3 on page 315 of [17]. It shows that the hyperbolic metric ρ is hyperbolically uniformly convex [14, page 541]. Since it also shows, in particular, that the CN inequality (*courbure négative*) [3, page 63] holds in the Hilbert ball $(\mathbb{B}, \rho)$, we see that $(\mathbb{B}, \rho)$ is a CAT(0) space [2, page 163].

Recall that the Möbius transformations of $\mathbb{B}$ [4, page 98] are biholomorphic mappings $M_a : \mathbb{B} \to \mathbb{B}$ of the form

$$M_a(z) = \big(\sqrt{(1 - |a|^2)}Q_a + P_a\big)m_a(z), \quad z \in \mathbb{B}, \tag{2.5}$$

where $a \in \mathbb{B}$, P_a is the orthogonal projection of H onto the one-dimensional subspace spanned by a, $Q_a = I - P_a$, and $m_a(z) := (z + a)/(1 + \langle z, a\rangle)$. Every Möbius transformation is an automorphism of $\mathbb{B}$ and hence a ρ-isometry. As a matter of fact, any automorphism of $\mathbb{B}$ is of the form $U \circ M_a$ for some unitary operator U on H and a point $a \in \mathbb{B}$ [4, Theorem 14.1].

To each $x \in \mathbb{B}$, we associate a Hilbert space H_x the elements of which are denoted by $\{[x, y] : y \in \mathbb{B}\}$ [16, page 638]. Both the vector space structure and the inner product of H_x are determined by the (surjective) mapping $i : H_x \to H$ defined by

$$i([x,y]) := \big(\rho(x,y)/|M_{-x}(y)|\big) M_{-x}(y) \tag{2.6}$$

when $y \neq x$ and by $i([x,y]) := 0$ when $y = x$. In particular, the inner product in H_x is given by

$$\langle [x,y],[x,z]\rangle = \frac{\rho(x,y)\cdot\rho(x,z)}{|M_{-x}(y)||M_{-x}(z)|}\langle M_{-x}(y), M_{-x}(z)\rangle, \tag{2.7}$$

where $y \neq x$ and $z \neq x$, and the norm of the element $[x,y] \in H_x$ is $\rho(x,y)$, that is, $|[x,y]|_{H_x} = \rho(x,y)$. The spaces H_x and H_y, where $x, y \in \mathbb{B}$, are isometric Hilbert spaces via, for example, the isometry $U_{x,y} : H_x \to H_y$ defined by

$$U_{x,y}[x,z] := [y, M_y(M_{-x}(z))], \quad z \in \mathbb{B}. \tag{2.8}$$

The vector $[x,y] \in H_x$ may be identified with the vector v in the tangent space at x for which $\exp_x(v) = y$, where $\exp_x$ is the exponential map at x.

The following "law of cosines" is Lemma 2.2 on page 638 of [16].

Lemma 2.3. *For any three points $u, v, w \in \mathbb{B}$,*

$$\rho^2(v,w) \geq \rho^2(u,v) + \rho^2(u,w) - 2\,\mathrm{Re}\langle [u,v],[u,w]\rangle. \tag{2.9}$$

Combining Lemmata 2.2 and 2.3, we obtain another useful inequality.

Lemma 2.4. *For any three points $u, v, w \in \mathbb{B}$ and any number $0 \leq t \leq 1$,*

$$\rho^2(tv \oplus (1-t)w, u) \leq t^2\rho^2(v,u) + (1-t)^2\rho^2(w,u) + 2t(1-t)\mathrm{Re}\langle [u,v],[u,w]\rangle. \tag{2.10}$$

We may also rewrite the "law of cosines" as follows.

Lemma 2.5.

$$|[u,v] - [u,w]|_{H_u} \leq \rho(v,w). \tag{2.11}$$

Now let K be a nonempty, ρ-closed and ρ-convex subset of $\mathbb{B}$. We denote by P_K the *nearest point projection* of $\mathbb{B}$ onto K defined by

$$P_K(p) = \{p_0 \in K \ : \ \rho(p,p_0) \leq \rho(p,q) \ \ \forall q \in K\}, \quad p \in \mathbb{B}.$$

Lemma 2.6. *For any point $p \in \mathbb{B}$, $P_K(p)$ is a singleton and the following inequality holds for all $q \in K$:*

$$\mathrm{Re}\langle [P_K(p), p], [P_K(p), q]\rangle \leq 0.$$

Proof. A proof of the existence and uniqueness of the point $P_K(p)$ can be found in [4, page 108].

Next, recall [16, page 642] that a self-mapping T of $\mathbb{B}$ is *firmly nonexpansive* (of the first kind) [4, page 124] if and only if

$$\mathrm{Re}\{\langle [Tx,Ty],[Tx,x]\rangle + \langle [Ty,Tx],[Ty,y]\rangle\} \leq 0$$

for all $x, y \in \mathbb{B}$. Since $P_K : \mathbb{B} \to \mathbb{B}$ is known to be firmly nonexpansive (of the first kind) [4, page 124], we may take $x = p \in \mathbb{B}$ and $y = q \in K$, and obtain that $P_K q = q$ and $\mathrm{Re}\langle [P_K(p), p], [P_K(p), q]\rangle \leq 0$, as claimed. □

Let $\{x_n\}_{n=0}^{\infty}$ be a ρ-bounded sequence in $\mathbb{B}$, and let K be a nonempty, ρ-closed and ρ-convex subset of $\mathbb{B}$. Consider the functional $g : \mathbb{B} \to [0, \infty)$ defined by

$$g(x) = \limsup_{n\to\infty} \rho(x_n, x), \quad x \in \mathbb{B}.$$

A point z in K is said to be an *asymptotic center* of the sequence $\{x_n\}_{n=0}^{\infty}$ with respect to K if $g(z) = \min\{g(x) : x \in K\}$. The minimum of g over K is called the *asymptotic radius* of $\{x_n\}_{n=0}^{\infty}$ with respect to K.

Proposition 2.7 ([4, page 116])**.** *Every ρ-bounded sequence in $(\mathbb{B}, \rho)$ has a unique asymptotic center with respect to any nonempty, ρ-closed and ρ-convex subset of $\mathbb{B}$.*

The asymptotic center of $\{x_n\}_{n=0}^{\infty}$ with respect to K is denoted by $A(K, \{x_n\})$ and its asymptotic radius by $r(K, \{x_n\})$. If $K = \mathbb{B}$ we shall write $A(\{x_n\})$ and $r(\{x_n\})$, respectively.

Lemma 2.8 ([4, page 116])**.** *If $\{x_n\} \subset K$, then $A(\{x_n\}) = A(K, \{x_n\})$.*

Proposition 2.9. [4, page 117] *If a ρ-bounded sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to x, then $x = A(\{x_n\})$.*

We say that a mapping $T : K \to K$ is *ρ-nonexpansive* (that is, 1-Lipschitz) if for any two points $x, y \in K$, the following inequality holds:

$$\rho(Tx, Ty) \le \rho(x, y).$$

It is known that every holomorphic self-mapping of $\mathbb{B}$ is ρ-nonexpansive [4, page 118].

Let $T : K \to K$ be a ρ-nonexpansive mapping. We shall call a sequence $\{y_n\}_{n=0}^{\infty} \subset K$ an *approximating sequence* for T if $\lim_{n\to\infty} \rho(y_n, Ty_n) = 0$.

Theorem 2.10 ([4, page 120])**.** *Let $T : K \to K$ be a ρ-nonexpansive mapping. The following statements are equivalent:*

(a) *T has a fixed point;*
(b) *There exists a point x in K such that the sequence of iterates $\{T^n x\}_{n=0}^{\infty}$ is ρ-bounded;*
(c) *The sequence of iterates $\{T^n x\}_{n=0}^{\infty}$ is ρ-bounded for each x in K;*
(d) *There exists a ρ-bounded approximating sequence for T.*

The asymptotic centers of the sequences in parts (b) and (d) are fixed points of T.

We also need the following result concerning the structure of the fixed point set of a ρ-nonexpansive mapping T.

Theorem 2.11 ([4, page 120])**.** *The fixed point set of a ρ-nonexpansive mapping $T : K \to K$ is ρ-closed and ρ-convex.*

Finally, we recall a lemma [18] (see also [13, Theorem 1]) regarding a certain recursive inequality.

Lemma 2.12. *Let $\{\beta_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two real sequences satisfying the following conditions:*

(i) $\{\beta_n\}_{n=0}^{\infty} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$*;*
(ii) $\limsup_{n\to\infty} b_n \le 0$.

Let $\{a_n\}_{n=0}^{\infty}$ *be a sequence of non-negative real numbers such that*

$$a_{n+1} \leq (1-\beta_n)a_n + \beta_n b_n, \quad n \geq 0.$$

Then $\lim_{n\to\infty} a_n = 0$.

3. An iterative algorithm

In this section we study an iterative procedure for approximating fixed points of ρ-nonexpansive self-mappings of $\mathbb{B}$. In particular, we state and prove our main result (Theorem 3.1 below). It is a strong convergence theorem for this algorithm.

Let a sequence $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1)$ satisfy the following three conditions:

$$\lim_{n\to\infty} \alpha_n = 0; \tag{3.1}$$

$$\sum_{n=0}^{\infty} \alpha_n = \infty; \tag{3.2}$$

$$\text{either} \quad \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \quad \text{or} \quad \lim_{n\to\infty} \alpha_{n-1}/\alpha_n = 1. \tag{3.3}$$

Given a ρ-nonexpansive mapping T of $\mathbb{B}$, a holomorphic mapping $f : \mathbb{B} \to \alpha\mathbb{B}$, where $0 \leq \alpha < 1$, and a point $x_0 \in \mathbb{B}$, we consider the sequence $\{x_n\}_{n=0}^{\infty}$ defined by the recursion

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, \qquad n = 0, 1, 2, \ldots. \tag{3.4}$$

Let $F = F(T) = \{x \in \mathbb{B} : x = Tx\}$ denote the fixed point set of a self-mapping T of $\mathbb{B}$.

Theorem 3.1. *Let* $(\mathbb{B}, \rho)$ *be the Hilbert ball equipped with the hyperbolic metric* $\rho : \mathbb{B} \times \mathbb{B} \to \mathbb{R}^+$ *and let* $T : \mathbb{B} \to \mathbb{B}$ *be a* ρ*-nonexpansive mapping with a fixed point. Let* $f : \mathbb{B} \to \alpha\mathbb{B}$ *be holomorphic, where* $0 \leq \alpha < 1$*, and let the sequence* $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1)$ *satisfy (3.1), (3.2) and (3.3). Then, given an arbitrary point* $x_0 \in \mathbb{B}$*, the sequence* $\{x_n\}_{n=0}^{\infty}$ *generated by algorithm (3.4) converges in norm to* $v \in \mathbb{B}$*, the unique solution of the equation* $z = P_F(f(z))$*, where* $P_F : \mathbb{B} \to F$ *is the nearest point projection of* $\mathbb{B}$ *onto the nonempty fixed point set* $F = F(T)$ *of* T*.*

Proof. The equation $z = P_F(f(z))$ has indeed a unique solution $v \in \mathbb{B}$ because the holomorphic mapping $f : \mathbb{B} \to \alpha\mathbb{B}$ is a strict ρ-contraction [7, Lemma 2.4], the nearest point mapping $P_F : \mathbb{B} \to F$ is ρ-nonexpansive [4, Theorem 19.2] and the metric space $(\mathbb{B}, \rho)$ is complete.

We divide the proof into four steps.

Step 1. The sequences $\{x_n\}_{n=0}^{\infty}$ and $\{Tx_n\}_{n=0}^{\infty}$ are ρ-bounded.

Given a point $y \in \mathbb{B}$ and a subset S of $\mathbb{B}$, let $h(y, S) := \sup\{\rho(y, s) : s \in S\}$. Now fix a point $x \in F$ and set $M := \max\{\rho(x_0, x), h(x, f(\mathbb{B}))\}$. Assuming that $\rho(x_n, x) \leq M$, we have, by Lemma 2.1,

$$\begin{aligned}\rho(x_{n+1}, x) &= \rho(\alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, x)\\ &\leq \alpha_n \rho(f(x_n), x) + (1-\alpha_n)\rho(Tx_n, x)\\ &\leq \alpha_n \rho(f(x_n), x) + (1-\alpha_n)\rho(x_n, x)\\ &\leq \alpha_n M + (1-\alpha_n)M = M.\end{aligned}$$

Hence $\rho(Tx_n, x) \leq \rho(x_n, x) \leq M$ for all $n \in \mathbb{N}$, as claimed.

Step 2. $\lim_{n\to\infty} \rho(x_{n+1}, x_n) = 0$.

For each $n \in \mathbb{N}$, consider the metric segment $[f(x_{n-1}), Tx_{n-1}]$ as the metric embedding of the real interval $[s, t]$ under $e : \mathbb{R} \to \mathbb{B}$. Namely, $e(s) = f(x_{n-1})$ and $e(t) = Tx_{n-1}$.

By Step 1, we know that there is a number $C \in \mathbb{R}^+$ such that $\rho(x_n, x_{n-1}) \leq C$ and $h(Tx_n, f(\mathbb{B})) \leq C$ for all $n \in \mathbb{N}$. We also know [7, Lemma 2.4] that the holomorphic mapping $f : \mathbb{B} \to \alpha\mathbb{B}$ is a strict ρ-contraction with a ρ-Lipschitz constant α. Therefore we have

$$\begin{aligned}\rho(x_{n+1}, x_n) &= \rho(\alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, \alpha_{n-1} f(x_{n-1}) \oplus (1-\alpha_{n-1})Tx_{n-1})\\ &\leq \rho(\alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, \alpha_n f(x_{n-1}) \oplus (1-\alpha_n)Tx_{n-1})\\ &\quad + \rho(\alpha_n f(x_{n-1}) \oplus (1-\alpha_n)Tx_{n-1}, \alpha_{n-1} f(x_{n-1}) \oplus (1-\alpha_{n-1})Tx_{n-1})\\ &\leq \alpha_n \rho(f(x_n), f(x_{n-1})) + (1-\alpha_n)\rho(Tx_n, Tx_{n-1})\\ &\quad + \rho(e(\alpha_n s + (1-\alpha_n)t), e(\alpha_{n-1} s + (1-\alpha_{n-1})t))\\ &\leq \alpha_n \alpha \rho(x_n, x_{n-1}) + (1-\alpha_n)\rho(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}||s - t|\\ &= (1-(1-\alpha)\alpha_n)\rho(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}|\rho(f(x_{n-1}), Tx_{n-1})\\ &\leq (1-(1-\alpha)\alpha_n)\rho(x_n, x_{n-1}) + C|\alpha_n - \alpha_{n-1}|.\end{aligned}$$

Thus, if $\lim_{n\to\infty} \alpha_{n-1}/\alpha_n = 1$, then we can at this point apply Lemma 2.12 to conclude that $\lim_{n\to\infty} \rho(x_{n+1}, x_n) = 0$, as claimed.

When $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, we first fix $1 \leq k \leq n$. We then have

$$\rho(x_{n+1}, x_n) \leq C \prod_{i=k}^{n} (1-(1-\alpha)\alpha_i) + C \sum_{i=k}^{n} |\alpha_i - \alpha_{i-1}|.$$

Since $\prod_{i=k}^{\infty}(1-(1-\alpha)\alpha_i) = 0$ for each $k \geq 1$, letting $n \to \infty$, we get

$$\limsup_{n\to\infty} \rho(x_{n+1}, x_n) \leq C \sum_{i=k}^{\infty} |\alpha_i - \alpha_{i-1}|.$$

Letting $k \to \infty$, we now see that in this case too we have $\lim_{n\to\infty} \rho(x_{n+1}, x_n) = 0$, as claimed.

Step 3. $\limsup_{n\to\infty} \operatorname{Re}\langle [v, f(v)], [v, Tx_n] \rangle \leq 0$.

By Step 1, the real sequence $\{\mathrm{Re}\langle[v, f(v)], [v, Tx_n]\rangle\}_{n=0}^{\infty}$ is bounded; hence its upper limit is finite. Clearly, there is a subsequence $\{z_k\}_{k=1}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$, $z_k := x_{n_k}$, $k \geq 1$, so that, defining

$$c_k := \mathrm{Re}\langle[v, f(v)], [v, Tz_k]\rangle,$$

we have

$$\limsup_{n\to\infty} \mathrm{Re}\langle[v, f(v)], [v, Tx_n]\rangle = \lim_{k\to\infty} c_k.$$

Since $\{x_n\}_{n=0}^{\infty}$ is ρ-bounded by Step 1, we may assume, without any loss of generality, that $z_k \rightharpoonup \bar{x}$ as $k \to \infty$ for some $\bar{x} \in \mathbb{B}$, where $\rightharpoonup$ denotes weak convergence. Then, by Proposition 2.9, $\bar{x}$ is the asymptotic center of $\{z_k\}_{k=1}^{\infty}$. Next we show that $\{z_k\}_{k=1}^{\infty}$ is an approximating sequence for T. Indeed,

$$\rho(z_k, Tz_k) = \rho(x_{n_k}, Tx_{n_k}) \leq \rho(x_{n_k}, x_{n_k+1}) + \rho(x_{n_k+1}, Tx_{n_k}).$$

By Step 2, $\lim_{k\to\infty} \rho(x_{n_k+1}, x_{n_k}) = 0$. Also, using the definition of our algorithm and the properties of metric segments, we see that

$$\begin{aligned}\rho(x_{n_k+1}, Tx_{n_k}) &= \rho(\alpha_{n_k} f(x_{n_k}) \oplus (1-\alpha_{n_k})Tx_{n_k}, Tx_{n_k}) \\ &= \alpha_{n_k}\rho(f(x_{n_k}), Tx_{n_k}),\end{aligned}$$

and since, by Step 1, the real sequence $\{\rho(f(x_{n_k}), Tx_{n_k})\}_{k=1}^{\infty}$ is bounded, it follows that

$$\lim_{k\to\infty} \rho(z_k, Tz_k) = 0. \tag{3.5}$$

Thus $\{z_k\}_{k=1}^{\infty}$ is indeed an approximating sequence for T and applying Theorem 2.10, we conclude that its asymptotic center $\bar{x}$ is a fixed point of T. Namely, $\bar{x} \in F$. From (3.5) it also follows that $z_k - Tz_k \to 0$ as $k \to \infty$ [4, page 91]. Hence $Tz_k \rightharpoonup \bar{x}$ as $k \to \infty$.

Using the definition of the inner product in the tangent Hilbert space, we see that without loss of generality we may write

$$\begin{aligned}c_k &= \mathrm{Re}\langle i([v, f(v)]), i([v, Tz_k])\rangle \\ &= \mathrm{Re}\langle i([v, f(v)]), \frac{\rho(v, Tz_k)}{|M_{-v}(Tz_k)|} M_{-v}(Tz_k)\rangle.\end{aligned}$$

If $\bar{x} = v$, then using the weak continuity of the Möbius transformation M_{-v} [4, page 116], we have

$$\mathrm{Re}\langle i([v, f(v)]), M_{-v}(Tz_k)\rangle \to 0$$

as $k \to \infty$. By Step 1, the real sequence $\{\rho(v, Tz_k)\}_{k=1}^{\infty}$ is bounded and so, if the real sequence $\{1/(|M_{-v}(Tz_k)|)\}_{k=1}^{\infty}$ is also bounded, then $\lim_{k\to\infty} c_k = 0$.

Assume there exists a subsequence $\{M_{-v}(Tz_{k_l})\}_{l=1}^{\infty}$ that tends to the origin as $l \to \infty$. Applying M_v, we get $Tz_{k_l} \to M_v(0) = v$ as $l \to \infty$. Hence in this case $\rho(v, Tz_{k_l}) \to 0$ as $l \to \infty$, the subsequence $\{(M_{-v}(Tz_{k_l}))/(|M_{-v}(Tz_{k_l})|)\}_{l=1}^{\infty}$ is obviously bounded, and once again we have $\lim_{l\to\infty} c_{k_l} = 0$.

Assume now that $\bar{x} \neq v$. In this case we see that

$$c_k = a_k \cdot b_k,$$

where

$$a_k := \frac{|M_{-v}(\bar{x})|}{\rho(v,\bar{x})} \cdot \frac{\rho(v,Tz_k)}{|M_{-v}(Tz_k)|}$$

and

$$b_k := \mathrm{Re}\langle i([v,f(v)]), \frac{\rho(v,\bar{x})}{|M_{-v}(\bar{x})|} M_{-v}(Tz_k)\rangle.$$

Combining the weak continuity of the Möbius transformation M_{-v} with Lemma 2.6, we see that

$$\lim_{k\to\infty} b_k = \mathrm{Re}\langle [v,f(v)],[v,\bar{x}]\rangle \le 0.$$

As for the sequence $\{a_k\}_{k=1}^{\infty}$, it is clearly non-negative.

We claim that it is also bounded. To see this, we need to make sure that the sequence $\{M_{-v}(Tz_k)\}_{k=1}^{\infty}$ is bounded away from the origin. As before, assume to the contrary that there exists a subsequence $\{M_{-v}(Tz_{k_l})\}_{l=1}^{\infty}$ which tends to the origin as $l\to\infty$. Applying M_v, we get $Tz_{k_l}\to M_v(0)=v$ as $l\to\infty$ and so $\bar{x}=v$. This, however, contradicts our assumption that $\bar{x}\ne v$, and so we see that the sequence $\{a_k\}_{k=1}^{\infty}$ is indeed bounded. We conclude that $\lim_{k\to\infty} c_k \le 0$, as claimed.

Step 4. $\lim_{n\to\infty}\rho(v,x_n)=0$.

Using Lemmata 2.2 – 2.5, we see that

$$\begin{aligned}
\rho^2(x_{n+1},v) &= \rho^2(\alpha_n f(x_n)\oplus(1-\alpha_n)Tx_n, v)\\
&\le \alpha_n\rho^2(f(x_n),v)+(1-\alpha_n)\rho^2(Tx_n,v)-\alpha_n(1-\alpha_n)\rho^2(f(x_n),Tx_n)\\
&\le \alpha_n^2\rho^2(f(x_n),v)+(1-\alpha_n)^2\rho^2(Tx_n,v)\\
&+2\alpha_n(1-\alpha_n)\mathrm{Re}\langle[v,f(x_n)],[v,Tx_n]\rangle\\
&\le (1-\alpha_n)^2\rho^2(x_n,v)+\alpha_n^2\rho^2(f(x_n),v)\\
&+2\alpha_n(1-\alpha_n)\mathrm{Re}\langle[v,f(x_n],[v,Tx_n]\rangle\\
&= (1-\alpha_n)^2\rho^2(x_n,v)+\alpha_n^2\rho^2(f(x_n),v)\\
&+2\alpha_n(1-\alpha_n)\mathrm{Re}\langle[v,f(x_n]-[v,f(v)],[v,Tx_n]\rangle\\
&+2\alpha_n(1-\alpha_n)\mathrm{Re}\langle[v,f(v)],[v,Tx_n]\rangle\\
&\le (1-\alpha_n)^2\rho^2(x_n,v)+\alpha_n^2\rho^2(f(x_n),v)\\
&+2\alpha_n(1-\alpha_n)\rho(f(x_n),f(v))\rho(v,Tx_n)\\
&+2\alpha_n(1-\alpha_n)\mathrm{Re}\langle[v,f(v)],[v,Tx_n]\rangle\\
&\le [(1-\alpha_n)^2+2\alpha_n(1-\alpha_n)\alpha]\rho^2(x_n,v)\\
&+\alpha_n^2\rho^2(f(x_n),v)+2\alpha_n(1-\alpha_n)\mathrm{Re}\langle[v,f(v)],[v,Tx_n]\rangle.
\end{aligned}$$

Setting $\beta_n := \alpha_n(2-\alpha_n-2\alpha+2\alpha_n\alpha)$, we see that

$$\rho^2(x_{n+1},v)\le(1-\beta_n)\rho^2(x_n,v)+\beta_n b_n,$$

where, by Step 3, $\limsup_{n\to\infty} b_n\le 0$. Therefore we can at this point invoke Lemma 2.12 and conclude that $\lim_{n\to\infty}\rho(v,x_n)=0$, as claimed.

Hence the sequence $\{x_n\}_{n=0}^{\infty}$ converges in norm to v [4, page 91], as asserted. This completes the proof of Theorem 3.1. $\square$

This theorem improves upon [7, Theorem 4.1] and provides positive answers to two questions raised on page 366 of [7]: the mapping $f : \mathbb{B} \to \alpha\mathbb{B}$ is no longer assumed to be compact and the assumptions imposed on the parameter sequence $\{\alpha_n\}_{n=0}^{\infty}$ are weaker than those in [7]. When f is a constant we obtain [15, Theorem 3.1]. Related results, established by employing other methods, can be found in [12].

4. Approximating curves

Given a ρ-nonexpansive self-mapping T of $\mathbb{B}$, a holomorphic mapping $f : \mathbb{B} \to \alpha\mathbb{B}$, where $0 \le \alpha < 1$, and a number $0 \le t < 1$, we define the point $z_t \in \mathbb{B}$ as the unique fixed point of the strict ρ-contraction $S : \mathbb{B} \to \mathbb{B}$ defined by

$$Sx := tf(x) \oplus (1-t)Tx, \quad x \in \mathbb{B}. \tag{4.1}$$

Note that S is indeed a strict ρ-contraction by Lemmata 2.4 and 2.5 in [7]. It has a (unique) fixed point because the metric space $(\mathbb{B}, \rho)$ is complete. In other words,

$$z_t = tf(z_t) \oplus (1-t)Tz_t, \quad 0 \le t < 1. \tag{4.2}$$

In this section we prove the following strong convergence theorem regarding the behavior of the *approximating curve* $\{z_t : 0 \le t < 1\}$.

Theorem 4.1. *Let $(\mathbb{B}, \rho)$ be the Hilbert ball equipped with the hyperbolic metric $\rho : \mathbb{B} \times \mathbb{B} \to \mathbb{R}^+$ and let $T : \mathbb{B} \to \mathbb{B}$ be ρ-nonexpansive. Let $f : \mathbb{B} \to \alpha\mathbb{B}$ be holomorphic, where $0 \le \alpha < 1$, and let z_t, $0 \le t < 1$, be defined by (4.2). If T has a fixed point, then the strong $\lim_{t\to 0^+} z_t = v$, the unique solution of the equation $z = P_F(f(z))$, where $P_F : \mathbb{B} \to F$ is the nearest point projection of $\mathbb{B}$ onto the nonempty fixed point set $F = F(T)$ of T.*

Proof. Using Lemmata 2.4 and 2.5, we obtain

$$\begin{aligned}
\rho^2(z_t, v) &= \rho^2(tf(z_t) \oplus (1-t)Tz_t, v) \le t^2\rho^2(f(z_t), v) + (1-t)^2\rho^2(Tz_t, v) \\
&\quad + 2t(1-t)\mathrm{Re}\langle [v, f(z_t)], [v, Tz_t]\rangle \\
&\le (1-t)^2\rho^2(z_t, v) + 2t(1-t)\mathrm{Re}\langle [v, f(v)], [v, Tz_t]\rangle \\
&\quad + 2t(1-t)\mathrm{Re}\langle [v, f(z_t)] - [v, f(v)], [v, Tz_t]\rangle + t^2\rho^2(f(z_t), v) \\
&\le (1-t)^2\rho^2(z_t, v) + 2t(1-t)\rho(f(z_t), f(v))\rho(v, Tz_t) \\
&\quad + 2t(1-t)\mathrm{Re}\langle [v, f(v)], [v, Tz_t]\rangle + t^2\rho^2(f(z_t), v). \\
&\le (1-t)^2\rho^2(z_t, v) + 2t(1-t)\alpha\rho^2(z_t, v) \\
&\quad + 2t(1-t)\mathrm{Re}\langle [v, f(v)], [v, Tz_t]\rangle + t^2\rho^2(f(z_t), v).
\end{aligned}$$

Hence

$$(2 - t - 2\alpha + 2\alpha t)\rho^2(z_t, v) \le t\rho^2(f(z_t), v) + 2(1-t)\mathrm{Re}\langle [v, f(v)], [v, Tz_t]\rangle. \tag{4.3}$$

Since

$$\rho(z_t, v) \le t\rho(f(z_t), v) + (1-t)\rho(Tz_t, v) \le t\rho(f(z_t), v) + (1-t)\rho(z_t, v)$$

by Lemma 2.1, we see that $\rho(z_t, v) \leq \rho(f(z_t), v)$. Consequently, the approximating curve $\{z_t : 0 \leq t < 1\}$, as well as the curve $\{Tz_t : 0 \leq t < 1\}$, are ρ-bounded and

$$\lim_{t\to 0^+} \rho(z_t, Tz_t) = \lim_{t\to 0^+} t\rho(f(z_t), Tz_t) = 0.$$

Therefore, applying $\limsup_{t\to 0^+}$ to both sides of (4.3) and using the arguments in Step 3 of the proof of Theorem 3.1, we may conclude that $\lim_{t\to 0^+} \rho(z_t, v) = 0$. Hence the strong $\lim_{t\to 0^+} z_t = v$ [4, page 91], as asserted. This completes the proof of Theorem 4.1. □

Since the mapping $f : \mathbb{B} \to \alpha\mathbb{B}$ is no longer assumed to be compact, this theorem improves upon [8, Theorem 3.12] and solves a problem that was left open on page 3193 of [8].

References

[1] H. H. Bauschke, E. Matoušková and S. Reich, *Projection and proximal point methods: convergence results and counterexamples*, Nonlinear Anal. **56** (2004), 715–738.

[2] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-positive Curvature*, Springer, Berlin, 1999.

[3] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local: I. Données radicielles valuées*, Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5–252.

[4] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.

[5] E. Kopecká and S. Reich, *Hyperbolic monotonicity in the Hilbert ball*, Fixed Point Theory Appl. **2006** (2006), Article ID 78104, 1–15.

[6] E. Kopecká and S. Reich, *Nonexpansive retracts in Banach spaces*, Banach Center Publications **77** (2007), 161–174.

[7] E. Kopecká and S. Reich, *A note on the approximation of fixed points in the Hilbert ball*, J. Nonlinear Convex Anal. **9** (2008), 361–367.

[8] E. Kopecká and S. Reich, *Asymptotic behavior of resolvents of coaccretive operators in the Hilbert ball*, Nonlinear Anal. **70** (2009), 3187–3194.

[9] E. Kopecká and S. Reich, *A mean ergodic theorem for nonlinear semigroups on the Hilbert ball*, J. Nonlinear Convex Anal. **11** (2010), 185–197.

[10] M. Levenshtein and S. Reich, *Approximating fixed points of holomoprhic mappings in the Hilbert ball*, Nonlinear Anal. **70** (2009), 4145–4150.

[11] A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl. **241** (2000), 46–55.

[12] B. Piątek, *Viscosity iteration in CAT(κ) spaces*, Numerical Functional Anal. Optimization **34** (2013), 1245–1264.

[13] S. Reich, *Constructive techniques for accretive and monotone operators*, in Applied Nonlinear Analysis, Academic Press, New York, 1979, pp. 335–345.

[14] S. Reich, I. Shafrir, *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Anal. **15** (1990), 537–558.

[15] S. Reich and L. Shemen, *A note on Halpern's algorithm in the Hilbert ball*, J. Nonlinear Convex Anal. **14** (2013), 853–862.

[16] I. Shafrir, *Coaccretive operators and firmly nonexpansive mappings in the Hilbert ball*, Nonlinear Anal. **18** (1992), 637–648.

[17] I. Shafrir, *Theorems of ergodic type for ρ-nonexpansive mappings in the Hilbert ball*, Annali Mat. Pura Appl. **163** (1993), 313–327.

[18] H.-K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl. **298** (2004), 279–291.

Manuscript received August 20, 2013
revised August 30, 2013

Eva Kopecká
Institute of Mathematics, Czech Academy of Sciences, Žitná 25, CZ-11567 Prague, Czech Republic; and Department of Mathematics, University of Innsbruck, Technikerstrasse 19a, A-6020 Innsbruck, Austria
E-mail address: Eva.Kopecka@uibk.ac.at

Simeon Reich
Department of Mathematics, The Technion – Israel Institute of Technology, 32000 Haifa, Israel
E-mail address: sreich@tx.technion.ac.il