Journal of Nonlinear and Convex Analysis
Volume 15, Number 4, 2014, 809–818

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ALTERNATING CQ-ALGORITHMS FOR CONVEX FEASIBILITY AND SPLIT FIXED-POINT PROBLEMS

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Abstract. Let H_1, H_2, H_3 be real Hilbert spaces, let $C \subset H_1$, $Q \subset H_2$ be two nonempty closed convex sets, let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. We first consider the following new convex feasibility problem

$$(SEP) \qquad \text{Find} \quad x \in C, y \in Q \ \text{ such that } \ Ax = By.$$

Given a sequence (γ_k) of positive parameters and two initial arbitrarily points $x_0 \in H_1$ and $y_0 \in H_2$, we then present and study the convergence of the following new alternating CQ-algorithm

$$(ACQA) \qquad \begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)); \\ y_{k+1} = P_Q(y_k + \gamma_k B^*(Ax_{k+1} - By_k)), \end{cases}$$

where A^* and B^* denote the adjoint operators of A and B respectively.
Note that, by taking $B = I$, in (SEP), we recover the convex feasibility problem originally introduced in Censor and Elfving [9] and used later in intensity-modulated radiation therapy. If in addition $\gamma_k = 1$, in (ACQA), we obtain the related CQ-algorithm introduced by Byrne [6] and applied to dynamic emission tomographic image reconstruction. An extension to a new split common fixed-point problem governed by firmly quasi-nonexpansive mappings is presented and some examples are also provided.

1. Introduction and Preliminaries

Due to their extraordinary utility and broad applicability in many areas of applied mathematics (most notably, fully-discretized models of problems in image reconstruction from projections, in image processing, and in intensity-modulated radiation therapy), algorithms for solving convex feasibility problems continue to receive great attention, see for instance [1,5,7,13] and also [12,17–20]. In this paper our interest is in the study of the convergence of an alternating algorithm for solving a new split feasibility problem (SEP). This general class allows asymmetric and partial relations between the variables x and y. The interest is to cover many situation, for instance in decomposition methods for PDE's, applications in game theory and in intensity-modulated radiation therapy (IMRT). In decision sciences, this allows to consider agents who interplay only via some components of their decision variables, for further details, the interested reader is referred to [3]. In (IMRT), this amounts to envisage a weak coupling between the vector of doses absorbed in all voxels and that of the radiation intensity, for further details, the interested reader is referred to [8].

2010 *Mathematics Subject Classification.* Primary 49J53, 65K10; Secondary 49M37, 90C25.
Key words and phrases. Feasibility problem, split fixed-point problem, CQ-algorithm, alternating algorithm, firmly quasi-nonexpansive operator.

To begin with, let us recall that the split feasibility problem originally introduced in Censor and Elfving [9] is to find a point

$$x \in C \text{ such that } Ax \in Q, \tag{1.1}$$

where C is a closed convex subset of a Hilbert space H_1, Q is a closed convex subset of a Hilbert space H_2, and $A : H_1 \to H_2$ is a bounded linear operator.
Assuming that the (SFP) is consistent (i.e (1.1) has a solution), it is no hard to see that $x \in C$ solves (1.1) if and only if it solves the fixed-point equation

$$x = P_C\big(I - \gamma A^*(I - P_Q)A\big)x,\ \ x \in C, \tag{1.2}$$

where P_C and P_Q are the (orthogonal) projection onto C and Q, respectively, $\gamma > 0$ is any positive constant and A^* denotes the adjoint of A.
To solve the (1.2), Byrne [6] proposed his CQ algorithm which generates a sequence (x_k) by

$$x_{k+1} = P_C\big(I - \gamma A^*(I - P_Q)A\big)x_k,\ \ k \in \mathbb{N}, \tag{1.3}$$

where $\gamma \in (0, 2/\lambda)$ with λ being the spectral radius of the operator A^*A.

In the present paper, we present our idea in the context of feasibility problems, provide an extension to the split common fixed-point context and state some special cases.

Denote the solution set of (SEP) by

$$\Gamma = \{x \in C, y \in Q;\ Ax = By\}.$$

Now x, y solves (SEP) means that there exists $x \in C, y \in Q$ such that $Ax - By = 0$. This motivates us to consider the distance function $(x, y) \mapsto \|Ax - By\|$ and the minimization problem

$$\min_{x\in C, y\in Q} \frac{1}{2}\|Ax - By\|^2.$$

Observe that by writing down the optimality conditions and by denoting by N_C, N_Q the normal cone to the convex sets C and Q, we obtain

$$\begin{cases} 0 \in A^*(Ax - By)) + N_C(x); \\ 0 \in -B^*(Ax - By)) + N_Q(y), \end{cases}$$

which implies

$$\begin{cases} x - \gamma A^*(Ax - By) \in x + \gamma N_C(x); \\ y + \gamma B^*(Ax - By) \in y + \gamma N_Q(y), \end{cases}$$

which in turn leads to the fixed point formulation

$$\begin{cases} x = (I + \gamma N_C)^{-1}(x - \gamma A^*(Ax - By)) = P_C(x - \gamma A^*(Ax - By)); \\ y = (I + \gamma N_Q)^{-1}(y + \gamma B^*(Ax - By)) = P_Q(y + \gamma B^*(Ax - By)). \end{cases}$$

This equation suggests the possibility of iterating and thus consider our alternating CQ-algorithms for solving problem (SEP), namely

$$(ACQA) \qquad \begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)); \\ y_{k+1} = P_Q(y_k + \gamma_k B^*(Ax_{k+1} - By_k)). \end{cases}$$

Clearly, by taking $B = I$ we recover the classical feasibility problem and obtain a method which resembles to the CQ-algorithm (1.3). If in addition $\gamma_k = 1$, the second equality in $(ACQA)$ reduces to $y_{k+1} = P_Q(Ax_{k+1})$ and thus the first equality gives

$$x_{k+1} = P_C(x_k - A^*(Ax_k - P_Q(Ax_k))) = P_C(x_k - A^*(I - P_Q)(Ax_k)),$$

which is exactly the CQ-algorithm (1.3) proposed by Byrne [6] with $\gamma = 1$.

2. Convergence of the alternating CQ-algorithm

Remember that the projection operators have very attractive properties that make them particularly well suited for iterative algorithms. For instance, P_C is firmly nonexpansive, namely for all x, y

$$\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|(I - P_C)(x) - (I - P_C)(y)\|^2. \tag{2.1}$$

Now, we are in a position to prove our convergence result.

Theorem 2.1. *Let H_1, H_2, H_3 be real Hilbert spaces, let $C \subset H_1$, $Q \subset H_2$ be two nonempty closed convex sets, let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Assume that the solution set Γ is nonempty, (γ_k) is a positive no-deacreasing sequence such that $\gamma_k \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$ for a small enough $\varepsilon > 0$, where λ_A, λ_B stand for the spectral radius of A^*A and B^*B respectively. Then, the sequence (x_k, y_k) generated by our algorithm weakly converges to a solution $(\bar{x}, \bar{y})$ of (SEP). Moreover $(Ax_k - By_k)$ strongly converges to 0 and both (x_k) and (y_k) are asymptotically regular.*

Proof. Taking $(x, y) \in \Gamma$ i.e., $x \in C$; $y \in Q$, $Ax = By$ and using the fact that the projection operator P_C is firmly nonexpansive, the first equality of the algorithm gives

$$\|x_{k+1} - x\|^2 \leq \|x_k - x - \gamma_k A^*(Ax_k - By_k)\|^2 - \|x_{k+1} - x_k + \gamma_k A^*(Ax_k - By_k)\|^2. \tag{2.2}$$

We also have

$$\begin{aligned} \|x_k - x - \gamma_k A^*(Ax_k - By_k)\|^2 &= \|x_k - x\|^2 + \gamma_k^2 \|A^*(Ax_k - By_k)\|^2 \\ &- 2\gamma_k \langle A^*(Ax_k - By_k), x_k - x\rangle. \end{aligned}$$

On the other hand, from the definition of λ_A it follows

$$\begin{aligned} \gamma_k^2 \|A^*(Ax_k - By_k)\|^2 &= \gamma_k^2 \langle Ax_k - By_k, AA^*(Ax_k - By_k)\rangle \\ &\leq \lambda_A \gamma_k^2 \langle Ax_k - By_k, Ax_k - By_k\rangle \\ &= \lambda_A \gamma_k^2 \|Ax_k - By_k\|^2. \end{aligned}$$

Now, by setting $\theta := 2\gamma_k \langle A^*(Ax_k - By_k), x_k - x\rangle$, we obtain

$$\theta = 2\gamma_k \langle Ax_k - By_k, Ax_k - Ax\rangle = 2\gamma_k (\|Ax_k - B_k)\|^2 + \langle Ax_k - By_k, By_k - Ax\rangle).$$

Combining the key inequalities above with relation (2.5), we derive

$$\begin{aligned} \|x_{k+1} - x\|^2 &\leq \|x_k - x\|^2 - 2\gamma_k \langle Ax_k - By_k, By_k - Ax\rangle \\ &- \gamma_k (2 - \gamma_k \lambda_A) \|Ax_k - By_k\|^2 \\ &- \|x_{k+1} - x_k + \gamma_k A^*(Ax_k - By_k)\|^2. \end{aligned}$$

Similarly, the second equality of the algorithm leads to

$$\|y_{k+1} - y\|^2 \leq \|y_k - y\|^2 + 2\gamma_k \langle Ax_{k+1} - By_k, Ax_{k+1} - By\rangle$$

$$
\begin{aligned}
&- \gamma_k(2-\gamma_k\lambda_B)\|Ax_{k+1}-By_k\|^2\\
&- \|y_{k+1}-y_k-\gamma_k B^*(Ax_{k+1}-By_k)\|^2.
\end{aligned}
$$

By adding the two last inequalities and by taking into account the assumptions on (γ_k), the fact that $Ax = By$ and the two key equalities

$$2\langle Ax_k - By_k, By_k - Ax\rangle = -\|Ax_k - By_k\|^2 - \|By_k - Ax\|^2 + \|Ax_k - Ax\|^2.$$

and

$$2\langle By_k - Ax_{k+1}, Ax_{k+1} - By\rangle = -\|By_k - Ax_{k+1}\|^2 - \|Ax_{k+1} - By\|^2 + \|Bx_k - By\|^2,$$

we finally obtain

$$
\begin{aligned}
\|x_{k+1}-x\|^2 + \|y_{k+1}-y\|^2 &\le \|x_k - x\|^2 + \|y_k - y\|^2 - \gamma_k\|Ax_k - Ax\|^2\\
&+ \gamma_{k+1}\|Ax_{k+1} - Ax\|^2 - \gamma_k(1-\gamma_k\lambda_A)\|Ax_k - By_k\|^2\\
&- \gamma_k(1-\gamma_k\lambda_B)\|Ax_{k+1} - By_k\|^2\\
&- \|x_{k+1} - x_k + \gamma_k A^*(Ax_k - By_k)\|^2\\
&- \|y_{k+1} - y_k - \gamma_k B^*(Ax_{k+1} - By_k)\|^2.
\end{aligned}
$$

Now, by setting $\Gamma_k(x,y) := \|x_k - x\|^2 + \|y_k - y\|^2 - \gamma_k\|Ax_k - Ax\|^2$, we obtain the following key inequality $(\star)$

$$
\begin{aligned}
\Gamma_{k+1}(x,y) &\le \Gamma_k(x,y) - \gamma_k(1-\gamma_k\lambda_A)\|Ax_k - By_k\|^2\\
&\quad -\gamma_k(1-\gamma_k\lambda_B)\|Ax_{k+1} - By_k\|^2\\
&\quad -\|x_{k+1} - x_k - \gamma_k A^*(Ax_k - By_k)\|^2\\
&\quad -\|y_{k+1} - y_k + \gamma_k B^*(Ax_{k+1} - By_k)\|^2.
\end{aligned}
$$

On the other hand, note that

$$\gamma_k\|Ax_k - Ax\|^2 = \gamma_k\langle x_k - x, A^*A(x_k - x)\rangle \le \gamma_k\lambda_A\|x_k - x\|^2.$$

Hence

$$\Gamma_k(x,y) \ge (1-\gamma_k\lambda_A)\|x_k - x\|^2 + \|y_k - y\|^2 \ge 0. \tag{2.3}$$

The sequence $(\Gamma_k(x,y))$ being decreasing and lower bounded by 0, consequently it converges to some finite limit, says $l(x,y)$, and by passing to the limit in $(\star)$, we obtain that

$$\lim_{k\to+\infty}\|Ax_k - By_k\| = \lim_{k\to+\infty}\|x_{k+1} - x_k + \gamma_k A^*(Ax_k - By_k)\| = 0,$$

and

$$\lim_{k\to+\infty}\|Ax_{k+1} - By_k\| = \lim_{k\to+\infty}\|y_{k+1} - y_k - \gamma_k B^*(Ax_{k+1} - By_k)\| = 0.$$

Since

$$\|x_{k+1} - x_k\| \le \|x_{k+1} - x_k + \gamma_k A^*(Ax_k - By_k)\| + \gamma_k\|A^*(Ax_k - By_k)\|$$

and thanks to the fact that (γ_k) is bounded, we infer that (x_k) is asymptotically regular, namely $\lim_k \|x_{k+1} - x_k\| = 0$. Similarly (y_k) is asymptotically regular too. Conditions on (γ_k) and relation (2.6) imply

$$\Gamma_k(x,y) \ge \varepsilon\lambda_A\|x_k - x\|^2 + \|y_k - y\|^2,$$

which ensures that both the sequences (x_k) and (y_k) are bounded, because $(\Gamma_k(x,y))$ convergences to a finite limit.

Let x^* and y^* be respectively weak cluster points of the sequences (x_k) and (y_k). Then there exist two subsequences of (x_k) and (y_k) (again labelled (x_k) and (y_k)) which converge weakly to x^* and y^*. By noting that the two equalities in $(ACQA)$ can be rewritten as

$$\left\{ \begin{array}{l} \frac{x_k - x_{k+1}}{\gamma_k} - A^*(Ax_k - By_k) \in N_C(x_{k+1}); \\ \frac{y_k - y_{k+1}}{\gamma_k} + B^*(Ax_{k+1} - By_k) \in N_Q(y_{k+1}), \end{array} \right. \tag{2.4}$$

that the graphs of the maximal monotone operators N_C, N_Q are weakly-strongly closed and by passing to the limit in the last inclusions, we obtain that

$$0 \in N_C(x^*) \text{ and } 0 \in N_Q(y^*), \text{ which assures that } x^* \in C \text{ and } y^* \in Q.$$

Furthermore, the weak convergence of $(Ax_k - By_k)$ to $Ax^* - By^*$ and lower semicontinuity of the squared norm imply

$$\|Ax^* - By^*\|^2 \leq \liminf_{k \to +\infty} \|Ax_k - By_k\|^2 = 0,$$

hence $(x^*, y^*) \in \Gamma$. To show the uniqueness of the weak cluster point, we will use the same idea as in the celebrate Opial Lemma. Indeed, let $\bar{x}, \bar{y}$ be other weak cluster points of (x_k) and (y_k) respectively, by passing to the limit in the relation

$$\begin{array}{rcl} \Gamma_k(x^*, y^*) & = & \Gamma_k(\bar{x}, \bar{y}) + \|x^* - \bar{x}\|^2 + \|y^* - \bar{y}\|^2 - \gamma_k \|Ax^* - A\bar{x}\|^2 \\ & + & 2\langle x_k - \bar{x}, \bar{x} - x^* \rangle + 2\langle y_k - \bar{y}, \bar{y} - y^* \rangle - 2\gamma_k \langle Ax_k - A\bar{x}, A\bar{x} - Ax^* \rangle, \end{array}$$

we obtain

$$l((x^*, y^*) = l(\bar{x}, \bar{y}) + \|x^* - \bar{x}\|^2 + \|y^* - \bar{y}\|^2 - \gamma^* \|Ax^* - A\bar{x}\|^2,$$

with $\gamma^* = \lim_{k \to +\infty} \gamma_k$. Reversing the role of (x^*, y^*) and $(\bar{x}, \bar{y})$, we also have

$$l(\bar{x}, \bar{y}) = l((x^*, y^*) + \|x^* - \bar{x}\|^2 + \|y^* - \bar{y}\|^2 - \gamma^* \|Ax^* - A\bar{x}\|^2.$$

By adding the two last equalities and having in mind that (γ_k) is an no-decreasing sequence satisfying $1 - \gamma_k \lambda_A > \varepsilon \lambda_A$, we obtain

$$\varepsilon \lambda_A \|x^* - \bar{x}\|^2 + \|y^* - \bar{y}\|^2 \leq 0.$$

Hence $x^* = \bar{x}$ and $y^* = \bar{y}$, this implies that the whole sequence (x_k, y_k) weakly converges to a solution of problem (1.1), which completes the proof. □

3. From convex feasibility to split fixed-point

The CQ algorithm (1.3) involves the computation of the projections P_C and P_Q onto the sets C and Q and is therefore implementable in the case where the projections have closed-form expressions. A generalization to split common fixed point problems (SCFP) was given by Censor and Segal [11]. This formalism is in itself at the core of the modeling of many inverse problems in various areas of mathematics and physical sciences and has been used to model significant real-world inverse problems in sensor networks, in radiation therapy treatment planning, in resolution enhancement, in wavelet-based denoising, in antenna design, in computerized tomography, in materials science, in watermarking, in data compression, in magnetic resonance imaging, in holography, in color imaging, in optics and neural networks, in graph matching, we refer to [10] for the exact references. Censor and Segal consider the following problem

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q, \tag{3.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be two nonexpansive operators with non empty fixed-point sets $FixU = C$ and $FixT = Q$. To solve (3.8), Censor and Segal [11] proposed and proved, in finite dimensional spaces, the convergence of the following algorithm

$$x_{k+1} = U(x_k + \gamma A^t(T - I)Ax_k),\ k \in \mathbb{N}, \tag{3.2}$$

where $\gamma \in (0, 2/\lambda)$ with λ being the largest eigenvalue of the matrix A^tA (t stands for matrix transposition).

This suggest in our case to introduce the following problem

$$\text{find } x^* \in FixU, y^* \in FixT \text{ such that } Ax^* = By^*, \tag{3.3}$$

and to consider the following alternating SCFP-algorithm

$$(SCFPA) \qquad \begin{cases} x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)); \\ y_{k+1} = T(y_k + \gamma_k B^*(Ax_{k+1} - By_k)). \end{cases}$$

By taking $B = I$ we recover clearly the classical (SCFP) and obtain a method which resembles to (3.9). If in addition $\gamma_k = 1$, the second equality in ($SCFPA$) reduces to $y_{k+1} = T(Ax_{k+1})$ and thus the first equality leads to

$$x_{k+1} = U(x_k - A^*(Ax_k - T(Ax_k))) = U(x_k - A^*(I - T)(Ax_k)),$$

which is exactly the algorithm (3.9) proposed by Censor and Segal with $\gamma = 1$.
Now, remember that a mapping T is firmly quasi-nonexpansive, if for all $x \in H$ and $y \in FixT$ one has

$$\|T(x) - y\|^2 \leq \|x - y\|^2 - \|x - T(x)\|^2. \tag{3.4}$$

An example is the sub-gradient projection operator, that is the driving behind the majority of the algorithms, that employ the subgradients of convex loss functions, in order to solve for instance nonsmooth minimization tasks.

Theorem 3.1. *Let H_1, H_2, H_3 be real Hilbert spaces, let $U : H_1 \to H_1$, $T : H_2 \to H_2$ be two firmly quasi-nonexpansive mappings such that $I - U, I - T$ are demiclosed at 0. Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Assume that the solution set Γ is nonempty, (γ_k) is a positive nodeacreasing sequence such that $\gamma_k \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$ for a small enough $\varepsilon > 0$, where λ_A, λ_B stand for the spectral radius of A^*A and B^*B respectively. Then, the sequence (x_k, y_k) generated by (SCFPA) weakly converges to a solution $(\bar{x}, \bar{y})$ of* (3.3).
Moreover $\lim_{k\to+\infty} \|Ax_k - By_k\| = 0$, $\lim_{k\to+\infty} \|x_k - x_{k+1}\| = 0$ *and* $\lim_{k\to+\infty} \|y_k - y_{k+1}\| = 0$.

Proof. Taking $(x, y) \in \Gamma$ and using the fact that U is firmly quasi-nonexpansive, the first equality of the algorithm exactly gives

$$\|x_{k+1} - x\|^2 \leq \|x_k - x - \gamma_k A^*(Ax_k - By_k)\|^2 - \|x_{k+1} - x_k + \gamma_k A^*(Ax_k - By_k)\|^2.$$

Thanks to the fact that T is firmly quasi-nonexpansive, we also obtain

$$\|y_{k+1}-y\|^2 \le \|y_k-y+\gamma_k B^*(Ax_{k+1}-By_k)\|^2 - \|y_{k+1}-y_k-\gamma_k B^*(Ax_{k+1}-By_k)\|^2.$$

Thus following the lines of the proof of Theorem 2.1, we obtain that the sequences (x_k) and (y_k) are both bounded, asymptotically regular, verify $\lim_k \|Ax_k - By_k\| = 0$ and

$$\lim_{k\to+\infty} \|x_{k+1}-x_k+\gamma_k A^*(Ax_k-By_k)\| = \lim_{k\to+\infty} \|y_{k+1}-y_k-\gamma_k B^*(Ax_{k+1}-By_k)\| = 0.$$

Now, let x^* and y^* be respectively weak cluster points of the sequences (x_k) and (y_k) and note that the two equalities in $(SCFPA)$ imply that

$$\begin{cases} x_k - x_{k+1} - \gamma_k A^*(Ax_k - By_k) = (I-U)(x_k - \gamma_k A^*(Ax_k - By_k)); \\ y_k - y_{k+1} + \gamma_k B^*(Ax_{k+1} - By_k) = (I-T)(y_{k+1} + \gamma_k B^*(Ax_{k+1} - By_k)). \end{cases}$$

By passing to the limit in the last two equalities and by taking into account the fact that $(x_k - x_{k+1} - \gamma_k A^*(Ax_k - By_k))$ converges strongly to 0, that $(x_k - \gamma_k A^*(Ax_k - By_k))$ converges weakly to x^* and that $I-U$ is demiclosed at zero, we obtain that $x^* \in FixU$ Likewise we obtain that $y^* \in FixT$. Again, the weak convergence of $(Ax_k - By_k)$ to $Ax^* - By^*$ and the lower semicontinuity of the squared norm amount us to obtain

$$\|Ax^* - By^*\|^2 \le \liminf_{k\to+\infty} \|Ax_k - By_k\|^2 = 0,$$

hence $(x^*, y^*) \in \Gamma$. The rest of the proof is analogous to that of Theorem 2.1. □

4. Interesting spacial cases

We now turn our attention to provide some applications relying on some convex and nonlinear analysis notions, see for example W. Takahashi [16].

4.1. **Convex optimization via proximity mappings.** Let $f : H_1 \to I\!R \cup \{+\infty\}$ be a proper convex lower semicontinuous function and remember that $x \in argminf$ means that x minimizes the function f. It is well-known that

$$x \in argminf \Leftrightarrow x = prox_{\mu f}(x) := argmin\{f(u) + \frac{1}{2\mu}\|u-x\|^2\}.$$

That is fixed point set of the proximity mapping is precisely the set of minimizers of f. Such properties allow us to envision the possibility of developing algorithms based on the proximity operators. Indeed, let $g : H_2 \to I\!R \cup \{+\infty\}$ be another proper convex lower semicontinuous function, by setting $U = prox_{\mu f}$, $T = prox_{\nu g}$, the problem under consideration is nothing but

$$\text{Find } x^* \in argminf, y^* \in argming \text{ such that } Ax^* = By^*, \tag{4.1}$$

hence (x^*, y^*) solves

$$\min_{x,y}\{f(x) + g(y) + \frac{1}{2}\|Ax - By\|^2\}$$

an optimization problem with weak coupling in the objective function as well as

$$\min_{x,y}\{f(x) + g(y); Ax = By\},$$

an optimization problem with weak coupling in the constraint. These problems attract many authors's attention due to the fact that they model Surface energy variational problems, Domain decomposition for PDE s, Optimal control problems, Potential Games, see the excellent communication [2] (or [4]) by H. Attouch where a different proximal approach is proposed. In this context, our algorithm takes the following equivalent form

$$(SCFPA) \qquad \begin{cases} x_{k+1} = argmin_u\{f(u) + \frac{1}{2\mu}\|u - (x_k - \gamma_k A^*(Ax_k - By_k))\|^2\}; \\ y_{k+1} = argmin_v\{g(v) + \frac{1}{2\nu}\|v - (y_k + \gamma_k B^*(Ax_{k+1} - By_k))\|^2\}. \end{cases}$$

4.2. **Null-point problems via resolvent mappings.** Given a maximal monotone operator $M : H_1 \to 2^{H_1}$, it is well-known that its associated resolvent mapping, $J^M_\mu(x) := (I+\mu M)^{-1}$, is firmly nonexpansive and we have $0 \in M(x) \Leftrightarrow x = J^M_\mu(x)$. In other words zeroes of M are exactly fixed-points of its resolvent mapping. By taking $U = J^M_\mu(x)$, $T = J^N_\nu(x)$, where $N : H_2 \to 2^{H_2}$ is another maximal monotone operator, the problem under consideration is nothing but

$$\text{find } x^* \in M^{-1}(0), y^* \in N^{-1}(0) \text{ such that } Ax^* = By^*, \tag{4.2}$$

and the algorithm takes the following equivalent form

$$(SCFPA) \qquad \begin{cases} x_{k+1} = J^M_\mu(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = J^N_\nu(y_k + \gamma_k B^*(Ax_{k+1} - By_k)). \end{cases}$$

4.3. **Equilibrium problems via resolvent mappings.** Having in mind the connection between monotone operators and equilibrium functions, we may consider the following problem

$$\text{find } x^* \in C, y^* \in Q;\ F(x^*, u) \geq 0, H(y^*, v) \geq 0 \text{ and } Ax^* = By^* \ \forall u, v, \tag{4.3}$$

with C, Q closed convex sets and F, H belong in the class of bifunctions G verifying the following usual conditions:

(A1) $G(x, x) = 0$ for all $x, y \in D$;
(A2) G is monotone, i.e., $G(x, y) + G(y, x) \leq 0$ for all $x, y \in D$;
(A3) $\limsup_{t\downarrow 0} G(tz + (1-t)x, y) \leq G(x, y)$ for any $x, y, z \in D$;
(A4) for each $x \in D, y \to G(x, y)$ is convex and lower-semicontinuous.

It is well-known; see for example [14], that the associated resolvent operator $S_{\lambda G} : H \to D$ defined by

$$S_{\lambda G}(x) = \{z \in D :\ G(z, y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0,\ \forall y \in D\},$$

is firmly nonexpansive and its fixed-points are exactly the equilibria of G, that is $H(y^*, v) \geq 0\ \forall u$.
By setting $U = S_{\mu F}$, $T = S_{\nu H}$, the problem under consideration is nothing but (4.14) and the algorithm takes the following equivalent form

$$(SCFPA) \qquad \begin{cases} x_{k+1} = S_{\mu F}(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = S_{\nu H}(y_k + \gamma_k B^*(Ax_{k+1} - By_k)). \end{cases}$$

In the present paper, we considered a new convex feasibility problem, presented our alternating method and established its convergence. We also extended our algorithm to an alternating method for solving a new split common fixed-point problem, highlighted its applicability in modeling significant real world problems. Finally, we provided some applications to convex optimization, null point and equilibrium problems.

Acknowledgment

I would like to thank N. Lehdili for his kind invitation to Jossigy. A thought also to D. Butnariu in July 4, in addition to his work, I very appreciated his human qualities. The author also thanks Professor W. Takahashi for his careful reading of the paper and for the valuable comments which improved the presentation of this manuscript.

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Manuscript received September 5, 2012
revised May 23, 2013

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