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EXISTENCE OF GENERALIZED BEST APPROXIMATIONS

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Abstract. In this paper we present further extension of the best approximations theorems obtained by Ky Fan, J. Prolla and A. Carbone.

1. Introduction

A short and simple proof of the Brouwer theorem was given in 1929 by Knaster, Kuratowski and Mazurkiewicz. This proof is based on one corollary of the Sperner's lemma which is known as the KKM lemma. The first infinite dimensional generalization of this statement was obtained by Ky Fan [5] in 1961. This statement, which is an infinite dimensional generalization of the classical KKM lemma, is known as the KKM principle. It has many applications in modern nonlinear functional analysis (see [10], [13] and [14]).

In 1969, Ky Fan [4] established the following famous best approximation theorem.

Theorem 1.1 ([4, Ky Fan]). *Let C be a nonempty, compact, convex subset of a normed linear space X. Then for any continuous mapping f from C to X, exists a point $x_0 \in C$ with*

$$||x_0 - f(x_0)|| = \inf_{x \in C} ||x - f(x_0)||.$$

This result has been generalized to other spaces X and other types of maps, see for example [8], [9], [13]. Prolla [12] and Carbone [1, 2] obtained a form of the best approximation theorem of Ky Fan using almost affine and almost quasi-convex maps in normed vector spaces.

The theory of measures of non-compactness has many applications in Topology, Functional analysis and Operator theory. There are many nonequivalent definitions of this notion on metric and topological spaces. The first was introduced by Kuratowski in 1930. In this paper we shall used definition of L. Pasicki [11].

In this paper we present a further extension of the best approximation theorems obtained by Ky Fan [4], J. Prolla [12] and A. Carbone [1, 2]. In our result conditions of almost-affinity, quasi-convexity and compactness are omitted.

2. Preliminaries

Let X and Y be non-empty sets; we denote by 2^X the family of all non-empty subsets of X, $\mathcal{F}(X)$ the family of all non-empty finite subsets of X and $\mathcal{P}(X)$ the family of all subsets of X. A multi-function G from X into Y is a map $G : X \longrightarrow 2^Y$.

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For a nonempty subset A of vector space X, let $co(A)$ denote the convex hull of A.

For a nonempty subset B of metric space Y, let $diam(B)$ denote the diameter of B.

Definition 2.1. Let C be a nonempty subset of a topological vector space X. A map $G : C \to 2^X$ is called KKM map if for every finite set $\{x_1, \dots, x_n\} \subset C$, we have

$$co(\{x_1, \dots, x_n\}) \subseteq \bigcup_{k=1}^{n} G(x_k).$$

Theorem 2.2. *Let X be a topological vector space, K be a nonempty subset of X and $G : K \to 2^X$ a KKM map with closed values. Then $\bigcap_{x\in A} G(x) \neq \emptyset$, for any $A \in \mathcal{F}(K)$.*

The next statement is Ky Fan's KKM principle.

Theorem 2.3 ([5])**.** *Let X be a topological vector space, K be a nonempty subset of X and $G : K \to 2^X$ a KKM map with closed values. If $G(x)$ is compact for at least one $x \in K$ then $\bigcap_{x\in K} G(x) \neq \emptyset$.*

Now we give L. Pasicki's [11] definition of measure of non-compactness.

Definition 2.4 ([11])**.** Let X be a metric space. Measure of non-compactness on X is an arbitrary function $\phi : \mathcal{P}(X) \to [0, \infty]$ which satisfies following conditions:
1) $\phi(A) = 0$ if and only if A is totally bounded set;
2) from $A \subseteq B$ follows $\phi(A) \leq \phi(B)$;
5) for each $A \subseteq X$ and $x \in X$ $\phi(A \cup \{x\}) = \phi(A)$.

Theorem 2.5 ([11])**.** *Let X be a complete metric space and ϕ measure of non-compactness on X. If $\{B_n\}_{n\in N}$ is a sequence of its nonempty closed subsets such that:*
1) *$B_{n+1} \subseteq B_n$ for any $n \in N$;*
2) *$\lim_{n\to\infty} \phi(B_n) = 0$;*
then $K = \bigcap_{n\in N} B_n$ is a nonempty, compact set.

The most important examples of measures of non-compactness on a metric space (X, d) are:
1) Kuratowski's measure
$\alpha(A) = \inf\{r > 0 \ : \ A \subseteq \bigcup_{i=1}^{n} S_i,\ S_i \subseteq X,\ diam(S_i) < r,\ 1 \leq i \leq n\}$;
2) Hausdorff's measure
$\chi(A) = \inf\{\varepsilon > 0 \ : \ A$ has a finite $\varepsilon-$ net in $X\}$;
3) measure of Istratescu
$I(A) = \inf\{\varepsilon > 0 \ : \ A$ contains no infinite $\varepsilon-$ discrete set in $A\}$.

Relations between this functions are given by following inequality, which are obtained by Danes [3]

$$\chi(A) \leq I(A) \leq \alpha(A) \leq 2\chi(A).$$

Definition 2.6. Let E a metric linear space, ϕ measure of non-compactness on E, and $X \subseteq E$. A multi-function $G : X \to 2^E$ is a *condensing* multi-function if for every $\varepsilon > 0$ there exist $n \in \mathcal{N}$ and $x_1, \dots, x_n \in X$ such that

$$\phi(G(x_1) \bigcap \cdots \bigcap G(x_n)) < \varepsilon.$$

A condensing multi-function $G : X \to 2^E$ is a *condensing KKM* multi-function if it is KKM multi-function.

Definition 2.7. Let X be a normed space and C a nonempty convex subset of X.

(i) A map $g : C \to X$ is almost affine if for all $x, y \in C$ and $u \in C$

$$||g(\lambda x + (1-\lambda)y) - u|| \leq \lambda ||g(x) - u|| + (1-\lambda)||g(y) - u||,$$

for each λ with $0 < \lambda < 1$.

(ii) A map $g : C \to X$ is almost quasi-convex if for all $x, y \in C$ and $u \in C$

$$||g(\lambda x + (1-\lambda)y) - u|| \leq \max\{||g(x) - u||, ||g(y) - u||\}$$

for each λ with $0 < \lambda < 1$.

3. Results

The next theorem is a generalization of a Theorem 2.3 and of recent results of Horvath [6], [7], which used Kuratowski's measure of non-compactness. The proof is essentially the same as in [7]. For the convenience of the reader we shall give it.

Theorem 3.1. *Let E be a complete metric linear space, ϕ measure of non-compactness on E, $X \in 2^E$ and let $G : X \to 2^E$ be a condensing KKM multi-function. If $G(x)$ is a closed set for each $x \in X$, then $\bigcap_{x \in X} G(x)$ is non-empty and compact set.*

Proof. For each $n \in \{1, 2, \dots\}$ there exists a finite set $F(n) \in 2^X$ such that $\phi(\bigcap_{x \in F(n)} G(x)) < \dfrac{1}{n}$. We define a sequence of sets B_n by:

$$B_1 = \bigcap_{x \in F(1)} G(x); \dots; B_{n+1} = B_n \bigcap (\bigcap_{x \in F(n+1)} G(x)); \dots$$

By Theorem 2.2 B_n is a nonempty set, for all $n \in \{1, 2, \dots\}$. Also, $\alpha(B_n) < \dfrac{1}{n}$. Other conditions of Theorem 2.5: B_n closed and $B_{n+1} \subseteq B_n$ trivially holds. So we have that $K = \bigcap_{n \in N} B_n$ is a nonempty, compact set. For every finite set $H \in 2^X$, by Theorem 2.2 we have that $C_{H,n} = (\bigcap_{x \in H} G(x)) \bigcap B_n$ is a nonempty closed set for any $n \in \mathcal{N}$. The sequence $\{C_{H,n}\}_{n \in N}$, also satisfies other the conditions of Theorem 2.5 and so $\bigcap_{n \in N} C_{H,n}$ is a nonempty closed subset of K. This implies that $(\bigcap_{x \in H} G(x)) \bigcap K$ is a nonempty set for any finite $H \in 2^X$. Since the family of closed sets $\{G(x) \bigcap K\}_{x \in X}$ has the finite intersection property, then

$$\bigcap_{x \in X} (G(x) \bigcap K) \neq \emptyset$$

because K is compact and so $\bigcap_{x\in X} G(x) \neq \emptyset$. This set is compact because it is a closed subset of compact set K. □

Now we present the following best approximation theorem in normed spaces, which is our main result in this paper.

Theorem 3.2. *Let X be a normed linear space, C a nonempty convex complete subset of X, ϕ measure of non-compactness on X, $f : C \to X$ and $g : C \to C$ continuous maps. If there exists an almost quasi-convex onto map $h : C \to C$ such that*

$$\|g(x) - f(x)\| \leq \|h(x) - f(x)\| \text{ for each } x \in C \tag{3.1}$$

and for each $t > 0$ there exists $y \in C$ such that

$$\phi(\{x \in C : \|g(x) - f(x)\| \leq \|h(y) - f(x)\|\}) \leq t, \tag{3.2}$$

then there exists a point $x_0 \in C$ such that

$$\|g(x_0) - f(x_0)\| = \inf_{x\in C} \|x - f(x_0)\|.$$

Proof. Let for every $y \in C$, $G : C \to 2^C$ be defined by

$$G(y) = \{x \in C : \|g(x) - f(x)\| \leq \|h(y) - f(x)\|\}.$$

From (3.2) it follows that G is a condensing multi-function. We have that $G(y)$ is nonempty for all $y \in C$, because $y \in G(y)$ for all $y \in C$. Since f and g are continuous maps, then $G(y)$ is closed for all $y \in C$. Now, we show that for each finite set $\{x_1, \ldots, x_n\} \subset C$,

$$co\{x_1, \ldots, x_n\} \subseteq \bigcup_{k=1}^{n} G(x_k). \tag{3.3}$$

Suppose that

$$co\{x_1, \ldots, x_n\} \nsubseteq \bigcup_{k=1}^{n} G(x_k) \text{ for some } \{x_1, \ldots, x_n\} \subset C.$$

Then there exists $y_0 \in co\{x_1, \ldots, x_n\}$ such that $y_0 \notin G(x_k)$ for each $k \in \{1, \ldots, n\}$. So, we have

$$\|g(y_0) - f(y_0)\| > \|h(x_k) - f(y_0)\| \text{ for each } k \in \{1, \ldots, n\}.$$

Therefore,

$$\|g(y_0) - f(y_0)\| > \max_k \|h(x_k) - f(y_0)\| \geq \|h(y_0) - f(y_0)\|.$$

This is a contradiction with condition (3.1). Hence, condition (3.3) is true for each finite $\{x_1, \ldots, x_n\} \subset C$ and map G is a condensing KKM map. Now, from Theorem 3.1 it follows that there exists $x_0 \in C$ such that

$$x_0 \in \bigcap_{y\in C} G(y).$$

Therefore,

$$\|g(x_0) - f(x_0)\| = \inf_{x\in C} \|x - f(x_0)\|.$$

□

When C is a compact set from Theorem 3.2 we obtained the next result.

Theorem 3.3. *Let X be a normed linear space, C a nonempty convex compact subset of X, $f : C \to X$ and $g : C \to C$ continuous maps. If there exists an almost quasi-convex onto map $h : C \to C$ such that*

$$\|g(x) - f(x)\| \leq \|h(x) - f(x)\| \text{ for each } x \in C, \tag{3.4}$$

then there exists a point $x_0 \in C$ such that

$$\|g(x_0) - f(x_0)\| = \inf_{x \in C} \|x - f(x_0)\|.$$

Example 3.4. Let $C = [0, 1]$ and define maps $f, g, h : C \to C$ by

$$f(x) = 0, \; h(x) = x,$$

$$g(x) = \begin{cases} x, & x \in [0, \frac{1}{4}); \\ -x + \frac{1}{2}, & x \in [\frac{1}{4}, \frac{1}{2}); \\ 2x - 1, & x \in [\frac{1}{2}, 1]. \end{cases}$$

Then map g is not almost quasi-convex and results of J. B. Prolla [12] and A. Carbone [1, 2] are not applicable. Note that the maps f, g and h satisfy all hypotheses of Theorem 3.3.

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