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THE SPLIT COMMON NULL POINT PROBLEM FOR MAXIMAL MONOTONE MAPPINGS IN HILBERT SPACES AND APPLICATIONS

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Abstract. Based on recent works by Byrne-Censor-Gibali-Reich [C. Byrne, Y. Censor, A. Gibali and S. Reich, The split common null point problem, J. Nonlinear Convex Anal. 13 (2012), 759–775] and the second author [W. Takahashi, Strong convergence theorems for maximal and inverse-strongly monotone mappings in Hilbert spaces and applications, J. Optim. Theory Appl. 157 (2013), 781–802], we study the split common null point problem for maximal monotone mappings in Hilbert spaces which is related to the split feasibility problem by Censor and Elfving [Y. Censor and T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numer. Algorithms 8 (1994), 221–239]. We first obtain some properties for resolvents of maximal monotone operators in Hilbert spaces. Then using these properties, we establish a strong convergence theorem for finding a solution of the split common null point problem which is characterized as a unique solution of the variational inequality of a nonlinear operator. As applications, we get two new strong convergence theorems which are connected with the split feasibility problem and an equilibrium problem.

1. Introduction

Let H be a real Hilbert space and let C be a non-empty, closed and convex subset of H. Let $U : C \to H$ be a mapping. The set of solutions of the variational inequality for U is defined by

$$V(C,U) = \{\hat{x} \in C : \langle U\hat{x}, y - \hat{x}\rangle \geq 0, \ \forall y \in C\}.$$

A mapping $U : C \to H$ is called inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ux - Uy\rangle \geq \alpha\|Ux - Uy\|^2, \quad \forall x, y \in C.$$

Such a mapping U is called α-inverse strongly monotone. Let H_1 and H_2 be two real Hilbert spaces. Given set-valued mappings $A_i : H_1 \to 2^{H_1}$, $1 \leq i \leq m$, and $B_j : H_2 \to 2^{H_2}$, $1 \leq j \leq n$, respectively, and bounded linear operators $T_j : H_1 \to H_2$, $1 \leq j \leq n$, the *split common null point problem* [4] is to find a point $z \in H_1$ such that

$$z \in \big(\cap_{i=1}^m A_i^{-1}0\big) \cap \big(\cap_{j=1}^n T_j^{-1}(B_j^{-1}0)\big),$$

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of A_i and B_j, respectively. Let C and Q be non-empty, closed and convex subsets of H_1 and H_2, respectively. Let $T : H_1 \to H_2$

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be a bounded linear operator. Then the *split feasibility problem* [5] is to find $z \in H_1$ such that $z \in C \cap T^{-1}Q$. Putting $A_i = \partial i_C$ for all i, $B_j = \partial i_Q$ for all j and $T_j = T$ for all j in the split common null point problem, we see that the split feasibility peoblem is a special case of the split common null point problem, where ∂i_C and ∂i_Q are the subdifferentials of the indicator functions i_C of C and i_Q of Q, respectively. Defining $U = T^*(I - P_Q)T$ in the split feasibility peoblem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator, where T^* is the adjoint operator of T and P_C and P_Q are the metric projections of H_1 onto C and H_2 onto Q, respectively. Furthermore, if $C \cap T^{-1}Q$ is non-empty, then $z \in C \cap T^{-1}Q$ is equivalent to $z = P_C(I - \lambda U)z$, where $\lambda > 0$. From [16] we also know an implicit strong convergence theorem for finding a common point of the set of null points of the addition of an inverse strongly monotone mapping and a maximal monotone operator and the set of null points of a maximal monotone operator which is related to an equilibrium problem in a Hilbert space; see also [9, 11].

In this paper, motivated by these definitions and results, we study the split common null point problem for maximal monotone mappings in Hilbert spaces. We first obtain some properties for resolvents of maximal monotone operators in Hilbert spaces. Then using these properties, we establish a strong convergence theorem for finding a solution of the split common null point problem which is characterized as a unique solution of the variational inequality of a nonlinear operator. As applications, we get two new strong convergence theorems which are connected with the split feasibility problem and an equilibrium problem.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We have from [15] that for any $x, y \in H$ and $\lambda \in \mathbb{R}$

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \tag{2.1}$$

and

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore, we have that for $x, y, u, v \in H$

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

Let C be a non-empty, closed and convex subset of a Hilbert space H and let $T: C \to H$ be a mapping. We denote by $F(T)$ be the set of fixed points of T. A mapping $T : C \to H$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to H$ is called firmly nonexpansive if $\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$ for all $x, y \in C$. If a mapping T is firmly nonexpansive, then it is nonexpansive. If $T : C \to H$ is nonexpansive, then $F(T)$ is closed and convex; see [15]. For a non-empty, closed and convex subset C of H, the nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such a mapping P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive; $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$

for all $x, y \in H$. Furthermore, $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [13].

Let B be a set-valued mapping of H into 2^H. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in H : Bx \neq \emptyset\}$. A set-valued mapping B is said to be a monotone operator on H if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1} \colon H \to \mathrm{dom}(B)$, which is called the resolvent of B for r. We denote by $A_r = \frac{1}{r}(I - J_r)$ the Yosida approximation of B for $r > 0$. We know from [14] that

$$A_r x \in BJ_r x, \quad \forall x \in H,\ r > 0. \tag{2.4}$$

Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is known that $B^{-1}0 = F(J_r)$ for all $r > 0$ and the resolvent J_r is firmly nonexpansive, i.e.,

$$\|J_r x - J_r y\|^2 \leq \langle x - y, J_r x - J_r y\rangle, \quad \forall x, y \in H. \tag{2.5}$$

Furthermore, we have that for $s, r \in \mathbb{R}$ with $s \geq r > 0$ and $x \in H$

$$\|x - J_s x\| \geq \|x - J_r x\|. \tag{2.6}$$

In fact, since $\frac{x - J_r x}{r} \in BJ_r x$, $\frac{x - J_s x}{s} \in BJ_s x$ and B is monotone, we have that

$$\Big\langle \frac{x - J_r x}{r} - \frac{x - J_s x}{s}, J_r x - J_s x\Big\rangle \geq 0$$

and hence

$$\frac{1}{r}\langle x - J_r x, J_r x - J_s x\rangle \geq \frac{1}{s}\langle x - J_s x, J_r x - J_s x\rangle.$$

Using (2.3), we have that

$$\begin{aligned} &\frac{1}{r}\big(\|x - J_s x\|^2 + \|J_r x - J_r x\|^2 - \|x - J_r x\|^2 - \|J_r x - J_s x\|^2\big) \\ &\quad \geq \frac{1}{s}\big(\|x - J_s x\|^2 + \|J_s x - J_r x\|^2 - \|x - J_r x\|^2 - \|J_s x - J_s x\|^2\big) \end{aligned}$$

and hence

$$\begin{aligned} &\Big(\frac{1}{r} - \frac{1}{s}\Big)\big(\|x - J_s x\|^2 - \|x - J_r x\|^2\big) \\ &\qquad \geq \frac{1}{r}\|J_r x - J_s x\|^2 + \frac{1}{s}\|J_s x - J_r x\|^2 \geq 0. \end{aligned}$$

Thus we have that

$$(s - r)\big(\|x - J_s x\|^2 - \|x - J_r x\|^2\big) \geq 0.$$

Therefore $\|x - J_s x\| \geq \|x - J_r x\|$ for all $s, r \in \mathbb{R}$ with $s \geq r > 0$ and $x \in H$. We also know the following lemma from [12].

Lemma 2.1. *Let H be a Hilbert space and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds:*

$$\frac{s - t}{s}\langle J_s x - J_t x, J_s x - x\rangle \geq \|J_s x - J_t x\|^2$$

for all $s, t > 0$ *and* $x \in H$.

From Lemma 2.1, we have that

$$\|J_s x - J_t x\| \leq (|s-t|/s)\|x - J_s x\| \tag{2.7}$$

for all $s, t > 0$ and $x \in H$; see also [7, 13].

3. Lemmas

For proving the main theorem, we need some lemmas. Let C be a non-empty, closed and convex subset of a Hilbert space H. A mapping $g : C \to H$ is a contraction if there exists $k \in (0,1)$ such that$\|g(x) - g(y)\| \leq k\|x - y\|$ for all $x, y \in C$. We call such a mapping g a k-contraction. A linear bounded self-adjoint operator $G : H \to H$ is called strongly positive if there exists $\overline{\gamma} > 0$ such that $\langle Gx, x\rangle \geq \overline{\gamma}\|x\|^2$ for all $x \in H$. In general, a mapping $T : C \to H$ is called strongly monotone if there exists $\overline{\gamma} > 0$ such that $\langle x-y, Tx-Ty\rangle \geq \overline{\gamma}\|x-y\|^2$ for all $x, y \in C$. Such T is also called $\overline{\gamma}$-strongly monotone. The following results were essentially obtained in [16]. However, for the sake of completeness, we give the proofs.

Lemma 3.1. *Let* H *be a Hilbert space. Let* g *be a* k*-contraction of* H *into itself and let* G *be a strongly positive bounded linear self-adjoint operator on* H *with coefficient* $\overline{\gamma} > 0$. *Take* $\gamma > 0$ *with* $\gamma < \frac{\overline{\gamma}}{k}$ *and* $t > 0$ *with* $t(\|G\| + \gamma\ k)^2 < 2(\overline{\gamma} - \gamma\ k)$ *and* $2t(\overline{\gamma} - \gamma\ k) < 1$. *Then*

$$0 < 1 - t\{2(\overline{\gamma} - \gamma\ k) - t(\|G\| + \gamma\ k)^2\} < 1$$

and $I - t(G - \gamma g)$ *is a contraction of* H *into itself.*

Proof. Taking $\gamma > 0$ with $\gamma < \frac{\overline{\gamma}}{k}$, we have that $G - \gamma g$ is $\overline{\gamma} - \gamma\ k$-strongly monotone. Furthermore, taking $t > 0$ with $t(\|G\| + \gamma\ k)^2 < 2(\overline{\gamma} - \gamma\ k)$ and $2t(\overline{\gamma} - \gamma\ k) < 1$, we have that

$$0 < 1 - t\{2(\overline{\gamma} - \gamma\ k) - t(\|G\| + \gamma\ k)^2\} < 1.$$

Then we have that for any $x, y \in H$

$$\begin{aligned}
&\|x - t(G-\gamma g)x - (y - t(G-\gamma g)y)\|^2 \\
&\quad = \|x-y\|^2 - 2t\langle x-y, (G-\gamma g)x - (G-\gamma g)y\rangle \\
&\qquad + \|t(G-\gamma g)x - t(G-\gamma g)y\|^2 \\
&\quad \leq \|x-y\|^2 - 2t(\overline{\gamma} - \gamma\ k)\|x-y\|^2 \\
&\qquad + t^2(\|G\|^2 + 2\|G\|\gamma\ k + (\gamma\ k)^2)\|x-y\|^2 \\
&\quad = \{1 - 2t(\overline{\gamma} - \gamma\ k) + t^2(\|G\| + \gamma\ k)^2\}\|x-y\|^2 \\
&\quad = \left(1 - t\{2(\overline{\gamma} - \gamma\ k) - t(\|G\| + \gamma\ k)^2\}\right)\|x-y\|^2.
\end{aligned}$$

This implies that $I - t(G - \gamma g)$ is a contraction. □

Lemma 3.2. *Let* H *be a Hilbert space and let* C *be a non-empty, closed and convex subset of* H. *Let* g *be a* k*-contraction of* H *into itself and let* G *be a strongly positive bounded linear self-adjoint operator on* H *with coefficient* $\overline{\gamma} > 0$. *Take* $\gamma > 0$ *with* $\gamma < \frac{\overline{\gamma}}{k}$ *and* $t > 0$ *with* $t(\|G\| + \gamma\ k)^2 < 2(\overline{\gamma} - \gamma\ k)$ *and* $2t(\overline{\gamma} - \gamma\ k) < 1$. *Let* $w \in C$. *Then the following are equivalent:*

(1) $w = P_C(I - t(G - \gamma g))w$;
(2) $\langle (G - \gamma g)w, w - q\rangle \leq 0, \quad \forall q \in C$;
(3) $w = P_C(I - G + \gamma g)w$.

Such $w \in C$ exists always and is unique.

Proof. We have that for $w \in C$

$$\begin{aligned} w = P_C(I - t(G - \gamma g))w &\Leftrightarrow \langle w - t(G - \gamma g)w - w, w - q\rangle \geq 0, \quad \forall q \in C \\ &\Leftrightarrow \langle (G - \gamma g)w, w - q\rangle \leq 0, \quad \forall q \in C \\ &\Leftrightarrow \langle w - Gw + \gamma gw - w, w - q\rangle \geq 0, \quad \forall q \in C \\ &\Leftrightarrow w = P_C(I - G + \gamma g)w. \end{aligned}$$

Then (1), (2) and (3) are equivalent. We also have from Lemma 3.1 that $I - t(G - \gamma g)$ is a contraction of H into itself. Then $P_C(I - t(G - \gamma g))$ is a contraction of C into itself. Therefore, such $w \in C$ exists always and is unique. □

In the proof of the main theorem, we also need properties of firmly nonexpansive mappings in a Hilbert space. Let C be a non-empty, closed and convex subset of a Hilbert space H. If a mapping $T : C \to H$ is firmly nonexpansive, i.e., $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle$ for all $x, y \in C$, then $I - T : C \to H$ is also firmly nonexpansive. In fact, put $S = I - T$. Since T is firmly nonexpansive, we have that

$$\|(I - S)x - (I - S)y\|^2 \leq \langle x - y, (I - S)x - (I - S)y\rangle$$

for all $x, y \in C$. Then we have that

$$\|x - y\|^2 - 2\langle x - y, Sx - Sy\rangle + \|Sx - Sy\|^2 \leq \|x - y\|^2 - \langle x - y, Sx - Sy\rangle$$

and hence $\|Sx - Sy\|^2 \leq \langle x - y, Sx - Sy\rangle$. This implies that

$$\|(I - T)x - (I - T)y\|^2 \leq \langle x - y, (I - T)x - (I - T)y\rangle. \tag{3.1}$$

Furthermore, we have the following result for maximal monotone mappings in a Hilbert space.

Lemma 3.3. *Let H be a Hilbert space and let A be a maximal monotone mapping on H such that $A^{-1}0$ is non-empty. Let $J_\lambda = (I + \lambda A)^{-1}$ be the resolvent of A for $\lambda > 0$. Then*

$$\langle x - J_\lambda x, J_\lambda x - y\rangle \geq 0$$

for all $x \in H$ and $y \in A^{-1}0$.

Proof. We know that J_λ is firmly nonexpansive and $J_\lambda y = y$ for all $y \in A^{-1}0$. Then we have that for all $x \in H$ and $y \in A^{-1}0$

$$\begin{aligned} \langle x - J_\lambda x, J_\lambda x - y\rangle &= \langle x - y + y - J_\lambda x, J_\lambda x - y\rangle \\ &= \langle x - y, J_\lambda x - y\rangle + \langle y - J_\lambda x, J_\lambda x - y\rangle \\ &\geq \|J_\lambda x - y\|^2 - \|J_\lambda x - y\|^2 \\ &= 0. \end{aligned}$$

This completes the proof. □

Using Lemma 3.3, we prove the following result.

Lemma 3.4. *Let H_1 and H_2 be Hilbert spaces and let A and B be maximal monotone mappings on H_1 and H_2 such that $A^{-1}0$ and $B^{-1}0$ are non-empty, respectively. Let $T : H_1 \to H_2$ be a bounded linear operator such that $A^{-1}0 \cap T^{-1}(B^{-1}0)$ is non-empty and let T^* be the adjoint operator of T. Let J_λ and Q_μ be the resolvents of A for $\lambda > 0$ and B for $\mu > 0$, respectively. Let $\lambda, \mu, \nu, r > 0$ and $z \in H$. Then the following are equivalent:*

(i) $z = J_\lambda(I - rT^*(I - Q_\mu)T)z$;
(ii) $0 \in T^*(I - Q_\nu)Tz + Az$;
(iii) $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$.

Proof. Since $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$, there exists $z_0 \in A^{-1}0$ such that $Tz_0 \in B^{-1}0$.

(i) $\Rightarrow$ (iii). From $z = J_\lambda(I - rT^*(I - Q_\mu)T)z$ and the definition of J_λ, we have

$$z - rT^*(I - Q_\mu)Tz \in z + \lambda Az$$

and hence $-rT^*(I - Q_\mu)Tz \in \lambda Az$. Since A is monotone and $0 \in Az_0$, we have

$$\left\langle -\frac{r}{\lambda}T^*(I - Q_\mu)Tz, z - z_0 \right\rangle \geq 0.$$

Then we have that $\langle T^*(I - Q_\mu)Tz, z - z_0\rangle \leq 0$ and hence

$$\langle Tz - Q_\mu Tz, Tz - Tz_0\rangle \leq 0. \tag{3.2}$$

On the other hand, we have from Lemma 3.3 that $\langle Tz - Q_\mu Tz, Q_\mu Tz - Tz_0\rangle \geq 0$ and hence

$$\langle Tz - Q_\mu Tz, Tz_0 - Q_\mu Tz\rangle \leq 0. \tag{3.3}$$

From (3.2) and (3.4) we have that

$$\|Tz - Q_\mu Tz\|^2 = \langle Tz - Q_\mu Tz, Tz - Q_\mu Tz\rangle \leq 0 \tag{3.4}$$

and hence $Tz = Q_\mu Tz$. This implies that $Tz \in B^{-1}0$. We also have from $z = J_\lambda(I - rT^*(I - Q_\mu)T)z$ that $z = J_\lambda(z - rT^*(I - Q_\mu)Tz) = J_\lambda z$. This implies that $z \in A^{-1}0$. Therefore $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$.

(ii) $\Rightarrow$ (iii). From $0 \in T^*(I - Q_\nu)Tz + Az$, we have $-T^*(I - Q_\nu)Tz \in Az$. Since A is monotone and $0 \in Az_0$, we have that

$$\langle -T^*(I - Q_\nu)Tz, z - z_0\rangle \geq 0.$$

Thus we have that $\langle T^*(I - Q_\nu)Tz, z - z_0\rangle \leq 0$ and hence

$$\langle Tz - Q_\nu Tz, Tz - Tz_0\rangle \leq 0. \tag{3.5}$$

As in the proof of (i) $\Rightarrow$ (iii), we have

$$\langle Tz - Q_\nu Tz, Q_\nu Tz - Tz_0\rangle \geq 0. \tag{3.6}$$

From (3.5) and (3.6) we have that

$$\|Tz - Q_\nu Tz\|^2 = \langle Tz - Q_\nu Tz, Tz - Q_\nu Tz\rangle \leq 0 \tag{3.7}$$

and hence $Tz = Q_\nu Tz$. This implies that $Tz \in B^{-1}0$. As in the proof of (i) $\Rightarrow$ (iii), we have $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$.

(iii) $\Rightarrow$ (i). From $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, we have that $Tz \in B^{-1}0$ and $z \in A^{-1}0$. This implies that $Tz = Q_\mu Tz$ and $z = J_\lambda z$. Thus we have

$$J_\lambda(I - rT^*(I - Q_\mu)T)z = J_\lambda z - rT^*0 = J_\lambda z = z.$$

(iii) ⇒ (ii). From $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, we have that $Tz = Q_\nu Tz$ and $0 \in Az$. Thus we have $0 \in T^*(I - Q_\nu)Tz + Az$. The proof is complete. □

We also have the following lemma.

Lemma 3.5. *Let H_1 and H_2 be Hilbert spaces and let $\alpha > 0$. Let $T : H_1 \to H_2$ be a bounded linear operator such that $T \neq 0$. Let $S : H_2 \to H_2$ be an α-inverse strongly monotone mapping. Then a mapping $T^*ST : H_1 \to H_1$ is $\frac{\alpha}{\|TT^*\|}$-inverse strongly monotone.*

Proof. Since S is α-inverse strongly monotone, we have that for all $x, y \in H_1$

$$\begin{aligned}
\frac{\alpha}{\|TT^*\|}\|T^*STx - T^*STy\|^2 &= \frac{\alpha}{\|TT^*\|}\langle T^*STx - T^*STy, T^*STx - T^*STy\rangle \\
&= \frac{\alpha}{\|TT^*\|}\langle TT^*\big(STx - STy\big), STx - STy\rangle \\
&\leq \frac{\alpha}{\|TT^*\|}\|TT^*\big(STx - STy\big)\|\|STx - STy\| \\
&\leq \frac{\alpha}{\|TT^*\|}\|TT^*\|\|STx - STy\|^2 \\
&= \alpha\|STx - STy\|^2 \\
&\leq \langle STx - STy, Tx - Ty\rangle \\
&= \langle T^*STx - T^*STy, x - y\rangle.
\end{aligned}$$

This implies that $T^*ST : H_1 \to H_1$ is $\frac{\alpha}{\|TT^*\|}$-inverse strongly monotone. □

Remark. If B is a maximal monotone mapping on H_2 and Q_μ is the resolvent of B for $\mu > 0$, then Q_μ is a firmly nonexpansive mapping. Using (3.1), we also have that $(I - Q_\mu)$ is firmly nonexpansive, i.e., 1-inverse strongly monotone. Thus we have that $T^*(I - Q_\mu)T$ is $\frac{1}{\|TT^*\|}$-inverse strongly monotone. This fact is used in the proof of our main theorem.

4. Strong convergence theorem

Let C be a non-empty, closed and convex subset of a Hilbert space H. Let $\alpha > 0$ and let U be an α-inverse strongly monotone mapping of C into H. If $0 < \lambda \leq 2\alpha$, then $I - \lambda U : C \to H$ is nonexpansive. In fact, we have that for all $x, y \in C$

$$\begin{aligned}
\|(I - \lambda U)x - (I - \lambda U)y\|^2 &= \|x - y - \lambda(Ux - Uy)\|^2 \\
&= \|x - y\|^2 - 2\lambda\langle x - y, Ux - Uy\rangle + (\lambda)^2\|Ux - Uy\|^2 \\
&\leq \|x - y\|^2 - 2\lambda\alpha\|Ux - Uy\|^2 + (\lambda)^2\|Ux - Uy\|^2 \\
&= \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ux - Uy\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

Thus $I - \lambda U : C \to H$ is nonexpansive. Now we can prove a strong convergence theorem of Browder's type [3] which solves the split common null point problem in Hilbert spaces. For proving the theorem, we need another lemma obtained by Marino and Xu [8].

Lemma 4.1 ([8]). *Let H be a Hilbert space and let G be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. If $0 < \gamma \leq \|G\|^{-1}$, then $\|I - \gamma G\| \leq 1 - \gamma\overline{\gamma}$.*

Theorem 4.2. *Let H_1 and H_2 be Hilbert spaces. Let A and F be maximal monotone mappings on H_1 and let B be a maximal monotone mapping on H_2 such that $A^{-1}0$, $F^{-1}0$ and $B^{-1}0$ are non-empty. Let $T : H_1 \to H_2$ be a bounded linear operator such that $A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0$ is non-empty. Let T^* be the adjoint operator of T. Let J_λ and T_r be the resolvents of A for $\lambda > 0$ and F for $r > 0$, respectively and let Q_μ be the resolvent of B for $\mu > 0$. Let $0 < k < 1$ and let g be a k-contraction of H_1 into itself. Let G be a strongly positive bounded linear self-adjoint operator on H_1 with coefficient $\overline{\gamma} > 0$. Let $0 < \gamma < \frac{\overline{\gamma}}{k}$. Assume that $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,\infty)$ and $\{r_n\} \subset (0,\infty)$ satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \quad 0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{\|TT^*\|},$$

$$\liminf_{n\to\infty} \mu_n > 0 \quad \text{and} \quad \liminf_{n\to\infty} r_n > 0.$$

Then the following hold:

(i) *For any $n \in \mathbb{N}$, define $T_n : H_1 \to H_1$ by*

$$T_n x = \alpha_n \gamma g(x) + (I - \alpha_n G) J_{\lambda_n}(I - \lambda_n T^*(I - Q_{\mu_n})T) T_{r_n} x, \quad \forall x \in H_1.$$

Then T_n has a unique fixed point x_n in H_1 and $\{x_n\}$ is bounded;

(ii) *the set $A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0$ is a non-empty, closed and convex subset of H_1 and $P_{A^{-1}0\cap T^{-1}(B^{-1}0)\cap F^{-1}0}(I - G + \gamma g)$ has a unique fixed point z_0 in $A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0$;*

(iii) *the sequence $\{x_n\}$ converges strongly to $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0$, where $\{z_0\} = VI(A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0, G - \gamma g)$.*

Proof. Let us prove (i). For any $n \in \mathbb{N}$, define $A_n = T^*(I - Q_{\mu_n})T$. Then $T_n : H_1 \to H_1$ is written by

$$T_n x = \alpha_n \gamma g(x) + (I - \alpha_n G) J_{\lambda_n}(I - \lambda_n A_n) T_{r_n} x, \quad \forall x \in H_1.$$

From $\lim_{n\to\infty} \alpha_n = 0$, we may have $\alpha_n \leq \|G\|^{-1}$. Then we have from Lemma 4.1 that for any $x, y \in H_1$

$$\begin{aligned}
\|T_n x - T_n y\| &= \|\alpha_n \gamma g(x) + (I - \alpha_n G) J_{\lambda_n}(I - \lambda_n A_n) T_{r_n} x \\
&\qquad - \{\alpha_n \gamma g(y) + (I - \alpha_n G) J_{\lambda_n}(I - \lambda_n A_n) T_{r_n} y\}\| \\
&\leq \alpha_n \gamma \|g(x) - g(y)\| \\
&\qquad + \|I - \alpha_n G\| \|J_{\lambda_n}(I - \lambda_n A_n) T_{r_n} x - J_{\lambda_n}(I - \lambda_n A_n) T_{r_n} y\| \\
&\leq \alpha_n \gamma\ k \|x - y\| + (1 - \alpha_n \overline{\gamma}) \|(I - \lambda_n A_n) T_{r_n} x - (I - \lambda_n A_n) T_{r_n} y\| \\
&\leq \alpha_n \gamma\ k \|x - y\| + (1 - \alpha_n \overline{\gamma}) \|T_{r_n} x - T_{r_n} y\| \\
&\leq \alpha_n \gamma\ k \|x - y\| + (1 - \alpha_n \overline{\gamma}) \|x - y\| \\
&= (\alpha_n \gamma\ k + 1 - \alpha_n \overline{\gamma}) \|x - y\| \\
&= (1 - \alpha_n (\overline{\gamma} - \gamma\ k)) \|x - y\|.
\end{aligned}$$

Since $0 < 1 - \alpha_n(\overline{\gamma} - \gamma\ k) < 1$, T_n is a $(1 - \alpha_n(\overline{\gamma} - \gamma\ k))$-contraction of H_1 into itself and hence T_n has a unique fixed point x_n in H_1. Next we show that $\{x_n\}$ is bounded. Let $u \in A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0$. We have from $F^{-1}0 = F(T_{r_n})$ and Lemma 3.4 that $T_{r_n}u = u$ and $J_{\lambda_n}(I - \lambda_n A_n)u = u$. Using $u = \alpha_n Gu + u - \alpha_n Gu$, we have that for all $n \in \mathbb{N}$

$$\begin{aligned}\|x_n - u\| &= \|T_n x_n - \alpha_n Gu - u + \alpha_n Gu\| \\ &= \|\alpha_n(\gamma g(x_n) - Gu) + (I - \alpha_n G)(J_{\lambda_n}(I - \lambda_n A_n)T_{r_n}x_n - u)\| \\ &\le \alpha_n\|\gamma g(x_n) - Gu\| + \|I - \alpha_n G\|\|J_{\lambda_n}(I - \lambda_n A_n)T_{r_n}x_n - u\| \\ &\le \alpha_n\gamma\ k\|x_n - u\| + \alpha_n\|\gamma g(u) - Gu\| + (1 - \alpha_n\overline{\gamma})\|x_n - u\|.\end{aligned}$$

Thus we have $\alpha_n(\overline{\gamma} - \gamma\ k)\|x_n - u\| \le \alpha_n\|\gamma g(u) - Gu\|$ and hence

$$(\overline{\gamma} - \gamma\ k)\|x_n - u\| \le \|\gamma g(u) - Gu\|.$$

Then we have $\|x_n - u\| \le \frac{\|\gamma g(u) - Gu\|}{\overline{\gamma} - \gamma\ k}$. This implies that $\{x_n\}$ is bounded.

Let us prove (ii). Since A, F and B are maximal monotone operators, we have from [15] that $A^{-1}0, F^{-1}0$ and $B^{-1}0$ are closed and convex. Furthermore, since T is a bounded linear operator from H_1 to H_2, it is obvious that $T^{-1}B^{-1}0$ is closed and convex. Therefore, $A^{-1}0 \cap T^{-1}B^{-1}0 \cap F^{-1}0$ is closed and convex. We also have from Lemma 3.2 that $P_{A^{-1}0\cap T^{-1}B^{-1}0\cap F^{-1}0}(I - G + \gamma g)$ has a unique fixed point z_0 in $A^{-1}0 \cap T^{-1}B^{-1}0 \cap F^{-1}0$.

Let us prove (iii). Put $y_n = J_{\lambda_n}(I - \lambda_n A_n)T_{r_n}x_n$ and $u_n = T_{r_n}x_n$ for all $n \in \mathbb{N}$. Since $\{x_n\}$ is bounded, $\{u_n\}$ and $\{y_n\}$ are bounded. Furthermore, $\{g(x_n)\}$ and $\{Gx_n\}$ are also bounded. Let $z \in A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0$. We have from $z \in T^{-1}(B^{-1}0)$ that $(I - Q_{\mu_n})Tz = 0$ and hence $A_n z = T^*(I - Q_{\mu_n})Tz = 0$. Furthermore, we have from Lemma 3.5 and $0 < \limsup_{n\to\infty}\lambda_n < \frac{2}{\|TT^*\|}$ that

$$\begin{aligned}\|y_n - z\|^2 &= \|J_{\lambda_n}(I - \lambda_n A_n)u_n - J_{\lambda_n}(I - \lambda_n A_n)z\|^2 \\ &\le \|(I - \lambda_n A_n)u_n - (I - \lambda_n A_n)z\|^2 \\ &= \|u_n - z - \lambda_n A_n u_n\|^2 \\ &= \|u_n - z\|^2 - 2\lambda_n\langle u_n - z, A_n u_n\rangle + (\lambda_n)^2\|A_n u_n\|^2 \\ &= \|u_n - z\|^2 - 2\lambda_n\langle Tu_n - Tz, (I - Q_{\mu_n})Tu_n\rangle + (\lambda_n)^2\|A_n u_n\|^2 \\ &\le \|u_n - z\|^2 - 2\lambda_n\|(I - Q_{\mu_n})Tu_n\|^2 + (\lambda_n)^2\|TT^*\|\|(I - Q_{\mu_n})Tu_n\|^2 \\ &= \|u_n - z\|^2 + \lambda_n(\lambda_n\|TT^*\| - 2)\|(I - Q_{\mu_n})Tu_n\|^2 \\ &\le \|u_n - z\|^2\end{aligned} \tag{4.1}$$

and hence $\|y_n - z\| \le \|u_n - z\|$. We also have that

$$\begin{aligned}\|u_n - y_n\| &\le \|u_n - x_n\| + \|x_n - y_n\| \\ &= \|u_n - x_n\| + \|\alpha_n\gamma g(x_n) + (I - \alpha_n G)y_n - y_n\| \\ &= \|u_n - x_n\| + \alpha_n\|\gamma g(x_n) - Gy_n\|.\end{aligned} \tag{4.2}$$

Furthermore, using (2.5) and (2.3), we get that

$$2\|u_n - z\|^2 = 2\|T_{r_n}x_n - T_{r_n}z\|^2$$

$$\leq 2\langle x_n - z, u_n - z\rangle$$
$$= \|x_n - z\|^2 + \|u_n - z\|^2 - \|u_n - x_n\|^2$$

and hence

$$\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|u_n - x_n\|^2. \tag{4.3}$$

Since $z = \alpha_n Gz + z - \alpha_n Gz$, we have from (2.1), (4.1) and (4.3) that

$$\begin{aligned}
\|x_n - z\|^2 &= \|(I - \alpha_n G)(y_n - z) + \alpha_n(\gamma g(x_n) - Gz)\|^2 \\
&\leq (1 - \alpha_n\overline{\gamma})^2 \|y_n - z\|^2 + 2\alpha_n\langle \gamma g(x_n) - Gz, x_n - z\rangle \\
&\leq (1 - \alpha_n\overline{\gamma})^2(\|u_n - z\|^2 + \lambda_n(\lambda_n\|TT^*\| - 2)\|(I - Q_{\mu_n})Tu_n\|^2) \\
&\quad + 2\alpha_n\langle \gamma g(x_n) - Gz, x_n - z\rangle \\
&= (1 - \alpha_n\overline{\gamma})^2 \|u_n - z\|^2 + (1 - \alpha_n\overline{\gamma})^2\lambda_n(\lambda_n\|TT^*\| - 2)\|(I - Q_{\mu_n})Tu_n\|^2 \\
&\quad + 2\alpha_n\langle \gamma g(x_n) - Gz, x_n - z\rangle \\
&\leq (1 - \alpha_n\overline{\gamma})^2(\|x_n - z\|^2 - \|x_n - u_n\|^2) \\
&\quad + (1 - \alpha_n\overline{\gamma})^2\lambda_n(\lambda_n\|TT^*\| - 2)\|(I - Q_{\mu_n})Tu_n\|^2 \\
&\quad + 2\alpha_n\gamma\ k\|x_n - z\|^2 + 2\alpha_n\|\gamma g(z) - Gz\|\|x_n - z\| \\
&= \{1 - 2\alpha_n(\overline{\gamma} - \gamma\ k) + \alpha_n^2\overline{\gamma}^2\}\ \|x_n - z\|^2 - (1 - \alpha_n\overline{\gamma})^2\|x_n - u_n\|^2 \\
&\quad + (1 - \alpha_n\overline{\gamma})^2\lambda_n(\lambda_n\|TT^*\| - 2)\|(I - Q_{\mu_n})Tu_n\|^2 \\
&\quad + 2\alpha_n\|\gamma g(z) - Gz\|\|x_n - z\| \\
&\leq \|x_n - z\|^2 + \alpha_n^2\overline{\gamma}^2\ \|x_n - z\|^2 - (1 - \alpha_n\overline{\gamma})^2\|x_n - u_n\|^2 \\
&\quad + (1 - \alpha_n\overline{\gamma})^2\lambda_n(\lambda_n\|TT^*\| - 2)\|(I - Q_{\mu_n})Tu_n\|^2 \\
&\quad + 2\alpha_n\|\gamma g(z) - Gz\|\|x_n - z\|
\end{aligned}$$

and hence

$$\begin{aligned}
(1 - \alpha_n\overline{\gamma})^2\lambda_n(2 - \lambda_n\|TT^*\|)\|(I - Q_{\mu_n})Tu_n\|^2 + (1 - \alpha_n\overline{\gamma})^2\|x_n - u_n\|^2 \\
\leq \alpha_n^2\overline{\gamma}^2\ \|x_n - z\|^2 + 2\alpha_n\|\gamma g(z) - Gz\|\|x_n - z\|.
\end{aligned}$$

Then we have that

$$\begin{aligned}
(1 - \alpha_n\overline{\gamma})^2\lambda_n(2 - \lambda_n\|TT^*\|)\|(I - Q_{\mu_n})Tu_n\|^2 \\
\leq \alpha_n^2\overline{\gamma}^2\ \|x_n - z\|^2 + 2\alpha_n\|\gamma g(z) - Gz\|\|x_n - z\|.
\end{aligned}$$

and

$$(1 - \alpha_n\overline{\gamma})^2\|x_n - u_n\|^2 \leq \alpha_n^2\overline{\gamma}^2\ \|x_n - z\|^2 + 2\alpha_n\|\gamma g(z) - Gz\|\|x_n - z\|.$$

From $\lim_{n\to\infty}\alpha_n = 0$ and $0 < \liminf_{n\to\infty}\lambda_n \leq \limsup_{n\to\infty}\lambda_n < \frac{2}{\|TT^*\|}$, we have

$$\|(I - Q_{\mu_n})Tu_n\| \to 0 \quad \text{and} \quad \|x_n - u_n\| \to 0. \tag{4.4}$$

We have from (4.2) and (4.4) that

$$\|Tu_n - Q_{\mu_n}Tu_n\| \to 0 \quad \text{and} \quad \|y_n - u_n\| \to 0. \tag{4.5}$$

Take $\lambda_0 \in (0, \frac{2}{\|TT^*\|})$. Putting $A_\mu = T^*(I - Q_\mu)T$ and $z_n = (I - \lambda_n A_n)u_n$, where $0 < \mu < \liminf_{n\to\infty} \mu_n$, we have from (2.6) and (2.7) that

$$\begin{aligned}
&\|J_{\lambda_0}(I-\lambda_0 A_\mu)u_n - y_n\| \\
&\quad \le \|J_{\lambda_0}(I - \lambda_0 A_\mu)u_n - J_{\lambda_0}(I - \lambda_n A_n)u_n\| + \|J_{\lambda_0}(I - \lambda_n A_n)u_n - y_n\| \\
&\quad \le \|(I - \lambda_0 A_\mu)u_n - (I - \lambda_n A_n)u_n\| + \|J_{\lambda_0} z_n - J_{\lambda_n} z_n\| \\
&\quad = \|\lambda_0 A_\mu u_n - \lambda_n A_n u_n\| + \|J_{\lambda_0} z_n - J_{\lambda_n} z_n\| \\
&\quad = \|\lambda_0 A_\mu u_n - \lambda_0 A_n u_n + \lambda_0 A_n u_n - \lambda_n A_n u_n\| + \|J_{\lambda_0} z_n - J_{\lambda_n} z_n\| \\
&\quad \le \lambda_0\|T\|\|(I - Q_\mu)Tu_n - (I - Q_{\mu_n})Tu_n\| \\
&\qquad + \|\lambda_0 A_n u_n - \lambda_n A_n u_n\| + \|J_{\lambda_0} z_n - J_{\lambda_n} z_n\| \\
&\quad \le \lambda_0\|T\|(\|(I - Q_\mu)Tu_n\| + \|(I - Q_{\mu_n})Tu_n\|) \\
&\qquad + \|\lambda_0 A_n u_n - \lambda_n A_n u_n\| + \|J_{\lambda_0} z_n - J_{\lambda_n} z_n\| \\
&\quad \le 2\lambda_0\|T\|\|(I - Q_{\mu_n})Tu_n\|) + \|\lambda_0 A_n u_n - \lambda_n A_n u_n\| + \|J_{\lambda_0} z_n - J_{\lambda_n} z_n\| \\
&\quad \le 2\lambda_0\|T\|\|(I - Q_{\mu_n})Tu_n\|) + |\lambda_n - \lambda_0|\|A_n u_n\| + \frac{|\lambda_n - \lambda_0|}{\lambda_0}\|J_{\lambda_0} z_n - z_n\|.
\end{aligned} \tag{4.6}$$

Furthermore, we have that

$$\|J_{\lambda_0}(I - \lambda_0 A_\mu)u_n - u_n\| \le \|J_{\lambda_0}(I - \lambda_0 A_\mu)u_n - y_n\| + \|y_n - u_n\|. \tag{4.7}$$

We will use these inequalities (4.6) and (4.7) later. We know from (ii) and Lemma 3.2 that there exists a unique $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0$ such that

$$\langle (G - \gamma g)z_0, q - z_0\rangle \ge 0, \quad \forall q \in A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0.$$

In order to show that $x_n \to z_0$, it suffices to show that if $\{x_{n_i}\}$ is any subsequence of $\{x_n\}$, then we can find a subsequence of $\{x_{n_i}\}$ converging strongly to z_0; see [15, p. 28]. Since $\{x_{n_i}\}$ is bounded and $\{\lambda_{n_i}\}$ is bounded, without loss of generality there exist a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ and a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ such that $x_{n_{i_j}} \rightharpoonup w$ and $\lambda_{n_{i_j}} \to \lambda_0$ for some $\lambda_0 \in (0, \frac{2}{\|TT^*\|})$. From $x_n - u_n \to 0$, we have $u_{n_{i_j}} \rightharpoonup w$. Using $\lambda_{n_{i_j}} \to \lambda_0$, (4.4) and (4.6), we have that

$$\|J_{\lambda_0}(I - \lambda_0 A_\mu)u_{n_{i_j}} - y_{n_{i_j}}\| \to 0.$$

Furthermore, we have from $\|y_{n_{i_j}} - u_{n_{i_j}}\| \to 0$ and (4.7) that

$$\|J_{\lambda_0}(I - \lambda_0 A_\mu)u_{n_{i_j}} - u_{n_{i_j}}\| \to 0.$$

Since $J_{\lambda_0}(I - \lambda_0 A_\mu)$ is nonexpansive, we have that $w = J_{\lambda_0}(I - \lambda_0 A_\mu)w$ and hence $w \in A^{-1}0 \cap T^{-1}(B^{-1}0)$ from Lemma 3.4. We show $w \in F^{-1}0$. Since F is a maximal monotone operator, we have from (2.4) that $A_{r_{n_{i_j}}} x_{n_{i_j}} \in FT_{r_{n_{i_j}}} x_{n_{i_j}}$, where A_r is the Yosida approximation of F for $r > 0$. Furthermore we have that for any $(u, v) \in F$

$$\left\langle u - u_{n_{i_j}}, v - \frac{x_{n_{i_j}} - u_{n_{i_j}}}{r_{n_{i_j}}} \right\rangle \ge 0.$$

Since $\liminf_{n\to\infty} r_n > 0$, $u_{n_{i_j}} \rightharpoonup w$ and $x_{n_{i_j}} - u_{n_{i_j}} \to 0$, we have

$$\langle u - w, v\rangle \ge 0.$$

Since F is a maximal monotone operator, we have from [15, Theorem 6.5.4] that $0 \in Fw$ and hence $w \in F^{-1}0$. Thus we have $w \in A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0$. Finally, we show $x_{n_{i_j}} \to z_0$. For showing $x_{n_{i_j}} \to z_0$, we follow the idea of [3]; see also the proof of [15, Theorem 6.3.1]. Since $z_0 = \alpha_n G z_0 + z_0 - \alpha_n G z_0$, we have that

$$x_n - z_0 = \alpha_n(\gamma g(x_n) - Gz_0) + (I - \alpha_n G)(y_n - z_0).$$

Using $\|y_n - z_0\| \leq \|u_n - z_0\| \leq \|x_n - z_0\|$, we have that

$$\begin{aligned}
\|x_n - z_0\|^2 &= \langle x_n - z_0, x_n - z_0\rangle \\
&= \alpha_n \langle \gamma g(x_n) - Gz_0, x_n - z_0\rangle + \langle (I - \alpha_n G)(y_n - z_0), x_n - z_0\rangle \\
&\leq \alpha_n \langle \gamma g(x_n) - Gz_0, x_n - z_0\rangle + \|I - \alpha_n G\|\|y_n - z_0\|\|x_n - z_0\| \\
&\leq \alpha_n \langle \gamma g(x_n) - Gz_0, x_n - z_0\rangle + (1 - \alpha_n \overline{\gamma})\|x_n - z_0\|^2.
\end{aligned}$$

Thus we have that $\alpha_n \overline{\gamma}\|x_n - z_0\|^2 \leq \alpha_n \langle \gamma g(x_n) - Gz_0, x_n - z_0\rangle$ and hence

$$\overline{\gamma}\|x_n - z_0\|^2 \leq \langle \gamma g(x_n) - Gz_0, x_n - z_0\rangle.$$

Then we have that

$$\begin{aligned}
\|x_n - z_0\|^2 &\leq \frac{1}{\overline{\gamma}}\langle \gamma g(x_n) - Gz_0, x_n - z_0\rangle \\
&= \frac{1}{\overline{\gamma}}\langle \gamma g(x_n) - \gamma g(z_0) + \gamma g(z_0) - Gz_0, x_n - z_0\rangle \\
&\leq \frac{1}{\overline{\gamma}}\gamma\ k\|x_n - z_0\|^2 + \frac{1}{\overline{\gamma}}\langle \gamma g(z_0) - Gz_0, x_n - z_0\rangle.
\end{aligned}$$

This implies that

$$\|x_n - z_0\|^2 \leq \frac{\langle \gamma g(z_0) - Gz_0, x_n - z_0\rangle}{\overline{\gamma} - \gamma\ k}.$$

In particular, we have that

$$\|x_{n_{i_j}} - z_0\|^2 \leq \frac{\langle \gamma g(z_0) - Gz_0, x_{n_{i_j}} - z_0\rangle}{\overline{\gamma} - \gamma\ k}.$$

From $x_{n_{i_j}} \rightharpoonup w$ we have that

$$\begin{aligned}
\limsup_{j\to\infty} \|x_{n_{i_j}} - z_0\|^2 &\leq \lim_{j\to\infty} \frac{\langle \gamma g(z_0) - Gz_0, x_{n_{i_j}} - z_0\rangle}{\overline{\gamma} - \gamma\ k} \\
&= \frac{\langle \gamma g(z_0) - Gz_0, w - z_0\rangle}{\overline{\gamma} - \gamma\ k}.
\end{aligned}$$

Furthermore, since $w \in A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0$ and

$$\langle (G - \gamma g)z_0, q - z_0\rangle \geq 0, \quad \forall q \in A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0,$$

we have that $\limsup_{j\to\infty} \|x_{n_{i_j}} - z_0\|^2 \leq 0$. Thus $x_{n_{i_j}} \to z_0$. Therefore, we have that $\{x_n\}$ converges strongly to a unique $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0$ such that

$$\langle (G - \gamma g)z_0, q - z_0\rangle \geq 0, \quad \forall q \in A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0.$$

We know that this z_0 is a unique fixed point of $P_{A^{-1}0 \cap T^{-1}(B^{-1}0) \cap F^{-1}0}(I - G + \gamma g)$. This completes the proof. $\square$

5. Applications

In this section, using Theorem 4.2, we obtain two new strong convergence theorems which are related to the split inverse problem and an equilibrium problem in Hilbert spaces. Let H be a Hilbert space and let f be a proper lower semicontinuous convex function of H into $(-\infty, \infty]$. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x\rangle \leq f(y),\ \forall y \in H\}$$

for all $x \in H$. From Rockafellar [10], we know that ∂f is a maximal monotone operator. Let C be a non-empty, closed and convex subset of H and let i_C be the indicator function of C, i.e.,

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then i_C is a proper, lower semicontinuous and convex function on H. Thus the subdifferential ∂i_C of i_C is a maximal monotone operator. We can define the resolvent J_λ of ∂i_C for $\lambda > 0$, i.e.,

$$J_\lambda x = (I + \lambda \partial i_C)^{-1} x$$

for all $x \in H$. We have that for any $x \in H$ and $u \in C$

$$\begin{aligned} u = & J_\lambda x \iff x \in u + \lambda \partial i_C u \iff x \in u + \lambda N_C u \\ & \iff x - u \in \lambda N_C u \\ & \iff \frac{1}{\lambda}\langle x - u, v - u\rangle \leq 0,\ \forall v \in C \\ & \iff \langle x - u, v - u\rangle \leq 0,\ \forall v \in C \\ & \iff u = P_C x, \end{aligned}$$

where $N_C u$ is the normal cone to C at u, i.e.,

$$N_C u = \{z \in H : \langle z, v - u\rangle \leq 0,\ \forall v \in C\}.$$

Theorem 5.1. *Let H_1 and H_2 be Hilbert spaces. Let C and D be non-empty, closed and convex subsets of H_1 and let Q be a non-empty, closed and convex subset of H_2. Let $T : H_1 \to H_2$ be a bounded linear operator such that $C \cap T^{-1}Q \cap D$ is non-empty. Let T^* be the adjoint operator of T. Let P_C and P_D be the metric projections of H_1 onto C and D, respectively and let P_Q be the metric projection of H_2 onto Q. Let $0 < k < 1$ and let g be a k-contraction of H_1 into itself. Let G be a strongly positive bounded linear self-adjoint operator on H_1 with coefficient $\overline{\gamma} > 0$. Let $0 < \gamma < \frac{\overline{\gamma}}{k}$. Assume that $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\} \subset (0, \infty)$ and $\{r_n\} \subset (0, \infty)$ satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \text{and} \quad 0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{\|TT^*\|}.$$

Then the following hold:

(i) *For any $n \in \mathbb{N}$, define $T_n : H_1 \to H_1$ by*

$$T_n x = \alpha_n \gamma g(x) + (I - \alpha_n G) P_C (I - \lambda_n T^*(I - P_Q)T) P_D x, \quad \forall x \in H_1.$$

Then T_n has a unique fixed point x_n in H_1 and $\{x_n\}$ is bounded;

(ii) *the set* $C \cap T^{-1}Q \cap D$ *is a non-empty, closed and convex subset of* H_1 *and* $P_{C\cap T^{-1}Q\cap D}(I - G + \gamma g)$ *has a unique fixed point* z_0 *in* $C \cap T^{-1}Q \cap D$*;*

(iii) *the sequence* $\{x_n\}$ *converges strongly to* $z_0 \in C \cap T^{-1}Q \cap D$*, where* $\{z_0\} = VI(C \cap T^{-1}Q \cap D, G - \gamma g)$*.*

Proof. Put $A = \partial i_C$, $F = \partial i_D$ and $B = \partial i_Q$ in Theorem 4.2. Then we have that for $\lambda_n > 0$, $r_n > 0$ and $\mu_n > 0$, $J_{\lambda_n} = P_C$, $T_{r_n} = P_D$ and $Q_{\mu_n} = P_Q$. Furthermore, we have $(\partial i_C)^{-1}0 = C$, $(\partial i_D)^{-1}0 = D$ and $(\partial i_Q)^{-1}0 = Q$. Thus we obtain the desired result by Theorem 4.2. □

Let H be a Hilbert space and let C be a non-empty, closed and convex subset of H. Let $f : C \times C \to \mathbb{R}$ be a bifunction. Then an equilibrium problem (with respect to C) is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{5.1}$$

The set of such solutions $\hat{x}$ is denoted by $EP(f)$, i.e.,

$$EP(f) = \{\hat{x} \in C : f(\hat{x}, y) \geq 0, \ \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction $f : C\times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$,

$$\limsup_{t\downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y);$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

We know the following lemma which appears in Blum and Oettli [2].

Lemma 5.2 ([2])**.** *Let* C *be a non-empty, closed and convex subset of* H *and let* f *be a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1) − (A4)*. Let* $r > 0$ *and* $x \in H$*. Then there exists* $z \in C$ *such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C.$$

The following lemma was also given in Combettes and Hirstoaga [6].

Lemma 5.3 ([6])**.** *Assume that* $f : C \times C \to \mathbb{R}$ *satisfies* (A1) − (A4)*. For* $r > 0$ *and* $x \in H$*, define a mapping* $T_r : H \to C$ *as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}$$

for all $x \in H$*. Then the following hold:*

(1) T_r *is single-valued;*

(2) T_r *is a firmly nonexpansive mapping, i.e., for all* $x, y \in H$*,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(f)$*;*

(4) $EP(f)$ *is closed and convex.*

We call such T_r the resolvent of f for $r > 0$. Using Lemmas 5.2 and 5.3, Takahashi, Takahashi and Toyoda [12] obtained the following lemma. See [1] for a more general result.

Lemma 5.4 ([12]). *Let H be a Hilbert space and let C be a non-empty, closed and convex subset of H. Let $f : C \times C \to \mathbb{R}$ satisfy $(A1) - (A4)$. Let A_f be a set-valued mapping of H into itself defined by*

$$A_f x = \begin{cases} \{z \in H : f(x,y) \geq \langle y - x, z\rangle, \ \forall y \in C\}, & \forall x \in C, \\ \emptyset, \quad \forall x \notin C. \end{cases}$$

Then $EP(f) = A_f^{-1}0$ and A_f is a maximal monotone operator with $\mathrm{dom}(A_f) \subset C$. *Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r of f coincides with the resolvent of A_f, i.e.,*

$$T_r x = (I + rA_f)^{-1}x.$$

Using Theorem 4.2, we can also prove a strong convergence theorem for finding solutions of equilibrium problems in Hilbert spaces.

Theorem 5.5. *Let H_1 and H_2 be Hilbert spaces. Let C and D be non-empty, closed and convex subsets of H_1 and let Q be a non-empty, closed and convex subset of H_2. Let f_1 and f_2 be bifunctions of $C \times C$ into $\mathbb{R}$ and $D \times D$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. Let f_3 be a bifunction of $Q \times Q$ into $\mathbb{R}$ satisfying $(A1) - (A4)$ such that $EP(f_1)$, $EP(f_2)$ and $EP(f_3)$ are non-empty. Let $T : H_1 \to H_2$ be a bounded linear operator such that $EP(f_1) \cap T^{-1}EP(f_3) \cap EP(f_2)$ is non-empty. Let T^* be the adjoint operator of T. Let J_λ and T_r be the resolvents of f_1 for $\lambda > 0$ and f_2 for $r > 0$, respectively and let Q_μ be the resolvent of f_3 for $\mu > 0$. Let $0 < k < 1$ and let g be a k-contraction of H_1 into itself. Let G be a strongly positive bounded linear self-adjoint operator on H_1 with coefficient $\overline{\gamma} > 0$. Let $0 < \gamma < \frac{\overline{\gamma}}{k}$. Assume that $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,\infty)$ and $\{r_n\} \subset (0,\infty)$ satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \quad 0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{\|TT^*\|},$$

$$\liminf_{n\to\infty} \mu_n > 0 \quad \textit{and} \quad \liminf_{n\to\infty} r_n > 0.$$

Then the following hold:

(i) *For any $n \in \mathbb{N}$, define $T_n : H_1 \to H_1$ by*

$$T_n x = \alpha_n \gamma g(x) + (I - \alpha_n G)J_{\lambda_n}(I - \lambda_n T^*(I - Q_{\mu_n})T)T_{r_n}x, \quad \forall x \in H_1.$$

Then T_n has a unique fixed point x_n in H_1 and $\{x_n\}$ is bounded;

(ii) *the set $EP(f_1) \cap T^{-1}EP(f_3) \cap EP(f_2)$ is a non-empty, closed and convex subset of H_1 and $P_{EP(f_1)\cap T^{-1}EP(f_3)\cap EP(f_2)}(I - G + \gamma g)$ has a unique fixed point z_0 in $EP(f_1) \cap T^{-1}EP(f_3) \cap EP(f_2)$;*

(iii) *the sequence $\{x_n\}$ converges strongly to $z_0 \in EP(f_1)\cap T^{-1}EP(f_3)\cap EP(f_2)$, where $\{z_0\} = VI(EP(f_1) \cap T^{-1}EP(f_3) \cap EP(f_2), G - \gamma g)$.*

Proof. For the bifunctions $f_1 : C \times C \to \mathbb{R}$, $f_2 : D \times D \to \mathbb{R}$ and $f_3 : Q \times Q \to \mathbb{R}$, we can define A_{f_1}, A_{f_2} and A_{f_3} in Lemma 5.4. Putting $A = A_{f_1}$, $F = A_{f_2}$ and $B = A_{f_3}$ in Theorem 4.2, we obtain from Lemma 5.4 that $J_{\lambda_n} = (I + \lambda_n A_{f_1})^{-1}$,

$T_{r_n} = (I + r_n A_{f_2})^{-1}$ and $Q_{\mu_n} = (I + \mu_n A_{f_3})^{-1}$ for all $\lambda_n > 0$, $r_n > 0$ and $\mu_n > 0$, respectively. Thus we obtain the desired result by Theorem 4.2. □

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