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AN EXTRAGRADIENT-LIKE METHOD FOR GENERALIZED MIXED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS

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Abstract. In this paper, we first introduce a new general iterative scheme based on the extragradient-like method for finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of solutions of a variational inequality and the set of common fixed points for a finite family of nonexpansive mappings in the framework of a Hilbert space. Then we show that the proposed iterative sequence converges strongly to a common element. The results obtained in this paper generalize and improve the recent ones announced by many others.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let P_C be the metric projection of H onto C. A mapping T of H into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in H$. We denote by $F(T)$ the set of fixed points of T(i.e. $F(T) = \{x \in H : Tx = x\}$). Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings with $\cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\varphi : C \to R \cup \{+\infty\}$ be a proper extended real-valued function and Θ be a bifunction from $C \times C$ to R, where R is the set of all real numbers.

Recently, Peng and Yao [8] introduced the following generalized mixed equilibrium problem of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle Fx, y - x \rangle \geq 0, \quad \forall\, y \in C, \tag{1.1}$$

where $F : C \to H$ is a nonlinear mapping. The set of solutions of (1.1) is denoted by $GMEP$. Very recently, Ceng and Yao [2] also considered this problem. Here some special cases of problem (1.1) are stated as follows:

If $F = 0$, then problem (1.1) reduces to the following mixed equilibrium problem of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall\, y \in C,$$

which was considered by Ceng and Yao [3] The set of solutions of this problem is denoted by MEP.

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If $\varphi = 0$, then problem (1.1) reduces to the following generalized equilibrium problem of finding $x \in C$ such that

$$\Theta(x, y) + \langle Fx, y - x\rangle \geq 0, \quad \forall\, y \in C,$$

which was studied by Takahashi and Takahashi [14].

If $\varphi = 0$ and $F = 0$, then problem (1.1) reduces to the following equilibrium problem of finding $x \in C$ such that

$$\Theta(x, y) \geq 0, \ \forall\, y \in C. \tag{1.2}$$

The set of solutions of problem (1.2) is denoted by EP.

If $\Theta = 0, \varphi = 0$ and $F = A$, then problem (1.1) reduces to the following classical variational inequality problem of finding $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \ \forall\, y \in C. \tag{1.3}$$

The set of solutions of problem (1.3) is denoted by $VI(C, A)$.

The problem (1.1) is very general in the sense that it includes, as special cases, numerous problems in physics, optimization, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games and others; see, for instant [1–5], [7–11], [13–18].

In 1999, Atsushiba and Takahashi [1] defined the mapping W_n as follows:

$$\begin{aligned}
&U_{n,1} = \lambda_{n,1}T_1 + (1 - \lambda_{n,1})I,\\
&U_{n,2} = \lambda_{n,2}T_2U_{n,1} + (1 - \lambda_{n,2})I,\\
&U_{n,3} = \lambda_{n,3}T_3U_{n,2} + (1 - \lambda_{n,3})I,\\
&\qquad\vdots\\
&U_{n,N-1} = \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1 - \lambda_{n,N-1})I,\\
&W_n = U_{n,N} = \lambda_{n,N}T_NU_{n,N-1} + (1 - \lambda_{n,N})I,
\end{aligned}$$

where $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}, \ldots, \{\lambda_{n,N}\} \subset [0, 1]$. This mapping is called the $W-$mapping generalized by $T_1, T_2, \ldots,\ T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. In 2000, Takahashi and Shimoji [15] proved that if X is a strictly convex Banach space, then $F(W_n) = \cap_{i=1}^{N} F(T_i)$, where $0 < \lambda_{n,i} < 1, i = 1, 2, \ldots, N$.

Very recently, A. Kangtunyakarn and S. Suantai [5] defined the mapping S_n as follows:

$$\begin{aligned}
U_{n,0} &= I\\
U_{n,1} &= \alpha_1^{n,1}T_1U_{n,0} + \alpha_2^{n,1}U_{n,0} + \alpha_3^{n,1}I,\\
U_{n,2} &= \alpha_1^{n,2}T_2U_{n,1} + \alpha_2^{n,2}U_{n,1} + \alpha_3^{n,2}I,\\
U_{n,3} &= \alpha_1^{n,3}T_3U_{n,2} + \alpha_2^{n,3}U_{n,2} + \alpha_3^{n,3}I,\\
&\ \ \vdots\\
U_{n,N-1} &= \alpha_1^{n,N-1}T_{N-1}U_{n,N-2} + \alpha_2^{n,N-1}U_{n,N-2} + \alpha_3^{n,N-1}I,\\
S_n &= U_{n,N} = \alpha_1^{n,N}T_NU_{n,N-1} + \alpha_2^{n,N}U_{n,N-1} + \alpha_3^{n,N}I,
\end{aligned} \tag{1.4}$$

where for each $n \in N$, and $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j},\ \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$ with $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$. This mapping S_n is called the S-mapping generalized by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Furthermore, they showed that if X is a strictly convex Banach space, then $F(S_n) = \cap_{i=1}^N F(T_i)$ if $\alpha_1^{n,j} \in (0,1)$ for all $j = 1, 2, \ldots, N-1, \alpha_1^{n,N} \in (0,1]$ and $\alpha_2^{n,j}, \alpha_3^{n,j} \in [0,1)$ for all $j = 1, 2, \ldots, N$. It is easy to see $S-$mapping is the generalization of $W-$mapping.

On the other hand, for finding a common element of $F(S)$ and $VI(C, A)$ for a monotone, Lipschitz continuous mapping, Ceng and Yao introduced an extragradient-like approximation method and established the following strong convergence theorem.

Theorem 1.1 ([4, Theorem 3.1])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contractive mapping with a contractive constant $\alpha \in (0,1), A : C \to H$ be a monotone, $L-$Lipschitz continuous mapping and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by*

$$(1.5)\qquad \begin{cases} x_0 = x \in C \\ y_n = (1-\gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n f(y_n) + \beta_n S P_C(x_n - \lambda_n A y_n), \end{cases}$$

where $n \geq 0, \{\lambda_n\}$ is a sequence in (0,1) with $\sum_{n=0}^{\infty} \lambda_n < \infty$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in [0,1] satisfying the conditions:

(i) $\alpha_n + \beta_n \leq 1$ *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequences $\{x_n\}, \{y_n\}$ converge strongly to the same point $q = P_{F(S)\cap VI(C,A)} f(q)$ if and only if $\{Ax_n\}$ is bounded and $\lim_{n\to\infty}\langle Ax_n, y - x_n\rangle \geq 0$ for all $y \in C$.

Recently, for finding a common element of $F(T) \cap VI(C, A) \cap EP$, Plubtieng and Punpaeng [10] introduced the following iterative scheme: $x_1 = u \in H$ and

$$\begin{cases} \Theta(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T P_C(y_n - \lambda_n A y_n), \quad \forall\, n \geq 1, \end{cases}$$

under suitable conditions, some strong convergence theorems are proved which extend some recent results of Yao and Yao [16].

For finding a common element of Ω, where $\Omega = \cap_{i=1}^N F(T_i) \cap VI(C, A) \cap MEP$, Peng and Yao [9] introduced the following iterative scheme: let v be an arbitrary point in C and

$$(1.6)\qquad \begin{cases} x_1 = x \in C \\ \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \\ \forall\, y \in C, \\ y_n = P_C(u_n - \gamma_n A u_n), \\ x_{n+1} = \alpha_n v + \beta_n x_n + (1-\alpha_n-\beta_n) W_n P_C(u_n - \gamma_n A y_n), \\ \forall\, n \geq 1, \end{cases}$$

under suitable conditions, some strong convergence theorems are proved which extend some recent results in [7], [8], [13], [16] and [18].

Innovated and inspired by the above work, we raise two questions: (I) for iterative sequence generated by (1.6), can we use S_n to replace the mapping W_n? Furthermore, can we consider generalized mixed equilibrium problems?(II) Can we use the extragradient-like method in [4] and viscosity approximation method to establish some strong convergence theorems? In this paper, we present a positive answer for the above questions.

2. Preliminaries

Let H be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$. Let C be a nonempty closed convex subset of H. Let symbols $\rightharpoonup$ and $\to$ denote weak and strong convergence, respectively. For every point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for all $y \in C$. We know that P_C is a firmly nonexpansive mapping of H onto C, i.e.,

$$\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2, \ \forall\ x, y \in H.$$

Moreover, $P_C x$, is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \le 0, \tag{2.1}$$

$$\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{2.2}$$

for all $x \in H,\ y \in C$. In a real Hilbert space H, it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. It is easy to see that the following is true:

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda A u),\ \forall\ \lambda > 0.$$

It is well known that H satisfies the Opial condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \ne x$.

A mapping $A : C \to H$ is called $k-$Lipschitz continuous if there exists some constant $k > 0$ such that

$$\|Ax - Ay\| \le k\|x - y\|,\ \forall\ x, y \in C.$$

A mapping A of C is called α-inverse-strongly monotone, if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \ge \alpha\|Ax - Ay\|^2,\ \forall\ x, y \in C.$$

It is easy to see that if A is $\alpha-$inverse-strongly monotone, then A is monotone and Lipschitz continuous. The converse is not true in general, see [9].

Let A be a strongly positive bounded linear operator on H, i.e., there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \ge \overline{\gamma}\|x\|^2,\ \forall\ x \in H.$$

A set-valued mapping $T \to 2^H$ is called monotone if, for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \ge 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if , for $(x, f) \in H \times H, \langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone, $k-$Lipschitz continuous mapping of C into H and let $N_C v$ be a normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v-u, w\rangle \geq 0, \forall\, u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$(see [11]).

For solving the $GMEP$, let us give the following assumptions for the bifunction Θ, φ and the set C:

(H1) $\Theta(x, x) = 0$ for all $x \in C$;

(H2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for any $x, y \in C$;

(H3) for each $y \in C, x \to \Theta(x, y)$ is weakly upper semicontinuous;

(H4) for each $x \in C, y \to \Theta(x, y)$ is convex and lower semicontinuous;

(A1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C/D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(A2) C is a bounded set.

Lemma 2.1 (see [3])**.** *Let C be a nonempty closed convex subset of H. Let $\Theta : C \times C \to R$ be a bifunction satisfying conditions (H1)-(H4) and let $\varphi : C \to R$ be a lower semicontinuous and convex function. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$T_r^{(\Theta,\varphi)}(x) = \Big\{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall\, y \in C\Big\}$$

for all $x \in H$. Assume that either (A1) or (A2) holds. Then the following results hold:

(i) *$T_r^{(\Theta,\varphi)}(x) \neq \emptyset$ for each $x \in H$ and T_r is single-valued;*

(ii) *$T_r^{(\Theta,\varphi)}$ is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y\|^2 \leq \langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y\rangle;$$

(iii) *$F(T_r^{(\Theta,\varphi)}) = MEP$;*

(iv) *MEP is closed and convex.*

We also need the following lemmas.

Lemma 2.2 (see [17], Lemma 2.5)**.** *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \delta_n)\alpha_n + \delta_n s_n + v_n,$$

where $\{\delta_n\}, \{s_n\}$ and $\{v_n\}$ satisfy the conditions:

(i)$\{\delta_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \delta_n = \infty$;

(ii) $\limsup_{n\to\infty} s_n \leq 0$;

(iii)$v_n \geq 0 (n \geq 0), \sum_{n=1}^{\infty} v_n < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.3. *In a real Hilbert space H, there holds the following inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \text{for all } x, y \in H.$$

Lemma 2.4 ([6])**.** *Assume that B is a strongly positive linear operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho < \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.5 ([12])**.** *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in [0,1] with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that

$$x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$$

for all integer $n \geq 0$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

3. Strong convergence theorems

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be an $\alpha-$inverse-strongly monotone mapping, and $f : C \to C$ be a contractive mapping with a contractive constant $\theta \in (0,1)$. Let $\Theta : C \times C \to R$ be a bifunction satisfying (H1)-(H4) and $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with assumptions (A1) or (A2). Let $A : C \to H$ be a monotone and $k-$Lipschitz continuous mapping. Let $T_1, T_2, \ldots, T_N$ be a family of finitely nonexpansive mapping of C into H such that $\Omega = (\cap_{i=1}^N F(T_i)) \cap VI(C,A) \cap GMEP \neq \emptyset$. Let B be a strongly positive bounded linear operator on H with efficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \overline{\gamma}/\theta$. For $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0,1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1, \{\alpha_1^{n,j}\}_{j=1}^{N-1} \subset [\eta_1, \theta_1]$ with $0 < \eta_1 \leq \theta_1 < 1, \{\alpha_1^{n,N}\} \subset [\eta_N, 1]$ with $0 < \eta_N \leq 1$ and $\{\alpha_2^{n,j}\}_{j=1}^N, \{\alpha_3^{n,j}\}_{j=1}^N \subset [0, \theta_3]$ with $0 \leq \theta_3 < 1$. For any $x_1 = x \in C$. Let $\{x_n\}, \{y_n\}$ and $\{u_n\}$ be sequences defined by*

$$\begin{cases} \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Fx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \\ \quad \forall\ y \in C, \\ y_n = (1-\xi_n)u_n + \xi_n P_C(u_n - \gamma_n A u_n), \\ x_{n+1} = \alpha_n \gamma f(y_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)S_n P_C(u_n - \gamma_n A y_n), \\ \quad \forall\ n \geq 1, \end{cases} \qquad (\star)$$

where $\{S_n : C \to C\}$ is the sequence defined by (1.4), $\{\alpha_n\}, \{\beta_n\}, \{\xi_n\}$ and $\{\gamma_n\}$ are four sequences in [0,1], $\{r_n\}$ is a sequence such that $\{r_n\} \subset (0, 2\alpha)$. If the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=1}^\infty \alpha_n = \infty$;
(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(C3) $\Sigma_{n=1}^\infty \gamma_n < \infty$;
(C4) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\alpha$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;
(C5) $|\alpha_1^{n+1,j} - \alpha_1^{n,j}| \to 0$ *and* $|\alpha_3^{n+1,j} - \alpha_3^{n,j}| \to 0 (n \to \infty)$, *for all* $j \in \{1, 2, \ldots, N\}$.

Then $\{x_n\}, \{y_n\}$ *and* $\{u_n\}$ *converge strongly to* $z \in \Omega$ *if and only if* $\{Au_n\}$ *is bounded and* $\liminf_{n\to\infty}\langle Au_n, y - u_n\rangle \geq 0$ *for all* $y \in C$, *where* z *is the unique solution of variational inequality*

$$\langle(\gamma f - B)z, w - z\rangle \leq 0,\ \forall\ w \in \Omega,$$

that is $z = P_\Omega(((I - B) + \gamma f)z)$.

Proof. ($\Longrightarrow$). Suppose that $\{x_n\}, \{y_n\}$ and $\{u_n\}$ converge strongly to $z \in \Omega$. Then from the Lipschitz continuity of A it follows that $\{Au_n\}$ is bounded. Further,

$$\begin{aligned} & |\langle Au_n, y - u_n\rangle - \langle Az, y - z\rangle| \\ \leq\ & |\langle Au_n, y - u_n\rangle - \langle Au_n, y - z\rangle| + |\langle Au_n, y - z\rangle - \langle Az, y - z\rangle| \\ \leq\ & \|Au_n\|\|u_n - z\| + \|Au_n - Az\|\|y - z\| \\ \leq\ & \|Au_n\|\|u_n - z\| + k\|u_n - z\|\|y - z\| \to 0 (n \to \infty), \end{aligned}$$

which implies that

$$\lim_{n\to\infty}\langle Au_n, y - u_n\rangle = \langle Az, y - z\rangle \geq 0, \forall\ \ y \in C$$

due to $z \in VI(C, A)$.

($\Longleftarrow$). Note that u_n can be rewritten as $u_n = T_{r_n}^{(\Theta,\varphi)}(x_n - r_nFx_n)$ for every $n = 1, 2, \ldots$. Put $t_n = P_C(u_n - \gamma_nAy_n),\ \forall\ n = 1, 2, \ldots$. By (C1) and (C2), we may assume, with no loss of generality, that $\alpha_n \leq (1-\beta_n)\|B\|^{-1}$ and $1-\alpha_n(\overline{\gamma} - \theta\gamma) > 0$ for all n. From Lemma 2.4, we have $\|(1 - \beta_n)I - \alpha_nB\| \leq 1 - \beta_n - \alpha_n\overline{\gamma}$.

Next we divide the proof in five steps.

Step 1. First we show that the sequence $\{x_n\}$ is bounded.

Let $p \in \Omega$, then $p = P_C(p - \gamma_nAp) = T_{r_n}^{(\Theta,\varphi)}(p - r_nFp)$. From $u_n = T_{r_n}^{(\Theta,\varphi)}(x_n - r_nFx_n)$ and F is $\alpha-$inverse-strongly monotone and $\{r_n\} \subset (0, 2\alpha)$, we know that for any $n \geq 1$

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}^{(\Theta,\varphi)}(x_n - r_nFx_n) - T_{r_n}^{(\Theta,\varphi)}(p - r_nFp)\|^2 \\ &\leq \|x_n - p - r_n(Fx_n - Fp)\|^2 \\ &= \|x_n - p\|^2 - 2r_n\langle Fx_n - Fp, x_n - p\rangle + r_n^2\|Fx_n - Fp\|^2 \\ &\leq \|x_n - p\|^2 - r_n(2\alpha - r_n)\|Fx_n - Fp\|^2 \\ &\leq \|x_n - p\|^2. \end{aligned} \tag{3.1}$$

From (2.2), the monotonicity of A, and $p \in VI(C, A)$, we have

$$\begin{aligned} & \|t_n - p\|^2 \\ \leq\ & \|u_n - \gamma_nAy_n - p\|^2 - \|u_n - \gamma_nAy_n - t_n\|^2 \\ =\ & \|u_n - p\|^2 - \|u_n - t_n\|^2 + 2\gamma_n\langle Ay_n, p - t_n\rangle \\ =\ & \|u_n - p\|^2 - \|u_n - t_n\|^2 - 2\gamma_n(\langle Ay_n - Ap, y_n - p\rangle \\ & +\langle Ap, y_n - p\rangle + \langle Ay_n, t_n - y_n\rangle) \\ \leq\ & \|u_n - p\|^2 - \|u_n - t_n\|^2 + 2\gamma_n\langle Ay_n, y_n - t_n\rangle \\ =\ & \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 \\ & -2\langle u_n - y_n, y_n - t_n\rangle + 2\gamma_n\langle Ay_n, y_n - t_n\rangle \end{aligned}$$

$$= \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle u_n - \gamma_n A y_n - y_n, t_n - y_n\rangle. \tag{3.2}$$

Since $y_n = (1 - \xi_n)u_n + \xi_n P_C(u_n - \gamma_n A u_n)$ and A is $k-$Lipschitz continuous, we have

$$\begin{aligned}
& 2\langle u_n - \gamma_n A y_n - y_n, t_n - y_n\rangle \\
\leq\ & 2\|u_n - \gamma_n A y_n - y_n\|\|t_n - y_n\| \\
\leq\ & \|u_n - \gamma_n A y_n - y_n\|^2 + \|t_n - y_n\|^2 \\
=\ & \|u_n - y_n\|^2 + \|t_n - y_n\|^2 + 2\gamma_n\langle A y_n, y_n - u_n\rangle + \gamma_n^2\|A y_n\|^2 \\
=\ & \|u_n - y_n\|^2 + \|t_n - y_n\|^2 \\
& +2\gamma_n\xi_n\langle A y_n, P_C(u_n - \gamma_n A u_n) - P_C(u_n)\rangle + \gamma_n^2\|A y_n\|^2 \\
\leq\ & \|u_n - y_n\|^2 + \|t_n - y_n\|^2 + 2\gamma_n^2\xi_n\|A y_n\|\|A u_n\| + \gamma_n^2\|A y_n\|^2 \\
\leq\ & \|u_n - y_n\|^2 + \|t_n - y_n\|^2 + \gamma_n^2(\|A u_n\| + \|A y_n\|)^2.
\end{aligned} \tag{3.3}$$

Since $\{Au_n\}$ is bounded, A is $k-$Lipschitz continuous and $\{\gamma_n\} \subset [0, 1]$, we have

$$\|A y_n - A u_n\| \leq k\|y_n - u_n\| \leq k\xi_n\|P_C(u_n - \gamma_n A u_n) - P_C(u_n)\| \leq k\|A u_n\|,$$

which implies that $\{Ay_n\}$ is bounded. Put $M_1 = \sup\{\|Au_n\| + \|Ay_n\| : n \geq 1\}$. Then it follows from (3.3) that

$$2\langle u_n - \gamma_n A y_n - y_n, t_n - y_n\rangle \leq \|u_n - y_n\|^2 + \|t_n - y_n\|^2 + \gamma_n^2 M_1^2.$$

This together with (3.2) implies that

$$\begin{aligned}
\|t_n - p\|^2 \leq\ & \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + \|u_n - y_n\|^2 \\
& +\|t_n - y_n\|^2 + \gamma_n^2 M_1^2 \\
=\ & \|u_n - p\|^2 + \gamma_n^2 M_1^2,
\end{aligned} \tag{3.4}$$

which implies that

$$\|t_n - p\| \leq \|u_n - p\| + \gamma_n M_1. \tag{3.5}$$

From $(\star)$, $p \in VI(C, A)$ and the monotonicity of A, we get

$$\begin{aligned}
\|y_n - p\| =\ & \|(1 - \xi_n)u_n + \xi_n P_C(u_n - \gamma_n A u_n) - p\| \\
\leq\ & (1 - \xi_n)\|u_n - p\| + \xi_n\|P_C(u_n - \gamma_n A u_n) - P_C(p - \gamma_n A p)\| \\
\leq\ & (1 - \xi_n)\|u_n - p\| + \xi_n\|u_n - p - \gamma_n(A u_n - A p)\| \\
\leq\ & \|u_n - p\| + \xi_n\gamma_n\|A u_n - A p\| \\
\leq\ & \|u_n - p\| + \gamma_n(M_1 + \|A p\|).
\end{aligned} \tag{3.6}$$

By $(\star)$, (3.1), (3.5) and (3.6), and by induction, we have

$$\begin{aligned}
\|x_{n+1} - p\| \leq\ & \alpha_n\|\gamma f(y_n) - B(p)\| + \beta_n\|x_n - p\| \\
& +\|((1 - \beta_n)I - \alpha_n B)(S_n t_n - p)\| \\
\leq\ & \alpha_n\gamma\|f(y_n) - f(p)\| + \alpha_n\|\gamma f(p) - B(p)\| + \beta_n\|x_n - p\| \\
& +((1 - \beta_n) - \alpha_n\overline{\gamma})\|S_n t_n - S_n p\| \\
\leq\ & \alpha_n\gamma\theta\|y_n - p\| + \alpha_n\|\gamma f(p) - B(p)\| \\
& +\beta_n\|x_n - p\| + ((1 - \beta_n) - \alpha_n\overline{\gamma})\|t_n - p\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\gamma\theta\|x_n-p\|+\alpha_n\gamma\theta\gamma_n(M_1+\|Ap\|)\\
&\quad +\alpha_n\|\gamma f(p)-B(p)\|+\beta_n\|x_n-p\|\\
&\quad +((1-\beta_n)-\alpha_n\overline{\gamma})(\|x_n-p\|+\gamma_n(M_1+\|Ap\|))\\
&\leq (1-\alpha_n(\overline{\gamma}-\gamma\theta))\|x_n-p\|+\alpha_n\|\gamma f(p)-B(p)\|+\gamma_n(M_1+\|Ap\|)\\
&\leq (1-\alpha_n(\overline{\gamma}-\gamma\theta))\|x_n-p\|\\
&\quad +\alpha_n(\overline{\gamma}-\gamma\theta)\frac{1}{(\overline{\gamma}-\gamma\theta)}\|\gamma f(p)-B(p)\|+\gamma_n(M_1+\|Ap\|)\\
&\leq \max\Big\{\|x_1-p\|,\frac{1}{(\overline{\gamma}-\gamma\theta)}\|\gamma f(p)-B(p)\|\Big\}+\sum_{j=1}^{n}\gamma_j(M_1+\|Ap\|).
\end{aligned}
$$

Thus by (C3), we know that $\{x_n\}$ is bounded. From (3.1), (3.5), (3.6) and the assumptions, we have $\{u_n\}$, $\{t_n\}$,$\{y_n\}$, $\{F(x_n)\}$, $\{f(x_n)\}$ and $\{S_n(x_n)\}$ are all bounded.

Step 2. We shall prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

$$
\begin{aligned}
\|t_{n+1}-t_n\| &= \|P_C(u_{n+1}-\gamma_{n+1}Ay_{n+1})-P_C(u_n-\gamma_nAy_n)\|\\
&\leq \|u_{n+1}-u_n-\gamma_{n+1}Ay_{n+1}+\gamma_nAy_n\|\\
&\leq \|u_{n+1}-u_n\|+\gamma_{n+1}\|Ay_{n+1}\|+\gamma_n\|Ay_n\|\\
&\leq \|u_{n+1}-u_n\|+(\gamma_{n+1}+\gamma_n)M_1. \qquad (3.7)
\end{aligned}
$$

Since $u_n=T_{r_n}^{(\Theta,\varphi)}(x_n-r_nFx_n)$ and $u_{n+1}=T_{r_{n+1}}^{(\Theta,\varphi)}(x_{n+1}-r_{n+1}Fx_{n+1})$, put $k_n=x_n-r_nFx_n$,we have $u_n=T_{r_n}^{(\Theta,\varphi)}k_n$ and $u_{n+1}=T_{r_{n+1}}^{(\Theta,\varphi)}k_{n+1}$. So by $(\star)$ for $\forall\ y\in C$ we have

$$\Theta(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-k_n\rangle\geq 0, \qquad (3.8)$$

$$\Theta(u_{n+1},y)+\varphi(y)-\varphi(u_{n+1})+\frac{1}{r_{n+1}}\langle y-u_{n+1},u_{n+1}-k_{n+1}\rangle\geq 0. \qquad (3.9)$$

Take $y=u_{n+1}$ in (3.8) and $y=u_n$ in (3.9), then add two inequalities. By using Condition (H2), we have

$$\Big\langle u_{n+1}-u_n,\frac{u_n-k_n}{r_n}-\frac{u_{n+1}-k_{n+1}}{r_{n+1}}\Big\rangle\geq 0$$

and hence

$$\Big\langle u_{n+1}-u_n,u_n-u_{n+1}+u_{n+1}-k_n-\frac{r_n}{r_{n+1}}(u_{n+1}-k_{n+1})\Big\rangle\geq 0.$$

From (C4), we assume that there exists a real number a such that $r_n>a>0$ for all $n\in N_+$. Thus,

$$
\begin{aligned}
\|u_{n+1}-u_n\|^2 &\leq \Big\langle u_{n+1}-u_n,u_{n+1}-k_n-\frac{r_n}{r_{n+1}}(u_{n+1}-k_{n+1})\Big\rangle\\
&\leq \|u_{n+1}-u_n\|\Big\{\|k_{n+1}-k_n\|+|r_n-r_{n+1}|\frac{1}{a}\|u_{n+1}-k_{n+1}\|\Big\}.
\end{aligned}
$$

And further,

$$\|u_{n+1}-u_n\|\leq\|k_{n+1}-k_n\|+|r_n-r_{n+1}|\frac{1}{a}\|u_{n+1}-k_{n+1}\|. \qquad (3.10)$$

Since F is $\alpha-$inverse-strongly monotone, similarly to the proof of (3.1), we get

$$\|(I-r_{n+1}F)x_{n+1}-(I-r_{n+1}F)x_n\|\leq\|x_{n+1}-x_n\|.$$

It follows the above inequality and (3.10) that

$$\begin{aligned}\|u_{n+1}-u_n\| &\leq \|(I-r_{n+1}F)x_{n+1}-(I-r_{n+1}F)x_n\|\\ &\quad +|r_n-r_{n+1}|\|F(x_n)\|+|r_n-r_{n+1}|\frac{1}{a}\|u_{n+1}-k_{n+1}\|\\ &\leq \|x_{n+1}-x_n\|+|r_n-r_{n+1}|\|F(x_n)\|\\ &\quad +|r_n-r_{n+1}|\frac{1}{a}\|u_{n+1}-k_{n+1}\|\\ &\leq \|x_{n+1}-x_n\|+|r_n-r_{n+1}|M_2, \qquad (3.11)\end{aligned}$$

where $M_2=\sup_{n\geq 1}\Big\{\|F(x_n)\|+\dfrac{1}{a}\|u_{n+1}-k_{n+1}\|\Big\}$. From (3.7) and (3.11), we have

$$\|t_{n+1}-t_n\|\leq\|x_{n+1}-x_n\|+|r_n-r_{n+1}|M_2+(\gamma_{n+1}+\gamma_n)M_1. \qquad (3.12)$$

Let $M_3=\sup_{n\geq 1}\{\|f(x_n)\|,\|S_n(t_n)\|\}$, and $x_{n+1}=\beta_n x_n+(1-\beta_n)z_n, \forall\ \ n\geq 1$. Then, we have

$$\begin{aligned}z_{n+1}-z_n &= \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}\\ &= \frac{\alpha_{n+1}\gamma f(y_{n+1})+[(1-\beta_{n+1})I-\alpha_{n+1}B]S_{n+1}t_{n+1}}{1-\beta_{n+1}}\\ &\quad -\frac{\alpha_n\gamma f(y_n)+[(1-\beta_n)I-\alpha_n B]S_n t_n}{1-\beta_n}\\ &= \Big(\frac{\alpha_{n+1}\gamma f(y_{n+1})}{1-\beta_{n+1}}-\frac{\alpha_n\gamma f(y_n)}{1-\beta_n}\Big)+(S_{n+1}t_{n+1}-S_n t_n)\\ &\quad +\Big(-\frac{\alpha_{n+1}BS_{n+1}t_{n+1}}{1-\beta_{n+1}}+\frac{\alpha_n BS_n t_n}{1-\beta_n}\Big). \qquad (3.13)\end{aligned}$$

Next we estimate $\|S_{n+1}t_{n+1}-S_n t_n\|$. Since S_n is nonexpansive and from the definition of S_n, for any $k\in\{2,3,\ldots,N\}$ we get

$$\begin{aligned}&\|U_{n+1,k}t_n-U_{n,k}t_n\|\\ =\ &\|\alpha_1^{n+1,k}T_kU_{n+1,k-1}t_n+\alpha_2^{n+1,k}U_{n+1,k-1}t_n+\alpha_3^{n+1,k}t_n\\ &-(\alpha_1^{n,k}T_kU_{n,k-1}t_n+\alpha_2^{n,k}U_{n,k-1}t_n+\alpha_3^{n,k}t_n)\|\\ \leq\ &\alpha_1^{n+1,k}\|T_kU_{n+1,k-1}t_n-T_kU_{n,k-1}t_n\|+|\alpha_1^{n+1,k}-\alpha_1^{n,k}|\|T_kU_{n,k-1}t_n\|\\ &+\alpha_2^{n+1,k}\|U_{n+1,k-1}t_n-U_{n,k-1}t_n\|\\ &+|\alpha_2^{n+1,k}-\alpha_2^{n,k}|\|U_{n,k-1}t_n\|+|\alpha_3^{n+1,k}-\alpha_3^{n,k}|\|t_n\|\\ \leq\ &\alpha_1^{n+1,k}\|U_{n+1,k-1}t_n-U_{n,k-1}t_n\|+|\alpha_1^{n+1,k}-\alpha_1^{n,k}|\|T_kU_{n,k-1}t_n\|\\ &+\alpha_2^{n+1,k}\|U_{n+1,k-1}t_n-U_{n,k-1}t_n\|+|\alpha_1^{n,k}+\alpha_3^{n,k}-\alpha_1^{n+1,k}-\alpha_3^{n+1,k}|\\ &\times\|U_{n,k-1}t_n\|+|\alpha_3^{n+1,k}-\alpha_3^{n,k}|\|t_n\|\\ \leq\ &\|U_{n+1,k-1}t_n-U_{n,k-1}t_n\|+|\alpha_1^{n+1,k}-\alpha_1^{n,k}|\end{aligned}$$

$$\times(\|T_kU_{n,k-1}t_n\|+\|U_{n,k-1}t_n\|)+|\alpha_3^{n+1,k}-\alpha_3^{n,k}|(\|t_n\|+\|U_{n,k-1}t_n\|).$$

Hence by induction we have

$$\begin{aligned}
&\|S_{n+1}t_n-S_nt_n\|\\
=\ &\|U_{n+1,N}t_n-U_{n,N}t_n\|\\
\le\ &\|U_{n+1,1}t_n-U_{n,1}t_n\|+\Sigma_{j=2}^N|\alpha_1^{n+1,j}-\alpha_1^{n,j}|(\|T_jU_{n,j-1}t_n\|+\|U_{n,j-1}t_n\|)\\
&+\Sigma_{j=2}^N|\alpha_3^{n+1,j}-\alpha_3^{n,j}|(\|t_n\|+\|U_{n,j-1}t_n\|)\\
\le\ &|\alpha_1^{n+1,1}-\alpha_1^{n,1}|(\|T_1t_n\|+\|t_n\|)+\Sigma_{j=2}^N|\alpha_1^{n+1,j}-\alpha_1^{n,j}|\\
&\times(\|T_jU_{n,j-1}t_n\|+\|U_{n,j-1}t_n\|)+\Sigma_{j=2}^N|\alpha_3^{n+1,j}-\alpha_3^{n,j}|(\|t_n\|+\|U_{n,j-1}t_n\|)\\
\le\ &M_4(|\alpha_1^{n+1,1}-\alpha_1^{n,1}|+\Sigma_{j=2}^N|\alpha_1^{n+1,j}-\alpha_1^{n,j}|+2\Sigma_{j=2}^N|\alpha_3^{n+1,j}-\alpha_3^{n,j}|),
\end{aligned}$$

where

$$M_4=\max\{\sup_{n\ge1}\{\|T_1t_n\|+\|t_n\|\},\sup_{n\ge1}\{\|T_jU_{n,j-1}t_n\|+\|U_{n,j-1}t_n\|,j=1,2,\dots,N\}\}.$$

This together with (C5), we obtain

$$\lim_{n\to\infty}\|S_{n+1}t_n-S_nt_n\|=0. \tag{3.14}$$

It follows from (3.12) that

$$\begin{aligned}
\|S_{n+1}t_{n+1}-S_nt_n\| &\le \|S_{n+1}t_{n+1}-S_{n+1}t_n\|+\|S_{n+1}t_n-S_nt_n\|\\
&\le \|t_{n+1}-t_n\|+\|S_{n+1}t_n-S_nt_n\|\\
&\le \|x_{n+1}-x_n\|+|r_n-r_{n+1}|M_2\\
&\quad +(\gamma_{n+1}+\gamma_n)M_1+\|S_{n+1}t_n-S_nt_n\|.
\end{aligned} \tag{3.15}$$

By (3.13) and (3.15), we have

$$\begin{aligned}
&\|z_{n+1}-z_n\|\\
\le\ &\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma\|f(y_{n+1})\|+\|B\|\|S_{n+1}t_{n+1}\|)\\
&+\frac{\alpha_n}{1-\beta_n}(\gamma\|f(y_n)\|+\|B\|\|S_nt_n\|)+\|S_{n+1}t_{n+1}-S_nt_n\|\\
\le\ &\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma\|f(y_{n+1})\|+\|B\|\|S_{n+1}t_{n+1}\|)\\
&+\frac{\alpha_n}{1-\beta_n}(\gamma\|f(y_n)\|+\|B\|\|S_nt_n\|)+\|x_{n+1}-x_n\|+|r_n-r_{n+1}|M_2\\
&+(\gamma_{n+1}+\gamma_n)M_1+\|S_{n+1}t_n-S_nt_n\|.
\end{aligned}$$

This together with (C1)–(C4) and (3.14), we get

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Hence by Lemma 2.5, we have $\lim_{n\to\infty}\|z_n-x_n\|=0$. Consequently

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\|=0. \tag{3.16}$$

Step 3. Next we prove that $\lim_{n\to\infty}\|S_ny_n-y_n\|=0$.

(a) First, we prove that $\lim_{n\to\infty}\|x_n - S_nt_n\| = 0$. Indeed, observe that

$$\begin{aligned}\|x_n - S_nt_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - S_nt_n\| \\ &\le \|x_n - x_{n+1}\| + \|\alpha_n\gamma f(y_n) + \beta_nx_n - \beta_nS_nt_n - \alpha_nBS_nt_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\gamma\|f(y_n)\| + \beta_n\|x_n - S_nt_n\| + \alpha_n\|B\|\|S_nt_n\|\end{aligned}$$

and hence

$$\begin{aligned}\|x_n - S_nt_n\| &\le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| \\ &\quad + \frac{\alpha_n}{1-\beta_n}(\gamma\|f(y_n)\| + \|B\|\|S_nt_n\|) \to 0 (n\to\infty). \qquad (3.17)\end{aligned}$$

(b) Then we prove that $\|u_n - y_n\| \to 0, \|t_n - y_n\| \to 0$ and $\|u_n - t_n\| \to 0 (n\to\infty)$. In fact, notice that $\|\cdot\|^2$ is convex, for $p \in \Omega$, it follows from (3.4) and Lemma 2.3 that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\beta_n(x_n - p) + (1-\beta_n)(S_nt_n - p) + \alpha_n(\gamma f(y_n) - BS_nt_n)\|^2 \\ &\le \|\beta_n(x_n - p) + (1-\beta_n)(S_nt_n - p)\|^2 \\ &\quad + 2\alpha_n\langle\gamma f(y_n) - BS_nt_n, x_{n+1} - p\rangle \\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|S_nt_n - p\|^2 \\ &\quad + 2\alpha_n\|\gamma f(y_n) - BS_nt_n\|\|x_{n+1} - p\| \\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|t_n - p\|^2 \\ &\quad + 2\alpha_n\|\gamma f(y_n) - BS_nt_n\|\|x_{n+1} - p\| \\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|u_n - p\|^2 + \gamma_n^2M_1^2 \\ &\quad + 2\alpha_n(\gamma\|f(y_n)\| + \|BS_nt_n\|)\|x_{n+1} - p\|. \qquad (3.18)\end{aligned}$$

This together with (3.1), we can conclude that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 + (1-\beta_n)(r_n^2 - 2\alpha r_n) \\ &\quad \times\|Fx_n - Fp\|^2 + \gamma_n^2M_1^2 + 2\alpha_n\|\gamma(\|f(y_n)\| + \|BS_nt_n\|)\|x_{n+1} - p\|.\end{aligned}$$

This implies that

$$\begin{aligned}(1-\beta_n)r_n(2\alpha - r_n)\|Fx_n - Fp\|^2 &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \gamma_n^2M_1^2 \\ &\quad + 2\alpha_n(\gamma\|f(y_n)\| + \|BS_nt_n\|)\|x_{n+1} - p\| \\ &\le \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) + \gamma_n^2M_1^2 \\ &\quad + 2\alpha_n(\gamma\|f(y_n)\| + \|BS_nt_n\|)\|x_{n+1} - p\|.\end{aligned}$$

Therefore, by the assumptions and (3.16), we deduce

$$\lim_{n\to\infty}\|Fx_n - Fp\| = 0. \qquad (3.19)$$

It follows from Lemma 2.1 and (3.1) that

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n}^{(\Theta,\varphi)}(x_n - r_nFx_n) - T_{r_n}^{(\Theta,\varphi)}(p - r_nFp)\|^2 \\ &\le \langle x_n - r_nFx_n - (p - r_nFp), u_n - p\rangle \\ &= \frac{1}{2}\{\|x_n - p - r_n(Fx_n - Fp)\|^2 + \|u_n - p\|^2 \\ &\quad - \|x_n - u_n - r_n(Fx_n - Fp)\|^2\}\end{aligned}$$

$$\begin{aligned}&\le \frac{1}{2}\{\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n\|^2\\&\quad +2r_n\langle Fx_n-Fp,x_n-u_n\rangle-r_n^2\|Fx_n-Fp\|^2\}.\end{aligned}$$

This implies that

$$\begin{aligned}\|u_n-p\|^2 &\le \|x_n-p\|^2-\|x_n-u_n\|^2\\&\quad +2r_n\|Fx_n-Fp\|\|x_n-u_n\|-r_n^2\|Fx_n-Fp\|^2. \qquad (3.20)\end{aligned}$$

Combining (3.20) and (3.18), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|x_n-p\|^2-(1-\beta_n)\|x_n-u_n\|^2\\&\quad +2r_n(1-\beta_n)\|Fx_n-Fp\|\|x_n-u_n\|\\&\quad +\gamma_n^2M_1^2+2\alpha_n(\gamma\|f(y_n)\|+\|BS_nt_n\|)\|x_{n+1}-p\|,\end{aligned}$$

and thus

$$\begin{aligned}(1-\beta_n)\|x_n-u_n\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2r_n(1-\beta_n)\|Fx_n-Fp\|\|x_n-u_n\|\\&\quad +\gamma_n^2M_1^2+2\alpha_n(\gamma\|f(y_n)\|+\|BS_nt_n\|)\|x_{n+1}-p\|\\&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+2r_n(1-\beta_n)\|Fx_n-Fp\|\\&\quad \times\|x_n-u_n\|+\gamma_n^2M_1^2+2\alpha_n(\gamma\|f(y_n)\|+\|BS_nt_n\|)\|x_{n+1}-p\|,\end{aligned}$$

which implies that $\|x_n-u_n\|\to 0 (n\to\infty)$. Since $u_n\in C$, we have $u_n=P_Cu_n$, and by (C3)

$$\begin{aligned}\|y_n-t_n\| &= \|(1-\xi_n)(P_Cu_n-P_C(u_n-\gamma_nAy_n))\\&\quad +\xi_n(P_C(u_n-\gamma_nAu_n)-P_C(u_n-\gamma_nAy_n))\|\\&\le (1-\xi_n)\|P_Cu_n-P_C(u_n-\gamma_nAy_n)\|\\&\quad +\xi_n\|P_C(u_n-\gamma_nAu_n)-P_C(u_n-\gamma_nAy_n)\|\\&\le (1-\xi_n)\gamma_n\|Ay_n\|+\xi_n\gamma_n\|Au_n-Ay_n\|\to 0(n\to\infty), \qquad (3.21)\\ \|u_n-y_n\| &= \xi_n\|u_n-P_C(u_n-\gamma_nAu_n)\|=\xi_n\|P_C(u_n)-P_C(u_n-\gamma_nAu_n)\|\\&\le \xi_n\gamma_n\|Au_n\|\to 0(n\to\infty). \qquad (3.22)\end{aligned}$$

Since

$$\begin{aligned}\|S_ny_n-y_n\| &\le \|S_ny_n-S_nt_n\|+\|S_nt_n-x_n\|+\|x_n-u_n\|+\|u_n-y_n\|\\&\le \|y_n-t_n\|+\|S_nt_n-x_n\|+\|x_n-u_n\|+\|u_n-y_n\|,\end{aligned}$$

then it follows from (3.17), (3.21), (3.22) and $\lim_{n\to\infty}\|x_n-u_n\|=0$ that

$$\lim_{n\to\infty}\|S_ny_n-y_n\|=0.$$

Observe that $P_\Omega((I-B)+\gamma f)$ is a contraction. Indeed, for any $x,y\in H$, we have

$$\begin{aligned}&\|P_\Omega((I-B)+\gamma f)(x)-P_\Omega((I-B)+\gamma f)(y)\|\\&\le \|((I-B)+\gamma f)(x)-((I-B)+\gamma f)(y)\|\\&\le \|I-B\|\|x-y\|+\gamma\|f(x)-f(y)\|\\&\le (1-\overline{\gamma})\|x-y\|+\gamma\theta\|x-y\|=(1-(\overline{\gamma}-\gamma\theta))\|x-y\|.\end{aligned}$$

Banach's contraction mapping principle guarantees that $P_\Omega((I-B)+\gamma f)$ has a unique fixed point, say $z\in H$. That is, $z=P_\Omega((I-B)+\gamma f)(z)$.

Step 4. Next we show that $\limsup_{n\to\infty}\langle\gamma f(z)-Bz, x_n-z\rangle\le 0$, where $z=P_\Omega((I-B)+\gamma f)(z)$. Indeed, we pick a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ so that

$$\lim_{i\to\infty}\langle\gamma f(z)-Bz, x_{n_i}-z\rangle=\limsup_{n\to\infty}\langle\gamma f(z)-Bz, x_n-z\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_i}\rightharpoonup w$. From $\|x_n-u_n\|\to 0$, we have $u_{n_i}\rightharpoonup w$. From $\|u_n-t_n\|\to 0, \|u_n-y_n\|\to 0$, we also have $t_{n_i}\rightharpoonup w, y_{n_i}\rightharpoonup w$. Since $\{u_{n_i}\}\subset C$ and C is closed and convex, we obtain $w\in C$.

Next we show that $w\in\Omega$.

(a) We first show that $w\in GMEP$. By $u_n=T_{r_n}^{(\Theta,\varphi)}(x_n-r_nFx_n)$, we know that

$$\Theta(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n, u_n-(x_n-r_nFx_n)\rangle\ge 0, \forall\ y\in C.$$

From (H2),

$$\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n, u_n-(x_n-r_nFx_n)\rangle\ge\Theta(y,u_n), \forall\ y\in C.$$

Hence for any $y\in C$,

$$\varphi(y)-\varphi(u_{n_i})+\frac{1}{r_{n_i}}\langle y-u_{n_i}, u_{n_i}-(x_{n_i}-r_{n_i}Fx_{n_i})\rangle\ge\Theta(y,u_{n_i}). \tag{3.23}$$

Put $z_t=ty+(1-t)w$ for all $t\in(0,1]$ and $y\in C$. Then we have $z_t\in C$. Taking $y=z_t$ in (3.23), we have

$$\begin{aligned}\langle z_t-u_{n_i}, Fz_t\rangle &\ge \langle z_t-u_{n_i}, Fz_t\rangle-\varphi(z_t)+\varphi(u_{n_i})\\ &\quad -\Big\langle F(x_{n_i}), z_t-u_{n_i}\rangle-\langle z_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\Big\rangle+\Theta(z_t,u_{n_i})\\ &= \langle z_t-u_{n_i}, Fz_t-F(u_{n_i})\rangle+\langle z_t-u_{n_i}, F(u_{n_i})-F(x_{n_i})\rangle-\varphi(z_t)\\ &\quad +\varphi(u_{n_i})-\Big\langle z_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\Big\rangle+\Theta(z_t,u_{n_i}).\end{aligned}\tag{3.24}$$

Since $\|u_{n_i}-x_{n_i}\|\to 0$, we have $\|F(u_{n_i})-F(x_{n_i})\|\to 0$. Further, from the monotonicity of F, we have

$$\langle z_t-u_{n_i}, Fz_t-F(u_{n_i})\rangle\ge 0.$$

So from (3.24), (H4), the weakly lower semicontinuity of φ, $\dfrac{u_{n_i}-x_{n_i}}{r_{n_i}}\to 0$ and $u_{n_i}\rightharpoonup w$, as $i\to\infty$, we have

$$\langle z_t-w, Fz_t\rangle\ge-\varphi(z_t)+\varphi(w)+\Theta(z_t,w).$$

From the above inequality and (H1), (H4), we obtain

$$\begin{aligned}0 &= \Theta(z_t,z_t)+\varphi(z_t)-\varphi(z_t)\\ &\le t\Theta(z_t,y)+(1-t)\Theta(z_t,w)+t\varphi(y)+(1-t)\varphi(w)-\varphi(z_t)\\ &= t(\Theta(z_t,y)+\varphi(y)-\varphi(z_t))+(1-t)(\Theta(z_t,w)+\varphi(w)-\varphi(z_t))\\ &\le t(\Theta(z_t,y)+\varphi(y)-\varphi(z_t))+(1-t)\langle z_t-w, Fz_t\rangle\\ &= t(\Theta(z_t,y)+\varphi(y)-\varphi(z_t))+(1-t)t\langle y-w, Fz_t\rangle\end{aligned}$$

and hence

$$0 \le \Theta(z_t, y) + \varphi(y) - \varphi(z_t) + (1-t)\langle y - w, Fz_t\rangle.$$

Let $t \to 0$, for each $y \in C$, from the above inequality and (H3) we have

$$\varphi(w) \le \Theta(w, y) + \varphi(y) + \langle y - w, Fw\rangle.$$

That is

$$\Theta(w, y) + \varphi(y) - \varphi(w) + \langle y - w, Fw\rangle \ge 0.$$

This implies that $w \in GMEP$.

(b) We prove that $w \in VI(C, A)$. For this purpose, we define a set-valued mapping $T : H \to 2^H$ by

$$Tw_1 = \begin{cases} Aw_1 + N_C w_1, & w_1 \in C, \\ \emptyset, & w_1 \notin C. \end{cases}$$

where $N_C w_1$ is the normal cone to C at $w_1 \in C$. Then T is maximal monotone and $0 \in Tw_1$ if and only if $w_1 \in VI(C, A)$. Let $(w_1, g) \in G(T)$. Then $Tw_1 = Aw_1 + N_C w_1$ and hence $g - Aw_1 \in N_C w_1$. So, we have $\langle w_1 - t, g - Aw_1\rangle \ge 0$ for all $t \in C$. Therefore, from $\liminf_{n\to\infty}\langle Au_n, y - u_n\rangle \ge 0$, we obtain

$$\begin{aligned} \langle w_1 - w, g\rangle &= \liminf_{i\to\infty}\langle w_1 - u_{n_i}, g\rangle \\ &\ge \liminf_{i\to\infty}\langle w_1 - u_{n_i}, Aw_1\rangle \\ &= \liminf_{i\to\infty}[\langle w_1 - u_{n_i}, Aw_1 - Au_{n_i}\rangle + \langle w_1 - u_{n_i}, Au_{n_i}\rangle] \\ &\ge \liminf_{i\to\infty}\langle w_1 - u_{n_i}, Au_{n_i}\rangle \ge 0. \end{aligned}$$

Hence we have $\langle w_1 - w, g\rangle \ge 0$. Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in VI(C, A)$.

(c) We next show that $w \in F(S_n) = \cap_{i=1}^N F(T_i)$. Suppose the contrary, $w \notin F(S_n)$. From the Opial condition and *step* 3, we have

$$\begin{aligned} \liminf_{i\to\infty}\|y_{n_i} - w\| &< \liminf_{i\to\infty}\|y_{n_i} - S_n w\| \\ &\le \liminf_{i\to\infty}\|y_{n_i} - S_n y_{n_i}\| + \liminf_{i\to\infty}\|S_n y_{n_i} - S_n w\| \\ &\le \liminf_{i\to\infty}\|S_n y_{n_i} - S_n w\| \\ &\le \liminf_{i\to\infty}\|y_{n_i} - w\|, \end{aligned}$$

which is a contraction. So we get $w \in F(S_n)$. This implies $w \in \Omega$. Therefore, by $z = P_\Omega(\gamma f(z) + (I - B)z)$ and (2.1), we have

$$\begin{aligned} \limsup_{n\to\infty}\langle \gamma f(z) - Bz, x_n - z\rangle &= \lim_{i\to\infty}\langle \gamma f(z) - Bz, x_{n_i} - z\rangle \\ &\le \langle \gamma f(z) - Bz, w - z\rangle \\ &= \langle \gamma f(z) + (I - B)z - z, w - z\rangle \le 0. \end{aligned} \tag{3.25}$$

Step 5. Finally we show that $x_n \to z$, $u_n \to z, y_n \to z$ and $t_n \to z$, where $z = P_\Omega(\gamma f(z) + (I - B)z)$. Indeed, by Lemma 2.3, (3.1), (3.4) and (3.6), we have

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
\le\ & \|\alpha_n\gamma f(y_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n B)S_n t_n - z\|^2\\
\le\ & \|((1-\beta_n)I-\alpha_n B)(S_n t_n - z)+\beta_n(x_n-z)\|^2\\
&+2\langle \alpha_n\gamma f(y_n)-\alpha_n Bz, x_{n+1}-z\rangle\\
=\ & \|\frac{(1-\beta_n)((1-\beta_n)I-\alpha_n B)(S_n t_n - z)}{(1-\beta_n)}+\beta_n(x_n-z)\|^2\\
&+2\alpha_n\gamma\langle f(y_n)-f(z), x_{n+1}-z\rangle+2\alpha_n\langle\gamma f(z)-Bz, x_{n+1}-z\rangle\\
\le\ & (1-\beta_n)\|\frac{((1-\beta_n)I-\alpha_n B)(S_n t_n - z)}{(1-\beta_n)}\|^2+\beta_n\|x_n-z\|^2\\
&+2\alpha_n\gamma\theta\|y_n-z\|\|x_{n+1}-z\|+2\alpha_n\langle\gamma f(z)-Bz, x_{n+1}-z\rangle\\
\le\ & (1-\beta_n)[\frac{(1-\beta_n-\alpha_n\overline{\gamma})\|t_n-z\|}{(1-\beta_n)}]^2+\beta_n\|x_n-z\|^2\\
&+2\alpha_n\gamma\theta\frac{\|y_n-z\|^2+\|x_{n+1}-z\|^2}{2}+2\alpha_n\langle\gamma f(z)-Bz, x_{n+1}-z\rangle\\
\le\ & \frac{((1-\beta_n)^2-2\alpha_n\overline{\gamma}(1-\beta_n)+\alpha_n^2\overline{\gamma}^2)(\|x_n-z\|^2+\gamma_n^2M_1^2)}{(1-\beta_n)}+\beta_n\|x_n-z\|^2\\
&+2\alpha_n\gamma\theta\frac{[\|x_n-z\|+\gamma_n(M_1+\|Az\|)]^2+\|x_{n+1}-z\|^2}{2}\\
&+2\alpha_n\langle\gamma f(z)-Bz, x_{n+1}-z\rangle\\
\le\ & \frac{((1-\beta_n)^2-2\alpha_n\overline{\gamma}(1-\beta_n)+\alpha_n^2\overline{\gamma}^2)(\|x_n-z\|^2+\gamma_n^2M_1^2)}{(1-\beta_n)}+\beta_n\|x_n-z\|^2\\
&+2\alpha_n\gamma\theta\frac{\|x_n-z\|^2+2\gamma_n(M_1+\|Az\|)\|x_n-z\|+\gamma_n^2(M_1+\|Az\|)^2}{2}\\
&+\alpha_n\gamma\theta\|x_{n+1}-z\|^2+2\alpha_n\langle\gamma f(z)-Bz, x_{n+1}-z\rangle\\
\le\ & (1-\alpha_n(2\overline{\gamma}-\gamma\theta))\|x_n-z\|^2+\alpha_n\gamma\theta\|x_{n+1}-z\|^2+\frac{\alpha_n^2\overline{\gamma}^2}{1-\beta_n}\|x_n-z\|^2\\
&+2\alpha_n\langle\gamma f(z)-Bz, x_{n+1}-z\rangle+\frac{(1+2\overline{\gamma}+\overline{\gamma}^2)\gamma_n^2M_1^2}{(1-\beta_n)}\\
&+\alpha_n\gamma\theta[2\gamma_n(M_1+\|Az\|)\|x_n-z\|+\gamma_n^2(M_1+\|Az\|)^2],
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-z\|^2\ \le\ & \frac{(1-\alpha_n(2\overline{\gamma}-\gamma\theta))}{1-\alpha_n\gamma\theta}\|x_n-z\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma\theta}\langle\gamma f(z)-Bz, x_{n+1}-z\rangle\\
&+\frac{(1+2\overline{\gamma}+\overline{\gamma}^2)\gamma_n^2M_1^2}{(1-\beta_n)(1-\alpha_n\gamma\theta)}+\frac{\alpha_n^2\overline{\gamma}^2}{(1-\beta_n)(1-\alpha_n\gamma\theta)}\|x_n-z\|^2\\
&+\frac{\alpha_n\gamma\theta}{1-\alpha_n\gamma\theta}[2\gamma_n(M_1+\|Az\|)\|x_n-z\|+\gamma_n^2(M_1+\|Az\|)^2]\\
\le\ & (1-\frac{2\alpha_n(\overline{\gamma}-\gamma\theta)}{1-\alpha_n\gamma\theta})\|x_n-z\|^2
\end{aligned}$$

$$
\begin{aligned}
& +\frac{2(\overline{\gamma}-\gamma\theta)\alpha_n}{(\overline{\gamma}-\gamma\theta)(1-\alpha_n\gamma\theta)}\langle \gamma f(z)-Bz, x_{n+1}-z\rangle \\
& +\frac{(1+2\overline{\gamma}+\overline{\gamma}^2)\gamma_n M_1^2}{(1-\beta_n)(1-\alpha_n\gamma\theta)}+\frac{2(\overline{\gamma}-\gamma\theta)\alpha_n^2\overline{\gamma}^2}{2(\overline{\gamma}-\gamma\theta)(1-\beta_n)(1-\alpha_n\gamma\theta)}\|x_n-z\|^2 \\
& +\frac{\gamma\theta}{1-\alpha_n\gamma\theta}[2(M_1+\|Az\|)\|x_n-z\|+(M_1+\|Az\|)^2]\gamma_n \\
= \; & (1-\delta_n)\|x_n-z\|^2+\delta_n s_n+v_n,
\end{aligned}
$$

where

$$\delta_n=\frac{2\alpha_n(\overline{\gamma}-\gamma\theta)}{1-\alpha_n\gamma\theta},$$

$$s_n=\frac{1}{\overline{\gamma}-\gamma\theta}\langle \gamma f(z)-Bz, x_{n+1}-z\rangle+\frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma}-\gamma\theta)(1-\beta_n)}\|x_n-z\|^2,$$

$$v_n=\left\{\frac{(1+2\overline{\gamma}+\overline{\gamma}^2)M_1^2}{(1-\beta_n)(1-\alpha_n\gamma\theta)}+\frac{\gamma\theta}{1-\alpha_n\gamma\theta}[2(M_1+\|Az\|)\|x_n-z\|+(M_1+\|Az\|)^2]\right\}\gamma_n.$$

Since $\delta_n \geq 2\alpha_n(\overline{\gamma}-\gamma\theta)$, we have $\Sigma_{n=1}^{\infty}\delta_n=\infty$. By Lemma 2.2, (C1)-(C3) and (3.25), we have $x_n \to z$. And it follows from (3.1) that $u_n \to z$. Therefore, from (3.5), (3.6) and (C3), we obtain $y_n \to z$ and $t_n \to z$. □

By Theorem 3.1, we can obtain some new and interesting strong convergence theorems. Now give some examples as follows:

Let $T_i = I$ for $i = 1, 2, \ldots, N$, by Theorem 3.1 we have the following result.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be an $\alpha-$inverse-strongly monotone mapping, and $f : C \to C$ be a contractive mapping with a contractive constant $\theta \in (0, 1)$. Let $\Theta : C \times C \to R$ be a bifunction satisfying (H1)-(H4) and $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with assumptions (A1) or (A2). Let $A : C \to H$ be a monotone and $k-$Lipschitz continuous mapping such that $\Omega = VI(C, A) \cap GMEP \neq \emptyset$. Let B be a strongly positive bounded linear operator on H with efficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \overline{\gamma}/\theta$. For any $x_1 = x \in C$. Let $\{x_n\}, \{y_n\}$ and $\{u_n\}$ be sequences defined by*

$$
\begin{cases}
\Theta(u_n, y)+\varphi(y)-\varphi(u_n)+\langle Fx_n, y-u_n\rangle+\dfrac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \\
\forall\ y \in C, \\
y_n=(1-\xi_n)u_n+\xi_n P_C(u_n-\gamma_n Au_n), \\
x_{n+1}=\alpha_n\gamma f(y_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n B)P_C(u_n-\gamma_n Ay_n),
\end{cases}
$$

where $n \geq 1, \{\alpha_n\}, \{\beta_n\},\{\xi_n\}$ and $\{\gamma_n\}$ are four sequences in [0,1] and $\{r_n\}$ is a sequence such that $\{r_n\} \subset (0, 2\alpha)$ satisfying the conditions (C1)-(C4) in Theorem 3.1. Then $\{x_n\}, \{y_n\}$ and $\{u_n\}$ converge strongly to $z \in \Omega$ if and only if $\{Au_n\}$ is bounded and $\liminf_{n\to\infty}\langle Au_n, y-u_n\rangle \geq 0$ for all $y \in C$, where z is the unique solution of variational inequality

$$\langle(\gamma f-B)z, w-z\rangle \leq 0,\ \forall\ w \in \Omega,$$

that is $z = P_\Omega(((I-B)+\gamma f)z)$.

Let $A = 0$, by Theorem 3.1, we obtain the following result.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be an $\alpha-$inverse-strongly monotone mapping, and $f : C \to C$ be a contractive mapping with a contractive constant $\theta \in (0,1)$. Let $\Theta : C \times C \to R$ be a bifunction satisfying (H1)-(H4) and $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with assumptions (A1) or (A2). Let $T_1, T_2, \ldots, T_N$ be a family of finitely nonexpansive mapping of C into H such that $\Omega = (\cap_{n=1}^{N} F(T_i)) \cap GMEP \neq \emptyset$. Let B be a strongly positive bounded linear operator on H with efficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \overline{\gamma}/\theta$. For $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0,1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1, \{\alpha_1^{n,j}\}_{j=1}^{N-1} \subset [\eta_1, \theta_1]$ with $0 < \eta_1 \le \theta_1 < 1, \{\alpha_1^{n,N}\} \subset [\eta_N, 1]$ with $0 < \eta_N \le 1$ and $\{\alpha_2^{n,j}\}_{j=1}^{N}, \{\alpha_3^{n,j}\}_{j=1}^{N} \subset [0, \theta_3]$ with $0 \le \theta_3 < 1$. For any $x_1 = x \in C$. Let $\{x_n\}$ and $\{u_n\}$ be sequences defined by*

$$\begin{cases} \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Fx_n, y - u_n \rangle + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \\ \forall\ y \in C, \\ x_{n+1} = \alpha_n \gamma f(u_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n B) S_n u_n, \quad \forall\ n \ge 1, \end{cases}$$

where $\{S_n : C \to C\}$ is the sequence defined by (1.4), $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0,1], $\{r_n\}$ is a sequence such that $\{r_n\} \subset (0, 2\alpha)$ satisfying the conditions (C1)-(C2) and (C4)-(C5) in Theorem 3.1. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in \Omega$, where z is the unique solution of variational inequality

$$\langle (\gamma f - B) z, w - z \rangle \le 0,\ \forall\ w \in \Omega,$$

that is $z = P_\Omega(((I - B) + \gamma f) z)$.

Let $\varphi = 0, F = 0$ and $\Theta(x, y) = 0$ for all $x, y \in C$ in Theorem 3.1, then $u_n = P_C x_n = x_n$. By Theorem 3.1, we obtain the following result.

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contractive mapping with a contractive constant $\theta \in (0,1)$. Let $A : C \to H$ be a monotone and $k-$Lipschitz continuous mapping. Let $T_1, T_2, \ldots, T_N$ be a family of finitely nonexpansive mapping of C into H such that $\Omega = (\cap_{n=1}^{N} F(T_i)) \cap VI(C, A) \neq \emptyset$. Let B be a strongly positive bounded linear operator on H with efficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \overline{\gamma}/\theta$. For $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0,1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1, \{\alpha_1^{n,j}\}_{j=1}^{N-1} \subset [\eta_1, \theta_1]$ with $0 < \eta_1 \le \theta_1 < 1, \{\alpha_1^{n,N}\} \subset [\eta_N, 1]$ with $0 < \eta_N \le 1$ and $\{\alpha_2^{n,j}\}_{j=1}^{N}, \{\alpha_3^{n,j}\}_{j=1}^{N} \subset [0, \theta_3]$ with $0 \le \theta_3 < 1$. For any $x_1 = x \in C$. Let $\{x_n\}$ and $\{y_n\}$ be sequences defined by*

$$\begin{cases} y_n = (1 - \xi_n) x_n + \xi_n P_C(x_n - \gamma_n A x_n), \\ x_{n+1} = \alpha_n \gamma f(y_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n B) S_n P_C(x_n - \gamma_n A y_n), \end{cases}$$

where $n \ge 1, \{S_n : C \to C\}$ is the sequence defined by (1.4), $\{\alpha_n\}, \{\beta_n\}, \{\xi_n\}$ and $\{\gamma_n\}$ are four sequences in [0,1] satisfying the conditions (C1)-(C3) and (C5) in Theorem 3.1. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $z \in \Omega$ if and only if $\{Ax_n\}$ is bounded and $\liminf_{n\to\infty} \langle Ax_n, y - x_n \rangle \ge 0$ for all $y \in C$, where z is the unique *solution of variational inequality*

$$\langle(\gamma f - B)z, w - z\rangle \leq 0, \ \forall\, w \in \Omega,$$

that is $z = P_\Omega(((I - B) + \gamma f)z)$.

Remark 3.1 Theorem 3.1 generalizes, extends and improves Theorem 3.1 of Ceng and Yao [4] in the following aspects:

(i) One nonexpansive mapping is replaced by $S-$mapping;

(ii) Our iterative process $(\star)$ is more general than (1.5) because it can be applied to solve the problem of finding a common element of the set of $GMEP$, the set of fixed point problems and the set of solutions of variational inequality problems.

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References

[1] S. Atsushiba and W. Takahashi, *Strong convergence theorems for a finite family of nonexpansive mappings and applications*, in: B.N. Prasad Birth Centenary Commemoration Volume, Indian J. Math. **41** (1999), 435–453.

[2] L. C. Ceng and J. C. Yao, *A relaxed extragradient-like method for a generalized mixed equilibrium problem, a general system of generalized equilibria and a fixed problem*, Nonlinear Anal. **72** (2010), 1922–1937.

[3] L. C. Ceng and J. C. Yao, *A hybrid iterative scheme for mixed equilibrium probems and fixed point problems*, J. Comput. Appl. Math. **214** (2008), 186–201.

[4] L. C. Ceng and J. C. Yao, *An extragradient-like approximation method for variational inequality problems and fixed point problems*, Appl. Math. Comput. **190** (2007), 205–215.

[5] A. Kangtunyakarn and S. Suantai, *Hybrid iterative scheme for generalized equilibrium problems and fixed point problems of finite family of nonexpansive mappings*, Nonlinear Anal. **3** (2009), 296–309.

[6] G. Marino and H. K. Xu, *A general iterative method for nonexpansive mappings in Hilbert spaces,* J. Math. Anal. Appl. **318** (2006), 43–52.

[7] N. Nadezhkina and W. Takahashi, *Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **128** (2006), 191–201.

[8] J. W. Peng and J. C. Yao, *A new hybrid-extragradient method for generalized mixed equilibrium problems and fixed point problems and variational inequality problems*, Taiwanese J. Math. **12** (2008), 1401–1432.

[9] J. W. Peng and J. C. Yao, *Strong convergence theorems of iterative scheme based on the extragradient method for mixed equilibrium problems and fixed point problems*, Math. Comput. Modelling **49** (2009), 1816–1828.

[10] S. Plubtieng and R. Punpaeng, *A new iterative method for equlibrium problems and fixed point problems of nonlinear mappings and monotone mappings*, Appl. Math. Comput. **197** (2008), 548–558.

[11] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75–88.

[12] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl. **305** (2005), 227–239.

[13] A. Tada and W. Takahashi, *Weak and strong convergence theorems for a nonexpansive mapping and an equlibrium problem*, J. Optim. Theory Appl. **133** (2007), 359–370.

[14] S. Takahashi and W. Takahashi, *Strong convergence theorem for a generalized equilibrium problem amd a nonexpansive mapping in a Hilbert space*, Nonlinear Anal. **69** (2008), 1025–1033.

[15] W. Takahashi and K. Shimoji, *Convergence theorems for nonexpansive mappings and feasibility problems*, Math. Comput. Modelling **32** (2000), 1463–1471.
[16] Y. H. Yao and J. C. Yao, *On modified iterative method for nonexpansive mappings and monotone mappings*, Appl. Math. Comput. **186** (2007), 1551–1558.
[17] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Society **66** (2002), 240–256.
[18] L. C. Zeng and J. C. Yao, *Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems*, Taiwan. J. Math. **10** (2006), 1293–1303.

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