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NECESSARY OPTIMALITY CONDITION AND STABILITY IN NONLINEAR SEMI-INFINITE OPTIMIZATION

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Abstract. Necessary optimality condition and the pseudo-Lipschitz property of solution map for parametric semi-infinite optimization problem (PSI for brevity) are studied in this paper. The main attention is devoted to developing new applications of some advanced tools of modern variational analysis and generalized differentiation to PSI from the view point of the coderivative approach to necessary optimality conditions and to robust stability of solution map under parameter perturbation. We present a Karush-Kuhn-Tucker (KKT) type first-order optimality condition for PSI without parameter perturbation, and then use this optimality condition to establish sufficient conditions for the pseudo-Lipschitz property of the solution map of PSI. The results obtained are also applied to some special classes of PSI involving smooth as well as convex optimization problems.

1. Introduction

Let P, X be Banach spaces, and let T be an arbitrary index set. Given a single-valued *cost mapping* $f\colon P \times X \to \overline{\mathbb{R}}$, we consider the *parametric optimization* problem

$$\min f(p,x) \quad \text{subject to } \ x \in G(p) \tag{1.1}$$

with the set-valued *constraint mapping* $G\colon P \rightrightarrows X$ defined by

$$G(p) := \{x \in X \,|\, g_t(p,x) \le 0,\, t \in T\}, \tag{1.2}$$

where for each $t \in T$, $g_t : P \times X \to \overline{\mathbb{R}}$ is an extended real-valued function. Here "min" is taken with respect to the decision variable x and p is a perturbation parameter. We associate with the parametric optimization problem (1.1) the *marginal/optimal value function*

$$\mu(p) := \inf\{f(p,x) \,|\, x \in G(p)\} \tag{1.3}$$

and the *solution/argminimum map*

$$S(p) := \big\{x \in G(p) \,\big|\, \mu(p) = f(p,x)\big\}. \tag{1.4}$$

The above optimization is referred to as a (standard) semi-infinite optimization problem if X is finite-dimensional Banach space (say $X = \mathbb{R}^n$) and T is a compact metric space.

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Semi-infinite optimization problems became an active research topic in applied mathematics in recent years. Necessary conditions and stability in semi-infinite optimization have been studied extensively in recent years; see, e.g., [2, 3, 5, 6, 8, 10, 12, 14, 15, 25–28].

Necessary conditions for nondifferentiable convex semi-infinite optimization problems (CSI for brevity) are given in [15] under the Farkas-Minkowski qualification. More recently, a necessary and sufficient optimality condition for CSI is established in [26] under the Farkas-Minkowski type constraint qualification (FMCQ). For more details and discussions of optimality conditions and constraint qualifications with their relations, we refer the reader to, e.g., [3, 10, 14, 15, 26, 28] and references therein. As mentioned above, most of semi-infinite optimization problems under consideration are smooth or convex.

In [1], Aubin proposed and examined the pseudo-Lipschitz behavior of solution maps to perturbed convex minimization problems. Rockafellar's thorough investigation is addressed to the Lipschitz stability of general constraint systems including solution maps to parametric generalized equations in [22] and marginal functions to parametric optimization problems in [21]. Sufficient conditions for the Lipschitz stability of the solution map of (1.1) are given in [8, 25] when the objective function and constraint functions are twice differentiable. Another characterizations for the same property of the solution map was presented in [2] under consideration where the objective function and constraint functions are smooth. In the case when the objective function and constraint functions are locally Lipschitz, the Lipschitz property of the marginal function was obtained in [27]. In the framework of parametric convex optimization problems, sufficient conditions for the Lipschitz property of the solution map are presented in [12] under linear perturbation of the constraint functions only, and sufficient conditions for the metric regularity of the inverse of the solution map (and so is equivalent to the pseudo-Lipschitz property of the solution map) under continuous perturbations of the right-hand side of the constraints and linear perturbations of the objective function are obtained in [5]. In addition to these, as described in the papers mentioned above, sufficient conditions for the pseudo-Lipschitz property of solution maps and marginal functions are given under classical derivative or Clarke's derivative-like constructions in nonsmooth analysis in [7].

The *coderivative* of set-valued mappings, which was introduced by Mordukhovich [16], has been well recognized as a convenient tool to study many important issues in variational analysis and optimization. We refer the reader to the recent books [4, 13, 17, 18, 23, 24] with their commentaries and bibliographies. Imitating the definition of Mordukhovich's coderivative where the coderivative type is defined via the Clarke normal cone, Zheng and Yang [28] first establish Fritz John type first-order optimality condition for semi-infinite optimization problems where the objective function and constraint functions are locally Lipschitz at the reference point. Furthermore, the optimality condition still holds with Mordukhovich coderivative when the problem under consideration has a finite number of constraint functions and the objective space is a finite dimensional space. This motivates us to use Mordukhovich coderivative for investigating semi-infinite optimization problems.

In this paper, the necessary optimality condition and the pseudo-Lipschitz property of the solution map of (1.1) are studied under consideration where the parameter space P is a Banach space, the objective space X is a finite dimensional space, the index set T is a nonempty compact metric space, and the objective function and constraint functions are locally Lipschitz at the reference point. Namely, we present a KKT type first-order optimality condition for (1.1) without parameter perturbation, and then use this optimality condition to establish sufficient conditions for the pseudo-Lipschitz property of the solution map of (1.1). The results obtained are also applied to some special classes of (1.1) involving smooth as well as convex optimization problems.

The paper is organized as follows. In Section 2 we recall some basic definitions and preliminaries from set-valued analysis, variational analysis and generalized differentiation. Section 3 is devoted to presenting a KKT type necessary condition for (1.1) without parameter perturbation under a constraint qualification that covers the existing constraint qualifications of MFCQ and FMCQ types. Section 4 is devoted to establishing new sufficient conditions for the pseudo-Lipschitz property of (1.4). Examples are also discussed to illustrate sufficient conditions for the pseudo-Lipschitz property of (1.4). In Section 5 we consider some special class of (1.1) involving smooth as well as convex optimization problems.

2. Basic definitions and preliminaries

Let us recall some notions which are related to our problem. Throughout the paper we use standard notation of convex analysis, set-valued analysis, variational analysis and generalized differentiation. We refer the reader to the books [3, 11, 17, 20, 23] for more details and discussions.

Let X be a Banach space and X^* be its topological dual. We denote by $\|\cdot\|$ the norm of X and $\langle\cdot,\cdot\rangle$ the dual pair between X^* and X. Given a set-valued mapping $F\colon X \rightrightarrows X^*$, we denote by

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{ x^* \in X^* \Big|\ \exists \text{ sequences } x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in F(x_k) \text{ for all } k \in I\!N \Big\}$$

the *sequential Painlevé-Kuratowski upper/outer limit* with respect to the norm topology of X and the weak* topology of X^*, where $I\!N := \{1, 2, \ldots\}$.

Let $\Omega \subset X$. The closure of Ω will be denoted by $\mathrm{cl}(\Omega)$ and $\mathrm{int}(\Omega)$ stands for the interior of Ω. We will use $\mathcal{N}(x)$ to denote the set of all neighborhoods of $x \in X$. We will denote by $\mathrm{co}(\Omega)$ the convex hull of Ω, and $\mathrm{cone}(\Omega)$ the convex hull of the set $\{\lambda\Omega \,|\, \lambda \geq 0\}$. Write $\emptyset + \Omega = \emptyset$, $\mathrm{co}\,\emptyset = \emptyset$ and $\mathrm{cone}\,\emptyset = \{0\}$ and, where 0 is the zero vector of X.

Given $\Omega \subset X$ and $\varepsilon \geq 0$, define the collection of ε-*normals* to Ω at $\bar{x} \in \Omega$ by

$$\widehat{N}_\varepsilon(\bar{x};\Omega) := \Big\{ x^* \in X^* \Big|\ \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq \varepsilon \Big\}, \tag{2.1}$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. When $\varepsilon = 0$, the set $\widehat{N}(\bar{x};\Omega) := \widehat{N}_0(\bar{x};\Omega)$ in (2.1) is a cone called the *prenormal cone* or the *Fréchet normal cone* to Ω at $\bar{x}$.

The *limiting* or *Mordukhovich normal cone* $N(\bar{x};\Omega)$ is obtained from $\widehat{N}_\varepsilon(x;\Omega)$ by taking the sequential Painlevé-Kuratowski upper limit in the weak* topology of X^* as

$$N(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{\Omega} \bar{x} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x;\Omega), \tag{2.2}$$

where one can put $\varepsilon = 0$ when Ω is closed around $\bar{x}$ and the space X is *Asplund*, i.e., a Banach space whose separable subspaces have separable duals.

In the special case when X is a finite dimensional space and Ω is closed around $\bar{x}$, the *Clarke normal cone* $N^C(\bar{x};\Omega)$ [7] to Ω at $\bar{x}$ always coincides with the convex closure of (2.2), i.e.,

$$N^C(\bar{x};\Omega) = \text{clco} N(\bar{x};\Omega)$$

and $N(\cdot;\Omega)$ has the robust property at $\bar{x}$ which given in the following proposition [19, Proposition 3.4].

Proposition 2.1. *Let X be a finite dimensional space and Ω be closed subset of X. Let $\bar{x} \in \Omega$. Then, for any sequences $x_k \to \bar{x}$ and $x_k^* \to x^*$ with $x_k^* \in N(x_k;\Omega)$, $k = 1, 2, \ldots$, one has $x^* \in N(\bar{x};\Omega)$.*

Let $F : X \rightrightarrows Y$ be a multifunction between Banach spaces. The effective domain and the graph of F are given by the formulas

$$\text{dom} F := \{x \in X \mid F(x) \neq \emptyset\}, \;\; \text{gph} F := \{(x,y) \in X \times Y \mid y \in F(x)\}.$$

The *Mordukhovich normal coderivative* $D^*F(\bar{x},\bar{y})\colon Y^* \rightrightarrows X^*$ of F at $(\bar{x},\bar{y}) \in \text{gph}\, F$ is defined by

$$D^*F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big| \; (x^*,-y^*) \in N((\bar{x},\bar{y});\text{gph}\, F)\big\}, \quad \forall y^* \in Y^*. \tag{2.3}$$

The *Fréchet coderivative* at $(\bar{x},\bar{y}) \in \text{gph}\, F$ is defined by

$$\widehat{D}^*F(\bar{x},\bar{y})(y^*) := \{x^* \in X^* \mid (x^*,-y^*) \in \widehat{N}((\bar{x},\bar{y});\text{gph}\, F))\} \quad \forall y^* \in Y^*. \tag{2.4}$$

A single-valued mapping $f\colon X \to Y$ is said to be *strictly differentiable* at $\bar{x}$ if there is a linear continuous operator $\nabla f(\bar{x})\colon X \to Y$ such that

$$\lim_{x,u \to \bar{x}} \frac{f(x) - f(u) - \langle \nabla f(\bar{x}), x - u\rangle}{\|x - u\|} = 0.$$

We known that for such mappings one has

$$D^*f(\bar{x})(y^*) = \widehat{D}^*f(\bar{x})(y^*) = \{(\nabla f(\bar{x}))^* y^*\} \quad \forall y^* \in Y^*,$$

i.e., the Mordukhovich normal coderivative (resp., Fréchet coderivative) is a generalization of the adjoint operator to the classical Jacobian/strict derivative. For more details, we refer the reader to [17].

For an extended real-valued function $\varphi : X \to \bar{\mathbb{R}} := [-\infty,\infty]$, we define

$$\text{dom}\, \varphi = \{x \in X \mid |\varphi(x)| < \infty\}, \quad \text{epi}\, \varphi = \{(x,\mu) \in X \times \mathbb{R} \mid \mu \geq \varphi(x)\},$$

and say that φ is *lower semicontinuous* at $\bar{x} \in X$ if $\liminf_{x \to \bar{x}} \varphi(x) \geq \varphi(\bar{x})$. Here $\liminf_{x \to \bar{x}}$ denotes the lower limit of scalar functions in the classical sense.

The *limiting* or *Mordukhovich subdifferential* $\partial\varphi(\bar{x})$ of φ at $\bar{x} \in \mathrm{dom}\,\varphi$ is defined by

$$\partial\varphi(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); \mathrm{epi}\varphi)\}.$$

If $\bar{x} \notin \mathrm{dom}\varphi$ then one puts $\partial\varphi(\bar{x}) = \emptyset$. If $\bar{x}$ is a *local minimum* of φ, then

$$(0, -1) \in \widehat{N}((\bar{x}, \varphi(\bar{x})); \mathrm{epi}\,\varphi) \subset N((\bar{x}, \varphi(\bar{x})); \mathrm{epi}\,\varphi).$$

The *presubdifferential* or *Fréchet subdifferential* of φ at $\bar{x} \in \mathrm{dom}\,\varphi$ is denoted by

$$\widehat{\partial}\varphi(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in \widehat{N}((\bar{x}, \varphi(\bar{x})); \mathrm{epi}\varphi)\}.$$

It is well known that if X is a finite dimensional space and φ is locally Lipschitz at $\bar{x}$ then

$$\partial^C\varphi(\bar{x}) = \mathrm{clco}\partial\varphi(\bar{x}),$$

where $\partial^C\varphi(\bar{x})$ stands for the Clarke generalized gradient [7].

3. Necessary optimality condition

In this section we present a necessary optimality condition for (1.1) at the reference point without parameter perturbation.

Let T be a nonempty compact metric space. We consider the Banach space $Y = C(T)$ of continuous functions $y : T \to \mathbb{R}$, equipped with the sup-norm

$$||y|| = \max_{t \in T} |y(t)|,$$

and the cone $K \subset Y$ has the form

$$K = \{y \in C(T) \mid y(t) \leq 0 \text{ for all } t \in T\}. \tag{3.1}$$

By the Riesz representation theorem, the dual space Y^* of $Y = C(T)$ is norm isomorphic to, and so can be identified with, the space of finite signed (regular) Borel measures on $(T, \mathcal{B})$, where $\mathcal{B}$ is the Borel sigma-algebra of T, with the norm given by the total variation of the corresponding measure. Namely, if ν is a finite signed (regular) Borel measure on $(T, \mathcal{B})$ and $y \in Y$ then

$$\langle \nu, y \rangle = \int_T y(t) d\nu, \tag{3.2}$$

$$||\nu|| := |\nu|(T) := \nu^+(T) + \nu^-(T),$$

where for each $A \in \mathcal{B}$,

$$\nu^+(A) := \sup\{\nu(B) \mid B \subset A,\ B \in \mathcal{B}\}$$
$$\nu^-(A) := -\inf\{\nu(B) \mid B \subset A,\ B \in \mathcal{B}\},$$

denote the *positive* and the *negative variation* of ν, respectively. We recall that $|\nu|(A) := \nu^+(A) + \nu^-(A)$ is said to be the *total variation measure* of the measure ν. The *support* of ν, denoted by $\mathrm{supp}\,\nu$, is the smallest closed subset of T such that its complement has total variation measure zero. A Borel measure ν is said to be *nonnegative*, written $\nu \succcurlyeq 0$, if $\nu(A) \geq 0$ for any $A \in \mathcal{B}$.

For a function $y \in K$, we denote by

$$T(y) := \{t \in T \mid y(t) = 0\}$$

the set of contact points of y. It follows from [3, Example 2.63] that

$$N(y; K) = \{\nu \in C(T)^* \mid \nu \succeq 0,\ \operatorname{supp} \nu \subset T(y)\}.$$

Consider the special case when T is a finite set $\{t_1, t_2, \dots, t_m\}$. Then $C(T) = \mathbb{R}^m$. For such case, consider the discrete measure

$$\nu = \sum_{t \in T(y)} \lambda_t \delta(t)$$

where $\lambda_t \geq 0$ and $\delta(t)$ denotes the (Dirac) measure of mass one at the point $t \in T$. It follows that

$$N(y; K) \subset \mathbb{R}^m_+.$$

Let $\psi : X \to \bar{\mathbb{R}}$ be locally Lipschitz at $\bar{x}$. Then, for every $\mu \geq 0$ one has

$$D^*\psi(\bar{x})(\mu) = \mu \partial \psi(\bar{x}).$$

We recall an important result from [28] on the necessary optimality condition. Consider the following optimization problem

$$\text{(3.3)} \qquad \min \varphi(x) \quad \text{subject to} \quad \psi(x) \leq 0 \text{ and } x \in \Omega,$$

where X is a finite dimensional space, φ, ψ are real functions from X to $\mathbb{R}$.

We see that $Z := X \times \mathbb{R} \times C(T)$ is an Asplund space if X is a finite dimensional space and $|T| < \infty$. Here $|T|$ denotes the cardinality of T. The following theorem is immediate from [28, Theorem 3.2] and [28, p. 180, Remark].

Theorem 3.1. *For* (3.3)*, let $\bar{x}$ be a minimizer. If φ and ψ are locally Lipschitz at $\bar{x}$, then there exist $\lambda \geq 0$, $\mu \geq 0$ and $x^* \in N(\bar{x}; \Omega)$ such that*

$$\lambda + \mu + \|x^*\| = 1 \qquad \text{and} \qquad 0 \in \lambda \partial \varphi(\bar{x}) + \mu \partial \psi(\bar{x}) + x^*.$$

We recall that a multifunction $F\colon X \rightrightarrows Y$ is said to be *closed* at the point $x_0 \in X$ if, for all sequences $\{x_i\}$ in X and $\{y_i\}$ in Y satisfying $x_i \to x_0, y_i \to y_0, y_i \in F(x_i)$, one has $y_0 \in F(x_0)$. F is said to be *lower semicontinuous* (lsc for brevity) at $x_0 \in \operatorname{dom} F$ if, for any open set $V \subset Y$ satisfying $V \cap F(x_0) \neq \emptyset$ there exists $U \in \mathcal{N}(x_0)$ such that $V \cap F(x) \neq \emptyset$ for all $x \in U$. F is *upper semicontinuous* (usc for brevity) at $x_0 \in X$ if, for every open set V containing $F(x_0)$ there exists $U \in \mathcal{N}(x_0)$ such that $F(x) \subset V$ for all $x \in U$. It is well known that if X is a compact Hausdorff space, F is usc on X and $F(x)$ is a compact subset of Y for all $x \in X$ then $\bigcup_{x \in X} F(x)$ is compact.

We consider the following semi-infinite optimization problem

$$\text{(3.4)} \qquad \min h(x) \quad \text{subject to} \quad x \in \Omega \text{ and } h_t(x) \leq 0 \text{ for all } t \in T,$$

where X is a finite dimensional space, T is a nonempty compact metric space, Ω is a closed subset of X, h and h_t $(t \in T)$ are real functions from X to $\mathbb{R}$. Let $\bar{x}$ be an element belonging to the constraint set of (3.4). In what follows we use the set of *active constraints* defined by

$$T(\bar{x}) := \{t \in T \mid h_t(\bar{x}) = 0\}.$$

Denote by $\partial_{[T]}h_t(x)$ the set

$$\text{clco}\{\xi^* \in X \mid \xi_i^* \in \text{co}\partial h_{t_i}(x_i),\ x_i \to x,\ t_i \to t,\ t_i \in T,\ \xi^* \text{ is a cluster point of } \xi_i^*\}.$$

Now we establish a necessary optimality condition for (3.4) which is useful in the sequel.

Theorem 3.2. *For* (3.4), *let* $\bar{x}$ *be a minimizer. Suppose that* h, h_t $(t \in T)$ *are locally Lipschitz at* $\bar{x}$, *there exists* $U \in \mathcal{N}(\bar{x})$ *such that, for each* $x \in U$, *the mapping* $t \mapsto h_t(x)$ *is continuous on* T, *the mapping* $(t, x) \mapsto \partial h_t(x)$ *is usc on* $T \times U$, *and*

$$\text{(CQ)} \qquad 0 \notin \text{clco} \cup_{t\in T(\bar{x})} \partial h_t(\bar{x}) + N(\bar{x};\Omega)$$

holds. Then

$$-\partial h(\bar{x}) \cap \Big[\text{cone}\big(\bigcup_{t\in T(\bar{x})} \partial h_t(\bar{x})\big) + N(\bar{x};\Omega)\Big] \neq \emptyset.$$

Proof. Let $\bar{x}$ be a minimizer of (3.4). The assertion of the theorem is trivial if $T(\bar{x})$ is empty. Suppose that $T(\bar{x}) \neq \emptyset$. Let $\psi(x) := \sup_{t\in T} h_t(x)$. Since for every $t \in T$, h_t is locally Lipschitz at $\bar{x}$, so is ψ. It is well known that (3.4) is equivalent to the following optimization problem

$$\min h(x) \quad \text{subject to} \quad \psi(x) \leq 0,\ x \in \Omega.$$

It follows from Theorem 3.1 that there exist $\lambda \geq 0$, $\mu \geq 0$ and $x^* \in N(\bar{x}, \Omega)$ satisfying

$$\text{(3.5)} \qquad \lambda + \mu + \|x^*\| = 1 \quad \text{and} \quad 0 \in \lambda\partial h(\bar{x}) + \mu\partial\psi(\bar{x}) + x^*.$$

We claim that $\lambda \neq 0$ whenever

$$\text{(3.6)} \qquad 0 \notin \partial\psi(\bar{x}) + N(\bar{x};\Omega).$$

Indeed, if $\lambda = 0$ then, by (3.5),

$$\mu + \|x^*\| = 1 \quad \text{and} \quad 0 \in \mu\partial\psi(\bar{x}) + x^*.$$

Hence, $\mu \neq 0$, and so $0 \in \partial\psi(\bar{x}) + \frac{x^*}{\mu}$, contrary to (3.6). Therefore $\lambda \neq 0$.

It suffices to show that (3.6) is valid. By $T(\bar{x}) \neq \emptyset$, we have $\psi(\bar{x}) = 0$. From [7, Theorem 2.8.2] it follows that

$$\text{(3.7)} \qquad \partial\psi(\bar{x}) \subset \Big\{ \int_T \partial_{[T]}h_t(\bar{x})d\mu \mid \mu \in P[T(\bar{x})]\Big\},$$

where $P[T(\bar{x})]$ denotes the set of all probability Radon measures supported on $T(\bar{x})$. Since $t \mapsto h_t(\bar{x})$ is continuous, it follows that $T(\bar{x})$ is compact. We claim that

$$\text{(3.8)} \qquad \Big\{ \int_T \partial_{[T]}h_t(\bar{x})d\mu \mid \mu \in P[T(\bar{x})]\Big\} \subset \text{clco}\Big[\bigcup_{t\in T(\bar{x})} \text{co}\,\partial h_t(\bar{x})\Big].$$

Indeed, if our claim is not true, then there exists an element ξ^* of X belonging to the right-hand side of (3.7) such that

$$\xi^* \notin \text{clco}\Big[\bigcup_{t\in T(\bar{x})} \text{co}\,\partial h_t(\bar{x})\Big].$$

Clearly, the mapping $(t, x) \mapsto \text{co}\,\partial h_t(x)$ is also usc on $T \times U$. Since $T(\bar{x})$ and $\text{co}\,\partial h_t(\bar{x})$ $(t \in T)$ are compact, it follows that $\bigcup_{t\in T(\bar{x})} \text{co}\,\partial h_t(\bar{x})$ is compact. Hence,

$\mathrm{co}\Big[\bigcup_{t\in T(\bar{x})} \mathrm{co}\,\partial h_t(\bar{x})\Big]$ is a convex compact subset of X since X is a finite dimensional space. Applying the separation theorem, we can assert that there exist $v \in X$ and $\alpha \in \mathbb{R}$ such that

$$\langle \xi^*, v\rangle < \alpha < \langle \eta_t^*, v\rangle,$$

for all $\eta_t^* \in \mathrm{co}\,\partial h_t(\bar{x})$ and $t \in T(\bar{x})$. Combining the compactness of $T(\bar{x})$ and $\mathrm{co}\,\partial h_t(\bar{x})$ $(t \in T(\bar{x}))$ with the upper semicontinuity of the mapping $(t, x) \mapsto \mathrm{co}\partial h_t(x)$ we deduce that the set

$$\Big\{\langle \eta_t^*, v\rangle \mid \eta_t^* \in \mathrm{co}\partial h_t(\bar{x}),\ t \in T(\bar{x})\Big\} \text{ is compact.}$$

Hence,

$$\langle \xi^*, v\rangle \notin \mathrm{clco}\Big\{\langle \eta_t^*, v\rangle \mid \eta_t^* \in \mathrm{co}\partial h_t(\bar{x}),\ t \in T(\bar{x})\Big\}. \tag{3.9}$$

On the one hand, it follows from [7, Remark 2.8.3] that there exist a mapping $t \mapsto \xi_t^* \in \partial_{[T]} h_t(\bar{x})$ from T to X and an element $\mu \in P[T(\bar{x})]$ such that

$$\langle \xi^*, v\rangle = \int_T \langle \xi_t^*, v\rangle d\mu = \int_{\mathrm{supp}\,\mu} \langle \xi_t^*, v\rangle d\mu. \tag{3.10}$$

By [9, Theorem II.2 and Corollary II.8], we have

$$\begin{aligned}\int_{\mathrm{supp}\,\mu} \langle \xi_t^*, v\rangle d\mu \in & \mu(\mathrm{supp}\,\mu)\mathrm{clco}\{\langle \xi_t^*, v\rangle \mid t \in \mathrm{supp}\,\mu\}\\ =&\mu(T(\bar{x}))\mathrm{clco}\{\langle \xi_t^*, v\rangle \mid t \in T(\bar{x})\}\\ =&\mathrm{clco}\{\langle \xi_t^*, v\rangle \mid t \in T(\bar{x})\}.\end{aligned} \tag{3.11}$$

On the another hand, since the mapping $(t, x) \mapsto \mathrm{co}\partial h_t(x)$ is upper semicontinuous it follows that

$$\xi_t^* \in \mathrm{co}\partial h_t(\bar{x}) \ \text{ for all } \ t \in T(\bar{x}).$$

Combining this with (3.10) and (3.11) we obtain

$$\langle \xi^*, v\rangle \in \mathrm{clco}\{\langle \eta_t^*, v\rangle \mid \eta_t^* \in \mathrm{co}\partial h_t(\bar{x})\ t \in T(\bar{x})\},$$

which is contrary to (3.9). Hence (3.8) follows.

Obviously,

$$\mathrm{clco}\Big[\bigcup_{t\in T(\bar{x})} \mathrm{co}\,\partial h_t(\bar{x})\Big] \subset \mathrm{clco}\Big[\bigcup_{t\in T(\bar{x})} \partial h_t(\bar{x})\Big].$$

Therefore $\partial\psi(\bar{x}) \subset \mathrm{clco}\Big[\bigcup_{t\in T(\bar{x})} \partial h_t(\bar{x})\Big]$ by (3.7) and (3.8). This inclusion and (CQ) imply (3.6). The proof is complete. □

We remark that CQ covers the existing constraint qualifications of MFCQ and FMCQ types. The KKT type first-order optimality condition in Theorem 3.2 is a key to establish the pseudo-Lipschitz property of the solution map for the parametric semi-infinite optimization problem in next section.

4. Pseudo-Lipschitz property of the solution map

In the sequel, we assume that P is a Banach space, X is an n-dimensional Euclidean space, Ω is a closed subset of X and T is a nonempty compact metric space. Let f and g_t ($t \in T$) be real functions from $P \times X \to \mathbb{R}$. Let $(\bar{p}, \bar{x}) \in P \times X$. Denote by $\partial_x f(\bar{p}, \bar{x})$ the limiting subdifferential of f with respect to x at $(\bar{p}, \bar{x})$, i.e.,

$$\partial_x f(\bar{p}, \bar{x}) = \{x^* \in X \mid (x^*, -1) \in N((\bar{x}, f(\bar{p}, \bar{x})); \mathrm{epi}_{\bar{p}} f)\},$$

where $\mathrm{epi}_{\bar{p}} f := \{(x, r) \in X \times \mathbb{R} \,|\, r \geq f(\bar{p}, \bar{x})\}$. Put

$$T(\bar{p}, \bar{x}) := \{t \in T \,|\, g_t(\bar{p}, \bar{x}) = 0\}.$$

We now recall the concept of the prox-regularity which was introduced by Poliquin and Rockafellar (see [23]).

The function φ is *prox-regular* at $\bar{x}$ for $\bar{u} \in \partial\varphi(\bar{x})$, if φ is locally lower semicontinuous at $\bar{x}$ and there exist $\varepsilon > 0$ and $r \geq 0$ such that

$$\varphi(x') \geq \varphi(x) + \langle u, x' - x\rangle - \frac{r}{2}\|x' - x\|^2$$

whenever $\|x' - \bar{x}\| < \varepsilon$ and $\|x - \bar{x}\| < \varepsilon$ with $x' \neq x$ and $|\varphi(x) - \varphi(\bar{x})| < \varepsilon$, while $\|u - \bar{u}\| < \varepsilon$ with $u \in \partial\varphi(x)$. We say that φ is prox-regular at $\bar{x}$ if it is prox-regular at $\bar{x}$ for any $\bar{u} \in \partial\varphi(\bar{x})$.

A set C is called prox-regular at $\bar{x} \in C$ for $\bar{v} \in N(C; \bar{x})$ if C is locally closed at $\bar{x}$ and there exist $\varepsilon > 0$ and $\rho \geq 0$ such that $\langle v, x' - x\rangle \leq \frac{\rho}{2}\|x' - x\|^2$ for all $x' \in C \cap \mathbb{B}(\bar{x}, \varepsilon)$ when $v \in N(x; C)$, $\|v - \bar{v}\| < \varepsilon$ and $\|x - \bar{x}\| < \varepsilon$. When this holds for all $\bar{v} \in N(\bar{x}; C)$, C is said to be prox-regular at $\bar{x}$.

Note that the class of prox-regular functions includes all $C^{1,1}$ functions, all lower semicontinuous, proper, convex functions, all lower-C^2 functions, all primal-lower-nice functions, and all "strongly amenable functions" (convex functions composed with C^2 mappings). This list covers most of the objective functions in finite-dimensional optimization, including constrained optimization where constraints are incorporated into the objective via infinite penalties. For more details of the prox-regularity, we refer the reader to [23].

The distance from $x \in X$ to a subset M of X is defined by

$$d(x, M) := \inf \{\mathrm{dist}(x, y) \mid y \in M\},$$

where $\mathrm{dist}(x, y) := \|x - y\|$ denotes the distance between two points x and y, and $d(x, \varnothing) := +\infty$.

Definition 4.1. A multifunction $F\colon X \rightrightarrows Y$ is said to be *pseudo-Lipschitz* or *Aubin continuous* (also called *Lipschitz-like*) at $(x_0, y_0) \in \mathrm{gph}F$ if there exist $U \in \mathcal{N}(x_0)$ and $V \in \mathcal{N}(y_0)$ and a constant $\ell > 0$ such that

$$d(y_2, F(x_1)) \leq \ell d(x_1, x_2),$$

for all $x_1, x_2 \in U$, and all $y_2 \in V \cap F(x_2)$.

One says that F is *inner semicontinuous* at $(x_0, y_0) \in \mathrm{gph}\, F$ if for any sequence $x_k \to x_0$ as $k \to +\infty$, there exists a sequence $y_k \to y_0$ with $y_k \in F(x_k)$, $k = 1, 2, \ldots$ Clearly, for each $y \in F(x_0)$, F is inner semicontinuous at (x_0, y) whenever it is lower semi-continuous at x_0.

Consider the following assumptions:

(A1) f is locally Lipschitz at $(\bar{p}, \bar{x})$, i.e., there exist $U \in \mathcal{N}(\bar{x})$, $V \in \mathcal{N}(\bar{p})$ and $\ell > 0$ such that

$$|f(p_1, x_1) - f(p_2, x_2)| \leq \ell(\|p_1 - p_2\| + \|x_1 - x_2\|),\ \forall p_1,\ p_2 \in V,\ \forall\ x_1,\ x_2 \in U;$$

(A2) for every $t \in T$, g_t is locally Lipschitz with respect to (p, x) uniformly in t at $(\bar{p}, \bar{x})$, i.e., there exist $U \in \mathcal{N}(\bar{x})$, $V \in \mathcal{N}(\bar{p})$ and $\hbar > 0$ such that

$$|g_t(p_1, x_1) - g_t(p_2, x_2)| \leq \hbar(\|p_1 - p_2\| + \|x_1 - x_2\|),\ \forall p_1,\ p_2 \in V,\ \forall\ x_1,\ x_2 \in U,\ \forall t \in T;$$

(A3) the mapping $(p, x, t) \mapsto g_t(p, x)$ is continuous at $(\bar{p}, \bar{x}, \bar{t})$ with any $\bar{t} \in T$; there exist $U \in \mathcal{N}(\bar{x})$ and $V \in \mathcal{N}(\bar{p})$ such that the mapping $(t, p, x) \mapsto \partial_x g_t(p, x)$ is usc on $T \times V \times U$;
(A4) there exist $U \in \mathcal{N}(\bar{x})$ and $V \in \mathcal{N}(\bar{p})$ such that the mapping $(p, x) \mapsto \partial_x f(p, x)$ is usc on $V \times U$;
(A5) Ω is prox-regular at $\bar{x}$, there exists $V \in \mathcal{N}(\bar{p})$ such that for any $p \in V$, $f(p, \cdot)$ is prox-regular at $\bar{x}$, and for any $t \in T$, $g_t(p, \cdot)$ is prox-regular at $\bar{x}$ uniformly in t.

Now we state a sufficient condition for the pseudo-Lipschitz property of $\mathcal{S}$ at the reference point.

Theorem 4.2. *Let $\bar{p} \in P$ and $\bar{x} \in \mathcal{S}(\bar{p})$. Suppose that the conditions* (A1)–(A5) *and the following conditions hold:*
(i) $0 \notin \operatorname{cl} \operatorname{co} \cup_{t \in T(\bar{p}, \bar{x})} \partial_x g_t(\bar{p}, \bar{x}) + N(\bar{x}; \Omega)$*;*
(ii) *There is no $T_0 \subset T(\bar{p}, \bar{x})$ with $|T_0| < n$ satisfying*

$$-\partial_x f(\bar{p}, \bar{x}) \cap \Big[\operatorname{cone}\big(\bigcup_{i \in T_0} \partial_x g_t(\bar{p}, \bar{x})\big) + N(\bar{x}; \Omega)\Big] \neq \emptyset. \tag{4.1}$$

Then $\mathcal{S}$ is pseudo-Lipschitz at $(\bar{p}, \bar{x})$ whenever it is inner-semicontinuous at this point.

Note that in [6] the combination of both conditions (i) and (ii) of Theorem 4.2 is referred to as *extended Nürnberger condition* under consideration where the objective function and the constraint functions are convex; for more details and discussions we refer the reader to [5, 6].

Before proving the main result, we need to establish the following lemma.

Lemma 4.3. *Under the assumptions of Theorem 4.2, one can assert that for any $\{(p^k, x^k)\}_{k=1}^{\infty} \subset \operatorname{gph}\mathcal{S}$ which converges to $(\bar{p}, \bar{x}) \in \operatorname{gph}\mathcal{S}$, there exist $u^k \in \partial_x f(p^k, x^k)$, $t_i^k \in T(p^k, x^k)$, $u_i^k \in \partial_x g_{t_i^k}(p^k, x^k)$ and $\lambda_i^k > 0$ for $i \in \{1, 2, \dots, n\}$, such that*

$$-u^k - \sum_{i=1}^{n} \lambda_i^k u_i^k \in N(x^k; \Omega) \quad \textit{for } k \textit{ large enough}$$

and $\{u_1^k, \dots, u_n^k\}$ forms a basis of X.

Proof. It is easily seen that assumption (i) in Theorem 4.2 holds with (p, x) belonging to some neighborhood of $(\bar{p}, \bar{x})$. Let $\{(p^k, x^k)\}_{k=1}^{\infty}$ be a sequence of gph$\mathcal{S}$ such that $\{(p^k, x^k)\}$ converges to $(\bar{p}, \bar{x}) \in$ gph$\mathcal{S}$. Since $(p^k, x^k) \to (\bar{p}, \bar{x})$ as $k \to \infty$, it follows that assumption (i) of Theorem 3.2 is satisfied at (p^k, x^k) for k large enough. Applying Theorem 3.2, we can assert from the Carathéodory's theorem that, for k large enough, there exist $q \in \mathbb{N}$, $u^k \in \partial_x f(p^k, x^k)$, $t_i^k \in T(p^k, x^k)$, $u_i^k \in \partial_x g_{t_i^k}(p^k, x^k)$ and $\lambda_i^k > 0$ for $i \in \{1, 2, \ldots, q\}$, such that $q \leq n$,

$$-u^k - \sum_{i=1}^{q} \lambda_i^k u_i^k \in N(x^k; \Omega) \tag{4.2}$$

and $\{u_i^k \mid i = 1, \ldots, q\}$ is a linearly independent system.

It remains to show that $q = n$. Suppose that $q < n$. By the compactness of T, we can assume, by taking a subsequence if necessary, that $\{t_i^k\}$ converges to some $t_i \in T$ for each $i \in \{1, \ldots, q\}$. Since $t_i^k \in T(p^k, x^k)$ and the mapping $(p, x, t) \mapsto g_t(p, x)$ is continuous at $(\bar{p}, \bar{x}, \bar{t})$ with any $\bar{t} \in T$, it follows that $t_i \in T(\bar{p}, \bar{x})$.

We claim that for each $i \in \{1, \ldots, q\}$, there exists $\lambda_i \geq 0$ such that

$$\lim_{k \to \infty} \lambda_i^k = \lambda_i. \tag{4.3}$$

Indeed, if our claim is false, then by taking a subsequence if necessary, we can assume that there exists $i_0 \in \{1, \ldots, q\}$ such that

$$\lim_{k \to \infty} \lambda_{i_0}^k = +\infty.$$

Put $\mu^k := \sum_{i=1}^{q} \lambda_i^k$, $k \geq 1$. Then $\lim\limits_{k \to \infty} \mu^k = +\infty$ and there is no loss of generality in assuming that the sequence $\{\frac{\lambda_i^k}{\mu^k}\}_{k \geq k_0}$ converges to some $\mu_i \geq 0$ for each $i \in \{1, \ldots, q\}$. Dividing by μ^k in (4.2) and letting $k \to \infty$, we deduce from Proposition 2.1 and the compactness of $\partial_x f(\bar{p}, \bar{x})$ and $\partial_x g_{t_i}(\bar{p}, \bar{x})$ with (A3) and (A4) that $\lim_{k \to \infty} u^k = u \in \partial_x f(\bar{p}, \bar{x})$, $\lim_{k \to \infty} u_i^k = u_i \in \partial_x g_{t_i}(\bar{p}, \bar{x})$ and

$$-\sum_{i=1}^{q} \mu_i u_i \in N(\bar{x}; \Omega) \text{ with } \sum_{i=1}^{q} \mu_i = 1.$$

This means $0 \in \text{co}(\{\partial_x g_t(\bar{p}, \bar{x}) \mid t \in T(\bar{p}, \bar{x})\}) + N(\bar{x}; \Omega)$ which contradicts assumption (i) and (4.3) follows.

Letting $k \to \infty$ in (4.2), we get $u \in \partial_x f(\bar{p}, \bar{x})$, $u_i \in \partial_x g_{t_i}(\bar{p}, \bar{x})$ $(i = 1, 2, \ldots, q)$,

$$-u - \sum_{i=1}^{q} \mu_i u_i \in N(\bar{x}; \Omega) \text{ with } \{t_1, \ldots, t_q\} \subset T(\bar{p}, \bar{x}) \text{ and } q < n, \tag{4.4}$$

which contradicts assumption (ii). Thus $q = n$ and the proof is complete. □

Proof of Theorem 4.2. Let $\bar{p} \in P$ and $\bar{x} \in \mathcal{S}(\bar{p})$. Suppose, contrary to our claim, that there exist a sequence $\{x^k\}_{k=1}^{\infty} \subset X$ converging to $\bar{x}$, sequences $\{p^k\}_{k=1}^{\infty}$ and $\{\bar{p}^k\}_{k=1}^{\infty}$ belonging to P which both converge to $\bar{p}$ such that $x^k \in \mathcal{S}(p^k)$ and

$$d(x^k, \mathcal{S}(\bar{p}^k)) > kd(p^k, \bar{p}^k) \quad \text{for all } k \geq 1. \tag{4.5}$$

Since $\mathcal{S}$ is inner-semicontinuous at $(\bar{p}, \bar{x})$, it follows that there exist $\bar{x}^k \in \mathcal{S}(\bar{p}^k)$ satisfying $\bar{x}^k \to \bar{x}$ as $k \to +\infty$. It follows from (4.5) that, for each $k \geq 1$ and $x^k \neq \bar{x}^k$,

$$\frac{d(p^k, \bar{p}^k)}{\|x^k - \bar{x}^k\|} < \frac{1}{k}. \tag{4.6}$$

From Lemma 4.3 it follows that, for k large enough, there exist $u^k \in \partial_x f(p^k, x^k)$, $\bar{u}^k \in \partial_x f(\bar{p}^k, \bar{x}^k)$, $t_i^k \in T(p^k, x^k)$, $\bar{t}_i^k \in T(\bar{p}^k, \bar{x}^k)$, $u_i^k \in \partial_x g_{t_i^k}(p^k, x^k)$, $\bar{u}_i^k \in \partial_x g_{\bar{t}_i^k}(\bar{p}^k, \bar{x}^k)$, $\lambda_i^k > 0$, $\bar{\lambda}_i^k > 0$ for all $i \in \{1, 2, \ldots, n\}$ such that both $\{u_1^k, \ldots, u_n^k\}$ and $\{\bar{u}_1^k, \ldots, \bar{u}_n^k\}$ form bases of X and

$$-u^k - \sum_{i=1}^{n} \lambda_i^k u_i^k \in N(x^k; \Omega), \; -\bar{u}^k - \sum_{i=1}^{n} \bar{\lambda}_i^k \bar{u}_i^k \in N(\bar{x}^k; \Omega). \tag{4.7}$$

We can assume, by taking a subsequence if necessary, that for each $i \in \{1, \ldots, n\}$ the sequences $\{t_i^k\}_{k \geq k_0}$ and $\{\bar{t}_i^k\}_{k \geq k_0}$ converge to some t_i and $\bar{t}_i$, respectively. Since the mapping $(p, x, t) \mapsto g_t(p, x)$ is continuous at $(\bar{p}, \bar{x}, \bar{t})$ with any $\bar{t} \in T$, it follows that

$$t_i, \; \bar{t}_i \in T(\bar{p}, \bar{x}), \; i \in \{1, 2, \ldots, n\}.$$

As in the proof of Lemma 4.3, we can assume that $\{\lambda_i^k\}_{k \geq k_0}$ and $\{\bar{\lambda}_i^k\}_{k \geq k_0}$ converge to some λ_i and $\bar{\lambda}_i$, respectively. Since $\partial_x f(\cdot)$ and $\partial_x g_{(\cdot)}(\cdot)$ $(i = 1, 2, \ldots, n)$ are upper semicontinuous at $(\bar{p}, \bar{x})$, by (A3) and (A4), it follows from the compactness of $\partial_x f(\bar{p}, \bar{x})$ and $\partial_x g_i(\bar{p}, \bar{x})$, by taking a subsequence if necessary, that $\lim_{k\to\infty} u^k = u \in \partial_x f(\bar{p}, \bar{x})$, $\lim_{k\to\infty} \bar{u}^k = \bar{u} \in \partial_x f(\bar{p}, \bar{x})$, $\lim_{k\to\infty} \bar{u}_i^k = \bar{u}_i \in \partial_x g_{\bar{t}_i}(\bar{p}, \bar{x})$, $i \in \{1, 2, \ldots, n\}$. Letting $k \to \infty$ in (4.7), by Proposition 2.1, we have

$$-u - \sum_{i=1}^{n} \lambda_i u_i \in N(\bar{x}; \Omega), \; -\bar{u} - \sum_{i=1}^{n} \bar{\lambda}_i \bar{u}_i \in N(\bar{x}; \Omega). \tag{4.8}$$

By the Carathéodory's theorem and assumption (ii), we have $\lambda_i > 0, \bar{\lambda}_i > 0$ for all $i \in \{1, \ldots, n\}$, and both $\{u_1, \ldots, u_n\}$ and $\{\bar{u}_1, \ldots, \bar{u}_n\}$ form bases of X.

On one hand, since $t_i^k \in T(p^k, x^k)$ and $\bar{x}^k \in \mathcal{S}(\bar{p}_k) \subset G(\bar{p}_k)$ for every $i \in \{1, \ldots, n\}$, it follows that

$$g_{t_i^k}(p^k, x^k) = 0, \; g_{t_i^k}(\bar{p}^k, \bar{x}^k) \leq 0. \tag{4.9}$$

By (A5), we deduce that there exist $r \geq 0$ such that

$$\langle u_i^k, \bar{x}^k - x^k \rangle \leq g_{t_i^k}(p^k, \bar{x}^k) - g_{t_i^k}(p^k, x^k) + \frac{r}{2}\|\bar{x}^k - x^k\|^2$$

for k large enough. It follows from (4.9) and (A2) that

$$\begin{aligned} \langle u_i^k, \bar{x}^k - x^k \rangle &\leq g_{t_i^k}(p^k, \bar{x}^k) - g_{t_i^k}(\bar{p}^k, \bar{x}^k) + \frac{r}{2}\|\bar{x}^k - x^k\|^2 \\ &\leq \hbar \|\bar{p}^k - p^k\| + \frac{r}{2}\|\bar{x}^k - x^k\|^2. \end{aligned}$$

Therefore,

$$\langle u_i^k, \frac{\bar{x}^k - x^k}{\|x^k - \bar{x}^k\|} \rangle \leq \hbar \frac{\|\bar{p}^k - p^k\|}{\|x^k - \bar{x}^k\|} + \frac{r}{2}\|\bar{x}^k - x^k\|. \tag{4.10}$$

We can assume, by taking a subsequence if necessary, that $\{\frac{\bar{x}^k - x^k}{||x^k-\bar{x}^k||_n}\}_{k\geq k_0}$ converges to some $z \in X$ with $||z|| = 1$. Letting $k \to \infty$ in (4.10), we can assert that

$$\langle u_i, z\rangle \leq 0 \quad \forall i \in \{1, \dots, n\}. \tag{4.11}$$

From the first inclusion in (4.8) and the prox-regularity of Ω, it follows that there exists $\rho \geq 0$ such that, for k large enough,

$$\langle -u - \sum_{i=1}^{n} \lambda_i u_i, \bar{x}^k - x^k\rangle \leq \frac{\rho}{2}\|\bar{x}^k - x^k\|^2.$$

This implies

$$\langle -u, z\rangle \leq \sum_{i=1}^{n} \lambda_i \langle u_i, z\rangle \leq 0. \tag{4.12}$$

On the other hand, since $\bar{t}_i^k \in T_{\bar{p}^k}(\bar{x}^k)$ and $x^k \in \mathcal{S}(p_k) \subset G(p_k)$ for every $i \in \{1, \dots, n\}$, it follows that

$$g_{\bar{t}_i^k}(\bar{p}^k, \bar{x}^k) = 0,\ g_{\bar{t}_i^k}(p^k, x^k) \leq 0.$$

From this and the prox-regularity of $g_{\bar{t}_i}$ it follows that there exist $\bar{r}_i \geq 0$, $i \in \{1, 2, \dots, n\}$ such that, for k large enough,

$$\langle \bar{u}_i^k, \frac{x^k - \bar{x}^k}{\|x^k - \bar{x}^k\|}\rangle \leq \hbar\frac{\|\bar{p}^k - p^k\|}{\|x^k - \bar{x}^k\|} + \frac{\bar{r}_i}{2}\|x^k - \bar{x}^k\|^2.$$

By the same argument we can show that for every $i \in \{1, \dots, n\}$,

$$\langle \bar{u}_i, -z\rangle \leq 0.$$

This and the second inclusion in (4.8) with the prox-regularity of Ω imply

$$\langle -\bar{u}, -z\rangle \leq \sum_{i=1}^{n} \lambda_i \langle \bar{u}_i, -z\rangle \leq 0.$$

Hence,

$$\langle \bar{u}_i, z\rangle \geq 0,\ \langle -\bar{u}, z\rangle \geq 0. \tag{4.13}$$

It follows from the prox-regularity of f that there exists $r \geq 0$ such that, for k large enough,

$$\langle u_k, \bar{x}^k - x^k\rangle \leq f(p^k, \bar{x}^k) - f(p^k, x^k) + \frac{r}{2}\|\bar{x}^k - x^k\|^2$$

and

$$\langle \bar{u}_k, x^k - \bar{x}^k\rangle \leq f(\bar{p}^k, x^k) - f(\bar{p}^k, \bar{x}^k) + \frac{r}{2}\|x^k - \bar{x}^k\|^2.$$

From this we obtain

$$\begin{aligned}\langle u_k, \bar{x}^k - x^k\rangle + \langle \bar{u}_k, x^k - \bar{x}^k\rangle \leq & f(p^k, \bar{x}^k) - f(\bar{p}^k, \bar{x}^k) + f(\bar{p}^k, x^k) - f(p^k, x^k) \\ & + r\|\bar{x}^k - x^k\|^2 \\ \leq & 2\ell\|\bar{p}^k - p^k\| + r\|\bar{x}^k - x^k\|^2. \end{aligned} \tag{4.14}$$

Dividing both sides of (4.14) by $\|x^k - \bar{x}^k\|$ and letting $k \to \infty$, we have from (4.6) that

$$\langle u, z\rangle + \langle \bar{u}, -z\rangle \leq 0,$$

and so,

$$\langle -\bar{u}, z\rangle \leq \langle -u, z\rangle. \tag{4.15}$$

Combining (4.11), (4.12), (4.13) and (4.15) we conclude that, for every $i \in \{1, \ldots, n\}$

$$\langle \bar{u}_i, z\rangle = \langle u_i, z\rangle = 0.$$

Since both $\{u_1^k, \ldots, u_n^k\}$ and $\{\bar{u}_1^k, \ldots, \bar{u}_n^k\}$ are bases of X, it follows that $z = 0$ which is impossible. The proof is complete. $\square$

Let us examine the following examples.

Example 4.4. In problem (1.1), let $T = [0,1] \cup \{2\}$, $X = P := \mathbb{R}$ and $\Omega := \mathbb{R}$. Let f, g_t $(t \in T)$ be functions defined by $f(p,x) = -x^2 + 3x - 2 + p$,

$$g_t(p,x) = \begin{cases} x - 1 + pt & \text{if } \ t \in [0,1] \\ -x - 1 + p & \text{if } \ t = 2, \quad \forall x \in \mathbb{R}. \end{cases}$$

Let $\bar{p} = 0$ and let $\bar{x} = -1 \in \mathcal{S}(\bar{p})$. We have $N(\bar{x};\Omega) = \{0\}$. It is a simple matter to check that (A1)–(A5) are valid and $\mathcal{S}$ is inner semi-continuous at $(\bar{p},\bar{x})$. Clearly, $T(\bar{p},\bar{x}) = \{2\}$ and $\partial_x g_2(\bar{p},\bar{x}) = -1$ and so, assumption (i) of Theorem 4.2 is fulfilled. Let us examine assumption (ii). We have $\partial_x f(\bar{p},\bar{x}) = 5$, $T(\bar{p},\bar{x}) = \{2\}$. If there exists $T_0 \subset T(\bar{p},\bar{x})$ such that $|T_0| < 1$, then $T_0 = \emptyset$. Hence

$$-\partial_x f(\bar{p},\bar{x}) \cap \Big[\text{cone}\,\Big(\bigcup_{t\in T_0} \partial_x g_t(\bar{p},\bar{x})\Big) + N(\bar{x};\Omega)\Big] = \emptyset,$$

and assumption (ii) is fulfilled. Applying Theorem 4.2 we conclude that $\mathcal{S}$ is pseudo-Lipschitz at $(\bar{p},\bar{x})$.

The following examples show that the assertion of Theorem 4.2 may be false if one of both assumptions (i) and (ii) is violated.

Example 4.5. In problem (1.1), let $T = [0,1] \cup \{2,3,4\} \subset \mathbb{R}$, $X := \mathbb{R}^2$, $P := \mathbb{R}$ and $\Omega := \mathbb{R}^2$. Let $f,\ g_t\ t \in T$ be functions defined by

$$f(p,x) = (x_1)^3 + p,\ \forall x = (x_1,x_2) \in I\!R^2,$$

$$g_t(p,x) = \begin{cases} -x_1 + px_2 - t & \text{if } \ t \in [0,1] \\ -x_1 & \text{if } \ t = 2 \\ -x_2 & \text{if } \ t = 3 \\ x_1 + x_2 - 1 & \text{if } \ t = 4,\ \forall x = (x_1,x_2) \in I\!R^2. \end{cases}$$

Let $\bar{p} = 0$ and let $\bar{x} := (0,0) \in \mathcal{S}(\bar{p})$. We have $N(\bar{x},\Omega) = \{0\}$. It is a simple matter to check that (A1)–(A5) hold. We now check the assumptions (i) and (ii) of Theorem 4.2. Clearly, $T(\bar{p},\bar{x}) = \{0,2,3\}$, $\partial_x g_0(\bar{p},\bar{x}) = \partial_x g_2(\bar{p},\bar{x}) = (-1,0)$ and $\partial_x g_3(\bar{p},\bar{x}) = (0,-1)$. Hence, assumption (i) is fulfilled. However, assumption (ii) is violated. Indeed, letting $T_0 = \{2\} \subset T(\bar{p}, \bar{x})$, we have $|T_0| = 1 < 2$. It is easily seen that $\partial_x f(\bar{p}, \bar{x}) = (0, 0)$ and $\operatorname{cone}\big(\bigcup_{t\in T_0} \partial_x g_t(\bar{p}, \bar{x})\big) = -\mathbb{R}_+ \times \{0\}$. Hence,

$$-\partial_x f(\bar{p}, \bar{x}) \cap \Big[\operatorname{cone}\big(\bigcup_{t\in T_0} \partial_x g_t(\bar{p}, \bar{x})\big) + N(\bar{x}; \Omega)\Big] = \{0\}.$$

Next we examine the pseudo-Lipschitz property of $\mathcal{S}$ at $(\bar{p}, \bar{x})$. Let $\{p^k = \frac{1}{2k}\}_{k=1}^{\infty} \subset P$. Then

$$g_t(p^k, x) = \begin{cases} -x_1 + \frac{1}{2k}x_2 - t & \text{if } t \in [0, 1] \\ -x_1 & \text{if } t = 2 \\ -x_2 & \text{if } t = 3 \\ x_1 + x_2 - 1 & \text{if } t = 4. \end{cases}$$

We see that $\mathcal{S}$ is inner semi-continuous at $(\bar{p}, \bar{x})$. However, $\mathcal{S}$ is not pseudo-Lipschitz at $(\bar{p}, \bar{x})$. Indeed, taking $\bar{p}^k = 0$, $\bar{x}^k = (1, 0) \in \mathcal{S}(\bar{p}^k)$ and $x^k \in \mathcal{S}(p^k) = \{(0, 0)\}$, we have

$$1 = d(\bar{x}^k, \mathcal{S}(p^k)) > \frac{1}{2} \geq k d(p^k, \bar{p}^k) \quad \forall k \geq 1.$$

Thus, $\mathcal{S}$ is not pseudo-Lipschitz at $(\bar{p}, \bar{x})$.

Example 4.6. In problem (1.1), let $T = [0, 1] \cup \{2, 3\}$ and $X = P := \mathbb{R}$ and $\Omega := \mathbb{R}$. Let f, g be functions defined by

$$f(p, x) = -x^4 + x^2,$$

$$g_t(p, x) = \begin{cases} -p^2 x - t & \text{if } t \in [0, 1] \\ -x - 1 & \text{if } t = 2 \\ x - 1 - p^2 & \text{if } t = 3. \end{cases}$$

Let $\bar{p} = 0$ and let $\bar{x} = 1 \in \mathcal{S}(\bar{p})$. We have $N(\bar{x}; \Omega) = \{0\}$. It is a simple matter to check that (A1)–(A5) are satisfied and

$$\mathcal{S}(p) = \begin{cases} \{-1, 0, 1\} & \text{if } p = 0 \\ 1 + p^2 & \text{if } p \neq 0. \end{cases}$$

So, $\mathcal{S}$ is inner semi-continuous at $(\bar{p}, \bar{x})$. Now we check assumptions (i) and (ii) of Theorem 4.2. Clearly, $T(\bar{p}, \bar{x}) = \{0, 3\}$ and $\partial_x g_0(\bar{p}, \bar{x}) = 0$ and $\partial_x g_3(\bar{p}, \bar{x}) = 1$. Hence,

$$0 \in \operatorname{cl}\operatorname{co} \cup_{t\in T(\bar{p},\bar{x})} \partial_x g_t(\bar{p}, \bar{x}) + N(\bar{x}; \Omega)$$

and assumption (i) is violated. Let us examine assumption (ii). It is easily seen that $\partial_x f(\bar{p}, \bar{x}) = -2$. If there exists $T_0 \subset T(\bar{p}, \bar{x})$ such that $|T_0| < 1$ then $T_0 = \emptyset$. Hence,

$$-\partial_x f(\bar{p}, \bar{x}) \cap \Big[\operatorname{cone}\big(\bigcup_{t\in T_0} \partial_x g_t(\bar{p}, \bar{x})\big) + N(\bar{x}; \Omega)\Big] = \emptyset,$$

and so, assumption (ii) is fulfilled.

New we check the pseudo-Lipschitz property of $\mathcal{S}$ at $(\bar{p}, \bar{x})$. Let $\{p^k = \frac{1}{k}\}$. We have $\mathcal{S}(p^k) = \{1 + \frac{1}{k^2}\}$ for every $k \geq 1$. Thus $\mathcal{S}$ is not pseudo-Lipschitz at $(\bar{p}, \bar{x})$.

5. Special cases

In this section we apply the general results obtained in Section 4 to special classes of PSI involving smooth as well as convex optimization problems. In this section, let $\Omega := X$.

We first establish sufficient conditions for pseudo-Lipschitz property of the solution mapping of parametric smooth semi-infinite optimization problems at the reference point.

Proposition 5.1. *Let $\bar{p} \in P$ and $\bar{x} \in \mathcal{S}(\bar{p})$. Suppose that* (A1)–(A4) *hold, that $f(p,\cdot)$ and $g_t(p,\cdot)$, $p \in P$, $t \in T$, are continuously differentiable at $\bar{x}$ and their derivatives are Lipschitz at this point, and that the following conditions hold:*
(i) *there exists $\xi \in X$ such that $\langle \nabla_x g_t(\bar{p},\bar{x}), \xi \rangle < 0$ for all $t \in T(\bar{p},\bar{x})$;*
(ii) *There is no $T_0 \subset T(\bar{p},\bar{x})$ with $|T_0| < n$ satisfying*

$$-\nabla_x f(\bar{p},\bar{x}) \in \operatorname{cone}\Big(\bigcup_{t \in T_0} \nabla_x g_t(\bar{p},\bar{x}) \Big).$$

Then $\mathcal{S}$ is pseudo-Lipschitz at $(\bar{p},\bar{x})$ whenever it is inner-semicontinuous at this point.

Proof. Since $f(p,\cdot)$ and $g_t(p,\cdot)$ ($p \in P$, $t \in T$) are continuously differentiable at $\bar{x}$ and their derivatives are Lipschitz at this point, it follows from [23, Proposition 13.34] that the functions $f(p,\cdot)$ and $g_t(p,\cdot)$ ($t \in T$) are prox-regular at $\bar{x}$. Hence, (A5) is fulfilled. By the separation theorem, we can assert that assumption (i) in Theorem 4.2 is equivalent to

$$0 \notin \operatorname{clco}\{\nabla_x g_t(\bar{p},\bar{x}) \,|\, t \in T(\bar{p},\bar{x})\}.$$

Thus, the assertion of the corollary is immediate from Theorem 4.2. □

Before establishing sufficient conditions for pseudo-Lipschitz property of the solution mapping of parametric convex semi-infinite optimization problems, we recall a important result from [20] on the convergence of subdifferentials of convex functions.

Lemma 5.2 ([20, Theorem 24.5]). *Let φ and φ^k ($k \in \{1,2,\ldots\}$) be convex functions such that φ^k converges pointwise to φ on an open convex set $C \subset X$ as $k \to \infty$. Let $x \in C$ and $\{x^k\}_{k=1}^{\infty} \subset C$ converge to x. Then, for each $\varepsilon > 0$, there exists an index $k_0 \in \mathbb{N}$, such that*

$$\partial \varphi^k(x^k) \subset \partial \varphi(x) + \varepsilon B \text{ for all } k \geq k_0.$$

The following result gives sufficient conditions for pseudo-Lipschitz property of the solution mapping of parametric convex semi-infinite optimization problems at the reference point.

Proposition 5.3. *Let $\bar{p} \in P$ and $\bar{x} \in \mathcal{S}(\bar{p})$. Suppose that* (A1)–(A3) *hold, that $f(p,\cdot)$ and $g_t(p,\cdot)$, $p \in P$, $t \in T$, are convex functions on X and that the following conditions hold:*
(i) *the Slater condition for $G(\bar{p})$, i.e., there is $x^0 \in X$ satisfying $g_t(\bar{p},x^0) < 0$ for all $t \in T$;*

(ii) *There is no* $T_0 \subset T(\bar{p}, \bar{x})$ *with* $|T_0| < n$ *satisfying*

$$-\partial_x f(\bar{p}, \bar{x}) \cap \Big[\text{cone}\,\Big(\bigcup_{t \in T_0} \partial_x g_t(\bar{p}, \bar{x})\Big)\Big] \neq \emptyset. \tag{5.1}$$

Then $\mathcal{S}$ *is pseudo-Lipschitz at* $(\bar{p}, \bar{x})$.

Proof. Clearly, (A4) is valid by Lemma 5.2. Since $f(p, \cdot)$ and $g_t(p, \cdot)$, $t \in T$, are convex functions on X, it follows that they are prox-regular at $\bar{x}$, and so (A5) is valid. By [11, Theorem VI.4.4.2], we can assert that $G(\bar{p})$ satisfies the Slater condition if and only if $0 \notin \text{cl}\,\text{co} \cup_{t \in T(\bar{p}, \bar{x})} \partial_x g_t(\bar{p}, \bar{x})$. We claim that $G(\cdot)$ is lower semicontinuous at $\bar{p}$. Indeed, let W be an open set such that $W \cap G(\bar{p}) \neq \emptyset$. Since $G(\bar{p})$ satisfies the Slater condition and T is compact, there exist an element $\hat{x} \in G(\bar{p})$ and $\rho > 0$ such that

$$g_t(\bar{p}, \hat{x}) \leq -\rho \quad \forall t \in T. \tag{5.2}$$

Take any $\bar{x} \in W \cap G(\bar{p})$ and choose a number $r \in (0, 1]$ such that

$$x_r := \bar{x} + r(\hat{x} - \bar{x}) \in W.$$

By the convexity of $G(\bar{p})$, $x_r \in W \cap G(\bar{p})$. It follows from (5.2) that

$$g_t(\bar{p}, x_r) \leq (1 - r) g_t(\bar{p}, \bar{x}) + r g_t(\bar{p}, \hat{x}) \leq -r\rho \quad \forall t \in T. \tag{5.3}$$

By the continuity of g_t and the compactness of T, there exists a neighborhood U of $\bar{p}$ such that $g_t(p, x_r) \leq 0$ for all $p \in U$ and $t \in T$. Thus, $x_r \in G(p)$ and $W \cap G(p) \neq \emptyset$ for all $p \in U$. This means that G is lower semicontinuous at $\bar{p}$ as claimed.

It suffices to show that $\mathcal{S}$ is inner-semicontinuous at $(\bar{p}, \bar{x})$. We claim that $\mathcal{S}(\bar{p}) = \{\bar{x}\}$. Indeed, since $\bar{x} \in \mathcal{S}(\bar{p})$, it follows from Theorem 3.2, the Carathéodory's theorem and assumption (ii) that there exist $u \in \partial_x f(\bar{p}, \bar{x})$, $t_i \in T(\bar{p}, \bar{x})$, $u_i \in \partial_x g_{t_i}(\bar{p}, \bar{x})$ and $\lambda_i > 0$ for $i \in \{1, \ldots, n\}$ such that $\{u_1, \ldots, u_n\}$ is a basis of X and

$$-u = \sum_{i=1}^{n} \lambda_i u_i. \tag{5.4}$$

Suppose that there exists

$$y \in \mathcal{S}(\bar{p}).$$

Then, for each $i \in \{1, \ldots, n\}$, $g_{t_i}(\bar{p}, \bar{x}) = 0$ and

$$\langle u_i, y - \bar{x} \rangle \leq g_{t_i}(\bar{p}, y) - g_{t_i}(\bar{p}, \bar{x}) = g_{t_i}(\bar{p}, y).$$

Hence,

$$\langle u_i, y - \bar{x} \rangle \leq 0. \tag{5.5}$$

Besides, by (5.4),

$$\langle -u - \sum_{i=1}^{n} \lambda_i u_i, y - \bar{x} \rangle = 0.$$

Therefore, by (5.5),

$$0 = f(\bar{p}, y) - f(\bar{p}, \bar{x}) \geq \langle u, y - \bar{x} \rangle$$

$$= -\sum_{i=1}^{n} \lambda_i \langle u_i, y - \bar{x}\rangle \geq 0.$$

This implies that $\sum_{i=1}^{n} \lambda_i \langle u_i, y - \bar{x}\rangle = 0$ and so, $\langle u_i, y - \bar{x}\rangle = 0$ for every $i \in \{1, \ldots, n\}$. Since $\{u_1, \ldots, u_n\}$ is a basis of X, it follows that there must exist real numbers β_i, $i \in \{1, \ldots, n\}$ such that $y - \bar{x} = \sum_{i=1}^{n} \beta_i u_i$. Hence,

$$||y - \bar{x}||^2 = \langle y - \bar{x}, y - \bar{x}\rangle = \sum_{i=1}^{n} \beta_i \langle u_i, y - \bar{x}\rangle = 0.$$

Thus $y = \bar{x}$ and our claim follows. We next claim that there is $U(\bar{p}) \in \mathcal{N}(\bar{p})$ such that $\mathcal{S}(p) \neq \emptyset$ for all $p \in U(\bar{p})$. Indeed, if our claim is false, then there exists $\{p^k\}_{k=1}^{\infty} \in P$ converging to $\bar{p}$ such that

$$\mathcal{S}(p^k) = \emptyset \text{ for all } k \geq 1. \tag{5.6}$$

It follows from the lower semicontinuity of G at $\bar{p}$ that there exist $\lambda > 0$, $y^k \in G(p^k) \cap \operatorname{cl} \mathbb{B}(\bar{x}, \lambda)$ for all $k \geq 1$ such that $y^k \to \bar{x}$ as $k \to \infty$. For each $k = 1, 2, \ldots$, since $\mathcal{S}(p^k) = \emptyset$, it follows that there exists $z^k \in G(p^k) \setminus \mathbb{B}(\bar{x}, \lambda)$ such that

$$f(p^k, z^k) - f(p^k, y^k) < 0 \tag{5.7}$$

(otherwise, for all $z \in G(p^k) \setminus \mathbb{B}(x^0, \lambda)$, $f(p^k, z) \geq f(p^k, y^k)$. Then $f(p^k, \cdot)$ has a minimizer on $G(p^k) \cap \operatorname{cl} \mathbb{B}(x^0, \lambda)$).

Consider the following two possible cases:

(a) $\lim_{k\to\infty} z^k = z^0$. Letting $k \to \infty$ in (5.7), we obtain from the closedness of G at $\bar{p}$ that $z^0 \in G(\bar{p})$ and

$$f(\bar{p}, z^0) - f(\bar{p}, \bar{x}) \leq 0.$$

Hence, $z^0 = \bar{x}$ which contradicts to the fact that $z^0 \in G(\bar{p}) \setminus \mathbb{B}(\bar{x}, \lambda)$.

(b) $\lim_{k\to\infty} \|z^k\| = +\infty$. Without loss of generality we may assume that

$$\lim_{k\to\infty} \frac{z^k}{\|z^k\|} = \hat{z}, \ \|\hat{z}\| = 1.$$

By convexity of $f(p^k, \cdot)$, we have

$$f\Big(p^k, \frac{1}{\|z^k\|} z^k + (1 - \frac{1}{\|z^k\|}) y^k\Big) - f(p^k, y^k) \leq \frac{1}{\|z^k\|}(f(p^k, z^k) - f(p^k, y^k)).$$

It follows from (5.7) that

$$f\Big(p^k, \frac{1}{\|z^k\|} z^k + (1 - \frac{1}{\|z^k\|}) y^k\Big) - f(p^k, y^k) < 0. \tag{5.8}$$

Obviously, $\frac{1}{\|z^k\|} z^k + (1 - \frac{1}{\|z^k\|}) y^k \in G(p^k)$ for all $k \geq 1$. Letting $k \to \infty$ in (5.8), we have

$$\hat{z} + \bar{x} \in G(\bar{p}) \ \text{ and } \ f(\bar{p}, \hat{z} + \bar{x}) - f(\bar{p}, \bar{x}) \leq 0.$$

Hence $\hat{z} + \bar{x} = \bar{x}$, which is impossible. Combining these two cases gives our claim.

It remains to prove that $\mathcal{S}$ is inner-semicontinuous at $\bar{p}$. It suffices to show that, for every the sequence $\{p^k\}_{k=1}^{\infty}$ converging to $\bar{p}$ there exists a sequence $\{x^k\}_{k=1}^{\infty}$ such that $x^k \in \mathcal{S}(p^k)$ and $x^k \to \bar{x}$ as $k \to \infty$. Given any $\{p^k\}_{k=1}^{\infty} \subset P$ such that $p^k \to \bar{p}$ as $k \to \infty$, let

$$x^k \in \mathcal{S}(p^k).$$

Since G is lower semicontinuous at $\bar{p}$, it follows that for each $k = 1, 2, \ldots$ there exists $w^k \in G(p^k)$ such that $\lim_{k\to\infty} w^k = \bar{x}$. Clearly,

$$f(p^k, x^k) - f(p^k, w^k) \le 0. \tag{5.9}$$

We have that the set $\{x^k\}_{k=1}^{\infty}$ is bounded. Indeed, if $\lim_{k\to\infty} \|x^k\| = +\infty$, then, by taking a subsequence if necessary, we can assume that

$$\lim_{k\to\infty} \frac{x^k}{\|x^k\|} = \hat{x},\ \|\hat{x}\| = 1.$$

It follows from the convexity of $f(p^k, \cdot)$ that

$$f\Big(p^k, \frac{1}{\|x^k\|}x^k + (1 - \frac{1}{\|x^k\|})w^k\Big) - f(p^k, w^k) \le \frac{1}{\|x^k\|}(f(p^k, x^k) - f(p^k, w^k)).$$

Letting $k \to \infty$, we can assert from (5.9) that

$$\hat{x} + \bar{x} \in G(\bar{p}) \ \text{ and } \ f(\bar{p}, \hat{x} + \bar{x}) - f(\bar{p}, \bar{x}) \le 0.$$

This implies that $\hat{x} + \bar{x} = \bar{x}$, a contradiction. Hence, $\{x^k\}_{k=1}^{\infty}$ is bounded. Without loss of generality we can assume that $\lim_{k\to\infty} x^k = \tilde{x} \in G(\bar{p})$. It follows from (5.9) that $f(\bar{p}, \tilde{x}) - f(\bar{p}, \bar{x}) \le 0$. Thus, $\tilde{x} = \bar{x}$ and $\mathcal{S}$ is inner-semicontinuous at $\bar{p}$. The proof is complete. □

We now consider a special case of problem (1.1) which has the form

$$\begin{aligned} (\mathrm{CSI})_{(c,\,b)}: \quad & \min\ \varphi(x) + c^T x \\ & \text{s.t.}\quad \varphi_t(x) \le b_t,\ t \in T, \end{aligned}$$

where T is a compact metric space, $c \in \mathbb{R}^n$, c^T denotes the transpose of c, $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ and $\varphi_t\colon \mathbb{R}^n \to \mathbb{R}$ $(t \in T)$ are given convex functions such that $(x, t) \mapsto \varphi_t(x)$ is continuous on $\mathbb{R}^n \times T$ and $b \in C(T, \mathbb{R})$ (the set of all continuous functions on T). The set of feasible points of $(\mathrm{CSI})_{(c,\,b)}$ is denoted by $\Gamma(c, b)$. $\mathcal{S}(c, b)$ stands for the set of all solutions of $(\mathrm{CSI})_{(c,\,b)}$. The set of active constraints at $x \in \Gamma(c, b)$ is given by

$$T_{(c,\,b)}(x) := \{t \in T \mid \varphi_t(x) = b_t\}.$$

The following corollary is immediate from Proposition 5.3 by taking

$$p := (c, b),\ f(p, x) = \varphi(x) + c^T x,\ g_t(p, x) = \varphi_t(x) - b_t.$$

Corollary 5.4. [5, Theorem 10] *For $(\mathrm{CSI})_{(c,\,b)}$, let $\bar{c} \in \mathbb{R}^n$, $\bar{b} \in C(T, \mathbb{R})$ and $\bar{x} \in \mathcal{S}(\bar{c}, \bar{b})$. Suppose that the following conditions hold:*
(i) *$\Gamma(\bar{c}, \bar{b})$ satisfies the Slater condition;*
(ii) *There is no $T_0 \subset T_{(\bar{c},\,\bar{b})}(\bar{x})$ with $|T_0| < n$ satisfying*

$$-\big(\bar{c} + \partial\varphi(\bar{x})\big) \cap \operatorname{cone}\Big(\bigcup_{t\in T_0} \partial\varphi_t(\bar{x})\Big) \neq \emptyset.$$

Then $\mathcal{S}$ is pseudo-Lipschitz at $((\bar{c}, \bar{b}), \bar{x})$.

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