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STRONG CONVERGENCE THEOREMS FOR GENERALIZED NONEXPANSIVE MAPPINGS WITH THE SYSTEM OF EQUILIBRIUM PROBLEMS IN BANACH SPACES

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ABSTRACT. In this paper, by using the shrinking projection method, we prove strong convergence theorems for finding a common element of the fixed points set of a generalized nonexpansive mapping and the set of common solutions of a family of equilibrium problems with bifunctions defined on the dual space of a Banach space.

1. INTRODUCTION

Let E be a Banach space and C be a nonempty, closed and convex subset of E. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. Denote the set of all fixed points of T by $F(T)$, i.e., $F(T) = \{x \in C : Tx = x\}$.

In 2008, Takahashi, Takeuchi, and Kubota [19] introduced an iterative method called the shrinking projection as follows: $x_0 \in H, C_1 = C, x_1 = P_{C_1}x_0$ and

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \end{cases}$$

where $n \in \mathbb{N}$ and $\{\alpha_n\} \subset [0,1]$ and P_{C_n} is the metric projection of H onto C_n. Under the certain control conditions, they proved that the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$. Since then, many authors used the shrinking projection method for proving strong convergence theorems in Banach spaces. (See for instance, [10, 11, 12] and the references therein.) Later, in 2009, Kimura and Takahashi [14] proved strong convergence theorems for a family of relatively nonexpansive mappings in a Banach space by using the shrinking projection method.

Let f be a bifunction from $C \times C$ to $\mathbb{R}$. The equilibrium problem is to find

$$\hat{x} \in C \quad \text{such that } f(\hat{x}, y) \ge 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(f)$. Finding such a solution is an important way to solve numerous problems in optimization problems, variational inequalities, minimax problems, games theory and others. (See Blum and Oettli [3],

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Combettes and Hirstoaga [5] and Moudafi [17].) The *system of equilibrium problems* is to find $\hat{x} \in C$ such that

$$\hat{x} \in C \quad \text{such that } f_\lambda(\hat{x}, y) \geq 0, \quad \forall y \in C \text{ and } \forall \lambda \in \Lambda. \tag{1.2}$$

If Λ is a singleton, then the problem (1.2) reduces to the problem (1.1).

In 2008, Takahashi and Zembayashi [21] considered the equilibrium problem with a bifunction defined on the dual space of a Banach space as follows: Let E be a smooth Banach space with its dual E^*. Let C be a nonempty and closed subset of E such that JC is a closed and convex subset of E^*, where J is the duality mapping on E. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$. Then the equilibrium problem is to find

$$\hat{x} \in C \quad \text{such that } f(J\hat{x}, Jy) \geq 0, \quad \forall y \in C.$$

Consequently, the authors obtained a strong convergence theorem for finding a solution of the equilibrium problem. Since then, the problem has been studied by the others such as [6, 9, 22]. Recently, Takahashi and Yao [20] proved strong convergence theorems for nonlinear operators with this equilibrium problem in Banach spaces.

In this paper, motivated by [14] and [20], we prove strong convergence theorems for generalized nonexpansive mappings with the system of equilibrium problems with bifunctions defined on the dual of a Banach space.

2. Preliminaries

Let E be a real Banach space with its dual E^*. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. Let $B = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteuax differentiable* if for each $x, y \in B$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called *smooth*. The norm of E is said to be *uniformly Gâteuax differentiable* if for each $y \in B$, the limit (2.1) is attained uniformly for all $x \in B$. It is also said to be *Fréchet differentiable* if for each $x \in B$, the limit (2.1) is attained uniformly for all $y \in B$. We denote the value of x^* at x by $\langle x, x^* \rangle$. Then the duality mapping J on E defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. It is known that if the norm of E is uniformly Gâteuax differentiable, then the duality mapping J is single valued and uniformly norm to weak* continuous on each bounded subset of E. If the norm of E is Fréchet differentiable, then J is norm-to-norm continuous. Furthermore, if E is smooth, strictly convex and reflexive, then the duality mapping J is single-valued, one-to-one and onto; see [4, 18] for more details.

Let E be a smooth Banach space and consider the following function studied by Alber [1] and Kamimura and Takahashi [13]. Let $\phi : E \times E \to [0, \infty)$ be the mapping defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $(x, y) \in E \times E$. We know that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.2}$$

for each $x, y, z \in E$. By the fact that $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$, for all $x, y \in E$. Let $\phi_* : E^* \times E^* \to [0, \infty)$ be the mapping defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2$$

for all $(x^*, y^*) \in E^* \times E^*$. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx)$$

for all $x, y \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Leftrightarrow x = y.$$

As is well known, the authors [13] proved the following theroem.

Theorem 2.1 ([13])**.** *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

If $\{C_n\}$ is a sequence of nonempty, closed and convex subsets of a reflexive Banach space E, the set $s - Li_nC_n$ is called the set of limit points of $\{C_n\}$ and $w - Ls_nC_n$ is called the set of weak cluster points of $\{C_n\}$.That is, $x \in s - Li_nC_n$ if and only if there exists $\{x_n\} \subset E$ such that $x_n \in C_n$ for each $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$; and $y \in w - Ls_nC_n$ if and only if there exists $\{y_{n_i}\} \subset E$ such that $y_{n_i} \in C_{n_i}$ for each $i \in \mathbb{N}$ and $y_{n_i} \rightharpoonup y$ as $i \to \infty$. If C_0 satisfies that

$$s - Li_nC_n = C_0 = w - Ls_nC_n,$$

then we say that $\{C_n\}$ converges to C_0 in the sense of Mosco [16] and we denote by $C_0 = M - \lim_{n\to\infty} C_n$. If $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco; see [16] for more details.

In 2003, the following was proved by Ibaraki, Kimura, and Takahashi [7].

Theorem 2.2 ([7])**.** *Let E be a smooth Banach space such that E^* has a Fréchet differentiable norm. Let $\{C_n\}$ be a sequence of nonempty, closed and convex subsets of E. If $C_0 = M - \lim_{n\to\infty} C_n$ exists and nonempty, then for each $x \in E$, $\Pi_{C_n}x$ converges strongly to $\Pi_{C_0}x$, where Π_{C_n} and Π_{C_0} are the generalized projections of E onto C_n and C_0, respectively.*

Let C be a nonempty, closed and convex subset of a smooth Banach space E. A mapping $T : C \to C$ is a *generalized nonexpansive type* if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(x, Ty) + \phi(y, Tx)$$

for all $x, y \in C$.

A mapping $T : C \to E$ is called *generalized nonexpansive* [8] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y), \qquad \forall x \in C\ , \forall y\ \in F(T),$$

where $F(T)$ is the set of fixed points of T. A point $p \in C$ is said to be a generalized asymptotic fixed point of T if there exists a sequence $\{x_n\}$ in C such that $Jx_n \rightharpoonup Jp$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$; see [10]. We denote the set of all generalized asymptotic fixed points of T by $\check{F}(T)$.

Recently, Takahashi and Yao [20] proved the following:

Theorem 2.3 ([20])**.** *Let E be a smooth and reflexive Banach space such that E^* has a uniformly Gâteaux differentiable norm. Let C be a closed subset of E such that JC is closed and convex and let $T : C \to C$ be a generalized nonexpansive type mapping. Then, the followings hold:*

(1) $\check{F}(T) = F(T)$;
(2) $JF(T)$ *is closed and convex;*
(3) $F(T)$ *is closed.*

A nonempty and closed subset C of a smooth Banach space E is said to be a *sunny generalized nonexpansive retract of E* if there exists a sunny generalized nonexpansive retraction R from E onto C. We know the following lemmas from Ibaraki and Takahashi [8]:

Lemma 2.4 ([8])**.** *Let C be a nonempty, closed and convex subset of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E to C is uniquely determined.*

Lemma 2.5 ([8])**.** *Let C be a nonempty, closed and convex subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the followings hold:*

(1) $z = Rx$ *if and only if* $\langle x - z, Jy - Jz\rangle \leq 0$ *for all* $y \in C$;
(2) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

In 2007, Kohsaka and Takahashi [15] proved the following results:

Theorem 2.6 ([15])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let C_* be a nonempty, closed and convex subset of E^*. Suppose that Π_{C_*} is the generalized projection of E^* onto C_*. Then R defined by $R = J^{-1}\Pi_{C_*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C_*$.*

Theorem 2.7 ([15])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty and closed subset of E. Then the followings are equivalent:*

(1) *C is a sunny generalized nonexpansive retract of E;*
(2) *C is a generalized nonexpansive retract of E;*
(3) *JC is closed and convex.*

Lemma 2.8 ([15])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty and closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the followings are equivalent:*

(1) $z = Rx$;
(2) $\phi(x, z) = min_{y \in C} \phi(x, y)$.

We know some structures of the set of fixed points of a generalized nonexpansive mapping from the following results:

Theorem 2.9 ([10, 12]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that JC is closed and convex. If $T : C \to C$ is a generalized nonexpansive mapping such that $F(T) \neq \emptyset$, then $F(T)$ is closed and $JF(T)$ is closed and convex.*

Theorem 2.10 ([10, 12]). *Let E be a smooth, strictly convex and reflexive Banach space and C be a closed subset of E such that JC is closed and convex. If $T : C \to C$ is a generalized nonexpansive mapping such that $F(T) \neq \emptyset$, then $F(T)$ is a sunny generalized nonexpansive retract of E.*

Let C be a nonempty and closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex. For solving the equilibrium problem, let us assume that a bifunction f from $JC \times JC$ to $\mathbb{R}$ satisfying the following conditions:
(A1) $f(x^*, x^*) = 0$ for all $x^* \in JC$;
(A2) f is monotone, i.e. $f(x^*, y^*) + f(y^*, x^*) \leq 0$ for all $x^*, y^* \in JC$;
(A3) for all $x^*, y^*, z^* \in JC$,

$$\limsup_{t \downarrow 0} f(tz^* + (1-t)x^*, y^*) \leq f(x^*, y^*);$$

(A4) for all $x^* \in JC, f(x^*, \cdot)$ is convex and lower semicontinuous.
The followings are some results on such a bifunction :

Lemma 2.11 ([2, 3]). *Let C be a closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex, let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, let $r > 0$ and let $x \in E$. Then, there exists $z \in C$ such that*

$$f(Jz, Jy) + \frac{1}{r}\langle z - x, Jy - Jz \rangle \geq 0 \quad \textit{for all } y \in C.$$

Lemma 2.12 ([21]). *Let C be a closed subset of a uniformly smooth and strictly convex Banach space E such that JC is closed and convex, and let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1) - (A4)$. For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows :*

$$T_r(x) = \{z \in C : f(Jz, Jy) + \frac{1}{r}\langle z - x, Jy - Jz \rangle \geq 0 \textit{ for all } y \in C\}$$

for all $x \in E$. Then, the followings hold:
(1) *T_r is single-valued;*
(2) *for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle x - y, JT_r x - JT_r y \rangle;$$

(3) *$F(T_r) = EP(f)$;*
(4) *$JEP(f)$ is closed and convex.*

Lemma 2.13 ([21]). *Let C be a closed subset of a smooth, strictly convex, and reflexive Banach space E such that JC is closed and convex, let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(x, T_r x) + \phi(T_r x, q) \leq \phi(x, q).$$

3. Strong convergence theorems with a system of equilibrium problems

Motivated by [20] and [14], we obtain the following theorem.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty and closed subset of E such that JC is closed and convex. Let $\{f_\lambda : \lambda \in \Lambda\}$ be a family of bifunctions from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $T : C \to C$ be a generalized nonexpansive mapping such that $\breve{F}(T) = F(T)$ and $F(T) \cap F^* \neq \emptyset$, where $F^* = \bigcap_{\lambda \in \Lambda} EP(f_\lambda)$. Let $C_1 = C$ and $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} z_{n,\lambda} \in C \text{ such that } f_\lambda(Jz_{n,\lambda}, Jy) + \frac{1}{r_\lambda}\langle z_{n,\lambda} - x_n, Jy - Jz_{n,\lambda}\rangle \geq 0, \\ \hfill \forall y \in C \text{ and } \forall \lambda \in \Lambda, \\ y_{n,\lambda} = \alpha_n x_n + (1-\alpha_n) T z_{n,\lambda}, \quad \forall \lambda \in \Lambda, \\ C_{n+1} = \{z \in C_n : \sup_{\lambda \in \Lambda} \phi(y_{n,\lambda}, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x \end{cases}$$

for every $n \in \mathbb{N}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1]$ such that $0 \leq \alpha_n \leq a < 1$ and $\{r_\lambda\} \subset [0, \infty)$ such that $0 < b \leq r_\lambda$ for some $a, b \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $R_{F(T)\cap F^*} x$, where $R_{F(T)\cap F^*}$ is the sunny generalized nonexpansive retraction from E onto $F(T) \cap F^*$.

Proof. We first show that $\{x_n\}$ is well-defined. For each $n \in \mathbb{N}$, put $z_{n,\lambda} = T_{r_\lambda} x_n$ and let $z \in F(T) \cap F^*$. From $z_{n,\lambda} = T_{r_\lambda} x_n$ and Lemma 2.13, we have that for any $n \in \mathbb{N}$

$$\phi(z_{n,\lambda}, z) = \phi(T_{r_\lambda} x_n, z) \leq \phi(x_n, z).$$

From the assumption, we have $F(T) \cap F^* \subset C = C_1$. Suppose that $F(T) \cap F^* \subset C_k$, hence $z \in C_k$. From the definition of T, we have that for all $\lambda \in \Lambda$

$$\begin{aligned} \phi(y_{n,\lambda}, z) &= \phi(\alpha_n x_n + (1-\alpha_n) T z_{n,\lambda}, z) \\ &\leq \alpha_n \phi(x_n, z) + (1-\alpha_n)\phi(T z_{n,\lambda}, z) \\ &\leq \alpha_n \phi(x_n, z) + (1-\alpha_n)\phi(z_{n,\lambda}, z) \\ &\leq \alpha_n \phi(x_n, z) + (1-\alpha_n)\phi(x_n, z) \\ &= \phi(x_n, z). \end{aligned}$$

Thus, $\sup_{\lambda \in \Lambda} \phi(y_{n,\lambda}, z) \leq \phi(x_n, z)$. This implies that $z \in C_{k+1}$ and hence C_n is nonempty for all $n \in \mathbb{N}$. By the definition of ϕ, we obtain that

$$C_{n+1} = \{z \in C_n : \sup_{\lambda \in \Lambda} \phi(y_{n,\lambda}, z) \leq \phi(x_n, z)\}$$

$$\begin{aligned} &= \bigcap_{\lambda\in\Lambda}\{z \in C_n : \phi(y_{n,\lambda}, z) \le \phi(x_n, z)\} \\ &= \bigcap_{\lambda\in\Lambda}\{z \in C : 2\langle x_n - y_{n,\lambda}, Jz\rangle + \|y_{n,\lambda}\|^2 - \|x_n\|^2 \le 0\} \cap C_n, \end{aligned}$$

which implies that C_n is closed for all $n \in \mathbb{N}$. Since J is injective, we get that

$$\begin{aligned} JC_{n+1} &= J\left(\bigcap_{\lambda\in\Lambda}\{z \in C : 2\langle x_n - y_{n,\lambda}, Jz\rangle + \|y_{n,\lambda}\|^2 - \|x_n\|^2 \le 0\} \cap C_n\right) \\ &= \bigcap_{\lambda\in\Lambda}\{z^* \in JC : 2\langle x_n - y_{n,\lambda}, z^*\rangle + \|y_{n,\lambda}\|^2 - \|x_n\|^2 \le 0\} \cap JC_n, \end{aligned}$$

and thus JC_n is closed and convex for all $n \in \mathbb{N}$. By Theorem 2.7 and Lemma 2.4, there exists a unique sunny generalized retraction R_{C_n} of E onto C_n. In addition, from Theorem 2.6, we know that R_{C_n} is denoted by $J^{-1}\Pi_{JC_n}J$, where Π_{JC_n} is the generalized projection of E^* onto JC_n. Thus $\{x_n\}$ is well-defined.

Since $\{JC_n\}$ is a nonincreasing sequence of nonempty, closed and convex subsets of E^* with respect to inclusion, it follows that

$$M - \lim_{n\to\infty} JC_n = \bigcap_{n=1}^{\infty} JC_n. \tag{3.1}$$

Thus, $\emptyset \neq JF(T) \cap JF^* \subset \bigcap_{n=1}^{\infty} JC_n$. Put $C_0^* = \bigcap_{n=1}^{\infty} JC_n$. From Theorem 2.2, we have that $\{\Pi_{JC_n}Jx\}$ converges strongly to $x_0^* = \Pi_{C_0^*}Jx$. Since E^* has a Fréchet differentiable norm, J^{-1} is continuous. Then, we have

$$J^{-1}\Pi_{JC_n}Jx \to J^{-1}x_0^*.$$

Since $x_n = R_{C_n}x = J^{-1}\Pi_{JC_n}Jx$, we can conclude that $\{x_n\}$ converges strongly to $J^{-1}x_0^*$. We shall show that $J^{-1}x_0^* = R_{F(T)\cap F^*}x$.

Since $x_n = R_{C_n}x$ and $x_{n+1} = R_{C_{n+1}}x \subset C_{n+1} \subset C_n$, we have from Lemma 2.5 and (2.2) that

$$\begin{aligned} 0 &\le 2\langle x - x_n, Jx_n - Jx_{n+1}\rangle \\ &= \phi(x, x_{n+1}) - \phi(x, x_n) - \phi(x_n, x_{n+1}) \\ &\le -\phi(x, x_n) + \phi(x, x_{n+1}) \end{aligned}$$

which implies $\phi(x, x_n) \le \phi(x, x_{n+1})$ for all $n \in \mathbb{N}$. Further, since $x_n = R_{C_n}x$ and $z \in F(T) \cap F^*$, we have from Lemma 2.8 that

$$\phi(x, x_n) \le \phi(x, z).$$

Thus, $\{\phi(x, x_n)\}$ is nondecreasing and bounded which imply that $\lim_{n\to\infty} \phi(x, x_n)$ exists and hence $\{x_n\}$ is bounded. Therefore, $\{y_{n,\lambda}\}$, $\{z_{n,\lambda}\}$ and $\{Tz_{n,\lambda}\}$ are also bounded for all $\lambda \in \Lambda$. From Lemma 2.5, we have

$$\begin{aligned} \phi(x_n, x_{n+1}) &= \phi(R_{C_n}x, x_{n+1}) \\ &\le \phi(x, x_{n+1}) - \phi(x, R_{C_n}x) \end{aligned}$$

$$= \phi(x, x_{n+1}) - \phi(x, x_n) \to 0.$$

Then, we have

$$\lim_{n\to\infty} \phi(x_n, x_{n+1}) = 0. \tag{3.2}$$

Since $x_{n+1} \in C_{n+1}$, $\sup_{\lambda\in\Lambda} \phi(y_{n,\lambda}, x_{n+1}) \le \phi(x_n, x_{n+1})$.
Therefore, $\lim_{n\to\infty} \phi(y_{n,\lambda}, x_{n+1}) = 0$ for all $\lambda \in \Lambda$.
By using Theorem 2.1, we have

$$\lim_{n\to\infty} \|y_{n,\lambda} - x_{n+1}\| = \lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.3}$$

Then for all $\lambda \in \Lambda$, we have

$$\begin{aligned} \|y_{n,\lambda} - x_n\| &\le \|y_{n,\lambda} - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\to 0. \end{aligned}$$

Since $\|x_n - y_{n,\lambda}\| = \|x_n - (\alpha_n x_n + (1-\alpha_n)Tz_{n,\lambda})\| = (1-\alpha_n)\|x_n - Tz_{n,\lambda}\|$ and $0 \le \alpha_n \le a < 1$, we have that for all $\lambda \in \Lambda$

$$\lim_{n\to\infty} \|x_n - Tz_{n,\lambda}\| = 0. \tag{3.4}$$

Let $z \in F(T) \cap F^*$. Using $z_{n,\lambda} = T_{r_\lambda} x_n$ and Lemma 2.13, we have that for each $n \in \mathbb{N}$ and $\lambda \in \Lambda$,

$$\begin{aligned} \phi(x_n, z) &\ge \phi(x_n, T_{r_\lambda} x_n) + \phi(T_{r_\lambda} x_n, z) \\ &= \phi(x_n, z_{n,\lambda}) + \phi(z_{n,\lambda}, z), \end{aligned}$$

and hence $\phi(x_n, z_{n,\lambda}) \le \phi(x_n, z) - \phi(z_{n,\lambda}, z)$. Using the definition of ϕ again, we get that

$$\begin{aligned} \phi(y_{n,\lambda}, z) &= \phi((\alpha_n x_n + (1-\alpha_n)Tz_{n,\lambda}), z) \\ &\le \alpha_n \phi(x_n, z) + (1-\alpha_n)\phi(z_{n,\lambda}, z). \end{aligned}$$

Thus,

$$\phi(z_{n,\lambda}, z) \ge \frac{\phi(y_{n,\lambda}, z) - \alpha_n \phi(x_n, z)}{1-\alpha_n}$$

which implies that

$$\begin{aligned} \phi(x_n, z_{n,\lambda}) &\le \phi(x_n, z) - \frac{\phi(y_{n,\lambda}, z) - \alpha_n \phi(x_n, z)}{1-\alpha_n} \\ &= \frac{\phi(x_n, z) - \phi(y_{n,\lambda}, z)}{1-\alpha_n}. \end{aligned}$$

Consider

$$\begin{aligned} \phi(x_n, z) - \phi(y_{n,\lambda}, z) &= \|x_n\|^2 - 2\langle x_n, Jz\rangle + \|z\|^2 - \|y_{n,\lambda}\|^2 + 2\langle y_{n,\lambda}, Jz\rangle - \|z\|^2 \\ &= \|x_n\|^2 - \|y_{n,\lambda}\|^2 - 2\langle x_n - y_{n,\lambda}, Jz\rangle \\ &\le \left|\|x_n\|^2 - \|y_{n,\lambda}\|^2\right| + 2\left|\langle x_n - y_{n,\lambda}, Jz\rangle\right| \\ &\le (\|x_n\| + \|y_{n,\lambda}\|)\|x_n - y_{n,\lambda}\| + 2\|x_n - y_{n,\lambda}\|\|Jz\|. \end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - y_{n,\lambda}\| = 0$, $\lim_{n\to\infty} (\phi(x_n, z) - \phi(y_{n,\lambda}, z)) = 0$ for all $\lambda \in \Lambda$.
Since $0 \le \alpha_n \le a < 1$, we have $\lim_{n\to\infty} \phi(x_n, z_{n,\lambda}) = 0$ for all $\lambda \in \Lambda$. From Theorem 2.1 again, we get that

$$\lim_{n\to\infty} \|x_n - z_{n,\lambda}\| = 0. \tag{3.5}$$

Since $\|z_{n,\lambda} - Tz_{n,\lambda}\| \le \|z_{n,\lambda} - x_n\| + \|x_n - Tz_{n,\lambda}\|$ and from (3.4) and (3.5), we have

$$\lim_{n\to\infty} \|z_{n,\lambda} - Tz_{n,\lambda}\| = 0 \tag{3.6}$$

for all $\lambda \in \Lambda$. Since $Jx_n = \prod_{JC_n} Jx \to x_0^* = JJ^{-1}x_0^*$, we have $Jz_{n,\lambda} \to x_0^*$. Thus, from (3.6) and $\check{F}(T) = F(T)$, we can conclude that $J^{-1}x_0^* \in F(T)$.

Next, we show that $J^{-1}x_0^* \in F^*$. From $x_n \to J^{-1}x_0^*$ and (3.5), we have $z_{n,\lambda} \to J^{-1}x_0^*$ for all $\lambda \in \Lambda$. By the definition of $z_{n,\lambda}$, we have that for all $y \in C$ and for all $\lambda \in \Lambda$,

$$f_\lambda(Jz_{n,\lambda}, Jy) + \frac{1}{r_\lambda}\langle Jy - Jz_{n,\lambda}, z_{n,\lambda} - x_n\rangle \ge 0.$$

From $(A2)$, we have that for all $\lambda \in \Lambda$

$$\frac{1}{r_\lambda}\langle Jy - Jz_{n,\lambda}, z_{n,\lambda} - x_n\rangle \ge f_\lambda(Jy, Jz_{n,\lambda}).$$

From $0 < b \le r_\lambda$ and (3.5), we get

$$\lim_{n\to\infty} \frac{z_{n,\lambda} - x_n}{r_\lambda} = 0.$$

Therefore,

$$0 \ge f_\lambda(Jy, x_0^*). \tag{3.7}$$

For all $t \in (0, 1]$ and $y \in C$, put $z_t^* = tJy + (1-t)x_0^*$. Since JC is convex, $z_t^* \in JC$. By using $(A1), (A4)$ and (3.7), we have that for all $\lambda \in \Lambda$

$$\begin{aligned} 0 &= f_\lambda(Jz_t^*, Jz_t^*) \\ &\le tf_\lambda(z_t^*, Jy) + (1-t)f_\lambda(z_t^*, x_0^*) \\ &\le tf_\lambda(z_t^*, Jy). \end{aligned}$$

Dividing by t, we have $f_\lambda(z_t^*, Jy) \ge 0$ for all $y \in C$. Letting $t \to 0$, we have from $(A3)$ that

$$f_\lambda(x_0^*, Jy) \ge 0$$

for all $y \in C$. Therefore, we have $J^{-1}x_0^* \in EP(f_\lambda)$ for all $\lambda \in \Lambda$ and hence $J^{-1}x_0^* \in F^*$. This implies that $J^{-1}x_0^* \in F(T) \cap F^*$.

Finally, we show that $J^{-1}x_0^* = R_{F(T)\cap F^*}x$. From Theorem 2.9, we have that $F(T)$ is closed and $JF(T)$ is closed and convex. Moreover, we have from Lemma 2.12 that $EP(f_\lambda)$ is closed and $JEP(f_\lambda)$ is closed and convex for each $\lambda \in \Lambda$. Since $F^* = \bigcap_{\lambda\in\Lambda} EP(f_\lambda)$ and J is injective, $J(F(T) \cap F^*)$ is also closed and convex. Then, we have from Theorem 2.7 that $F(T) \cap F^*$ is a sunny generalized nonexpansive retract of E. Let $z_0 = R_{F(T)\cap F^*}x$. Then $z_0 \in C_{n+1}$. Since $x_{n+1} = R_{C_{n+1}}x$, we have

$$\phi(x, x_{n+1}) \le \phi(x, z_0).$$

Therefore,

$$\phi(x, J^{-1}x_0^*) = \lim_{n\to\infty} \phi(x, x_n) \leq \phi(x, z_0).$$

Thus, $z_0 = J^{-1}x_0^*$ and hence we can complete the proof. □

By Theorem 3.1, we have the following result.

Corollary 3.2. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty and closed subset of E such that JC is closed and convex. Let $\{f_\lambda : \lambda \in \Lambda\}$ be a family of bifunctions from $JC \times JC$ to $\mathbb{R}$ satisfying (A1)−(A4). Let $T : C \to C$ be a generalized nonexpansive type mapping and $F(T) \cap F^* \neq \emptyset$, where $F^* = \bigcap_{\lambda\in\Lambda} EP(f_\lambda)$. Let $C_1 = C$ and $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} z_{n,\lambda} \in C \text{ such that } f_\lambda(Jz_{n,\lambda}, Jy) + \frac{1}{r_\lambda}\langle z_{n,\lambda} - x_n, Jy - Jz_{n,\lambda}\rangle \geq 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall y \in C \text{ and } \forall \lambda \in \Lambda, \\ y_{n,\lambda} = \alpha_n x_n + (1-\alpha_n)Tz_{n,\lambda}, \quad \forall \lambda \in \Lambda, \\ C_{n+1} = \{z \in C_n : \sup_{\lambda\in\Lambda} \phi(y_{n,\lambda}, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x \end{cases}$$

for every $n \in \mathbb{N}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1]$ such that $0 \leq \alpha_n \leq a < 1$ and $\{r_\lambda\} \subset [0,\infty)$ such that $0 < b \leq r_\lambda$ for some $a, b \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $R_{F(T)\cap F^*}x$, where $R_{F(T)\cap F^*}$ is the sunny generalized nonexpansive retraction from E onto $F(T) \cap F^*$.

Proof. By the assumption, we have $F(T) \neq \emptyset$ and hence T is a generalized nonexpansive mapping. From Theorem 2.3, we can use Theorem 3.1 to complete the proof. □

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