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p-QUASI-CONTRACTION MAPS AND FIXED POINTS IN METRIC SPACES

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Abstract. In this paper, we first introduce p-quasi-contraction maps in metric spaces for each $p \in \mathbb{N}$ and then we give a fixed point result for such maps. An example is given to support our result.

1. Introduction and Preliminaries

The well known Banach's fixed point theorem asserts that if (X, d) is a complete metric space and $T : X \to X$ is a map such that

$$d(Tx, Ty) \leq cd(x, y), \text{ for each } x, y \in X,$$

where $0 \leq c < 1$, then f has a unique fixed point $\overline{x} \in X$ and for any $x \in X$ the sequence $\{T^n x\}$ converges to $\overline{x}$.

In recent years some generalizations of the above Banach's contraction principle have appeared. Of all these, the following generalization of Ćirić [2] stands at the top.

Theorem 1.1. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a quasi-contraction map, that is, there exists $0 \leq c < 1$ such that*

$$d(Tx, Ty) \leq c \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for each $x, y \in X$. Then T has a unique fixed point $\overline{x} \in X$ and for any $x \in X$ the sequence $\{T^n x\}$ converges to $\overline{x}$.

In this paper, we first introduce p-quasi-contraction maps and then we prove a fixed point theorem for such maps in metric spaces.

2. Main results

We now introduce the concept of a *p-quasi-contraction* map in metric spaces for each $p \in \mathbb{N}$.

Definition 2.1. Let (X, d) be a metric space and let $p \in \mathbb{N}$. The self-map $T : X \to X$ is said to be a *p-quasi-contraction* if there exists $0 \leq c < 1$ such that

$$d(T^p x, T^p y) \leq c \max\{d(T^i u, T^j v) : u, v \in \{x, y\}, 0 \leq i, j \leq p \text{ and } i + j < 2p\}$$

for any $x, y \in X$.

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Notice that for $p = 1$ our definition reduces to the above mentioned definition of a quasi-contraction map.

Let (X, d) be a metric space and let $T : X \to X$ be a mapping. For $A \subseteq X$ let $\delta(A) = \sup\{d(a, b) : a, b \in A\}$, and for each $x \in X$, let

$$\begin{aligned} O(x, n) &= \{x, Tx, \ldots, T^n x\},\ n \in \mathbb{N}, \\ O(x, \infty) &= \{x, Tx, \ldots\}. \end{aligned}$$

Now, we are ready to state our main result.

Theorem 2.2. *Let (X, d) be a complete metric space and let $p \in \mathbb{N}$. Let $T : X \to X$ be a p-quasi-contraction map such that $T^m : X \to X$ is continuous for some $m \in \mathbb{N}$. Then T has a unique fixed point $\overline{x} \in X$ and for any $x \in X$ the sequence $\{T^n x\}$ converges to $\overline{x}$.*

Proof. Let $x \in X$. We first show that $\{\delta[O(x, n)]\}_n$ is a bounded sequence. On the contrary, assume that

$$\delta[O(x, \infty)] = \sup\{\delta[O(x, n)] : n \in \mathbb{N}\} = \infty.$$

Notice that since the sequence $\{\delta[O(x, n)]\}$ is non-decreasing then all of it's subsequences are also unbounded. Let n be a positive integer with $n \geq p$, and let $i, j \in \{p, p + 1, \ldots, n\}$. Since T is a p-quasi-contraction map then, we have

$$\begin{aligned} d(T^i x, T^j x) &= d(T^p T^{i-p} x, T^p T^{j-p} x) \\ &\leq c \max\{d(T^k T^{i-p} x, T^l T^{j-p} x) : 0 \leq k, l \leq p \text{ and } k + l < 2p\} \\ &\leq c\delta[O(x, n)]. \end{aligned} \tag{2.1}$$

Thus for sufficiently large $n \in \mathbb{N}$ there exist positive integers k, l with $k < p$ and $p \leq l \leq n$ such that

$$d(T^k x, T^l x) = \delta[O(x, n)], \tag{2.2}$$

(note that if $l < p$ for infinitely many $n \in \mathbb{N}$ then by (2.2) the sequence $\{\delta[O(x, n)]\}$ has a bounded subsequence which is a contradiction). From (2.1) and (2.2), we get

$$\begin{aligned} d(T^k x, T^l x) &\leq d(T^k x, T^p x) + d(T^p x, T^l x) \\ &\leq d(T^k x, T^p x) + c\delta[O(x, n)] \\ &= d(T^k x, T^p x) + cd(T^k x, T^l x). \end{aligned} \tag{2.3}$$

Therefore from (2.3), for sufficiently large $n \in \mathbb{N}$, we get

$$\begin{aligned} \delta[O(x, n)] &= d(T^k x, T^l x) \\ &\leq \frac{1}{1 - c} d(T^k x, T^p x) \\ &\leq \frac{1}{1 - c} \max\{d(T^i x, T^p x) : 1 \leq i \leq p\}. \end{aligned}$$

Thus

$$\delta[O(x, \infty)] \leq \frac{1}{1 - c} \max\{d(T^i x, T^p x) : 1 \leq i \leq p\} < \infty,$$

a contradiction.

Now, we show that the sequence $\{T^n x\}$ is a Cauchy sequence. Let n and m be any positive integers with $p \le n < m$. From (2.1), we get

$$d(T^n x, T^m x) = d(T^p T^{n-p} x, T^{m-n+p} T^{n-p} x) \le c\delta[O(T^{n-p} x, m-n+p)]. \tag{2.4}$$

By (2.2), there exist positive integers k_1 and l_1, $k_1 < p$ and $p \le l_1 \le m-n+p$, such that

$$\delta[O(T^{n-p} x, m-n+p)] = d(T^{k_1} T^{n-p} x, T^{l_1} T^{n-p} x). \tag{2.5}$$

Then we have

$$\begin{aligned} d(T^{k_1} T^{n-p} x, T^{l_1} T^{n-p} x) &= d(T^{k_1+p} T^{n-2p} x, T^{l_1+p} T^{n-2p} x) \\ &\le c\delta[O(T^{n-2p} x, l_1+p)] \\ &\le c\delta[O(T^{n-2p} x, m-n+2p)]. \end{aligned} \tag{2.6}$$

Therefore from (2.4), (2.5) and (2.6), we have

$$d(T^n x, T^m x) \le c\delta[O(T^{n-p} x, m-n+p)] \le c^2\delta[O(T^{n-2p} x, m-n+2p)].$$

Proceeding in this manner, we obtain

$$d(T^n x, T^m x) \le c^{[\frac{n}{p}]}\delta\Big[O\Big(T^{n-[\frac{n}{p}]p} x, m-n+\Big[\frac{n}{p}\Big]p\Big)\Big] \le c^{[\frac{n}{p}]}\delta[O(x, m+p)].$$

Then

$$d(T^n x, T^m x) \le c^{[\frac{n}{p}]}\delta[O(x, \infty)]. \tag{2.7}$$

Since $\lim_{n\to\infty} c^{[\frac{n}{p}]} = 0$, then from (2.7) we get that $\{T^n x\}$ is a Cauchy sequence. Since (X, d) is a complete metric space then $\lim_{n\to\infty} T^n x$ exists for each $x \in X$. Now we show that

$$\lim_{n\to\infty} T^n x = \lim_{n\to\infty} T^n y \text{ for each } x, y \in X. \tag{2.8}$$

To show the claim let $\lim_{n\to\infty} T^n x = x_0$ and $\lim_{n\to\infty} T^n y = y_0$. Since T is p-quasi-contraction then for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(T^{np} x, T^{np} y) &= d(T^p(T^{(n-1)p} x), T^p(T^{(n-1)p} y)) \\ &\le c\max\{d(T^{(n-1)p+i} x, T^{(n-1)p+j} x), d(T^{(n-1)p+i} y, T^{(n-1)p+j} y), \\ &\qquad d(T^{(n-1)p+i} x, T^{(n-1)p+j} y) : 0 \le i, j \le p \text{ and } i+j < 2p\}. \end{aligned} \tag{2.9}$$

Since

$$\lim_{n\to\infty} d(T^{(n-1)p+i} x, T^{(n-1)p+j} x) = \lim_{n\to\infty} d(T^{(n-1)p+i} y, T^{(n-1)p+j} y) = 0$$

and $\lim_{n\to\infty} d(T^{(n-1)p+i} x, T^{(n-1)p+j} y) = d(x_0, y_0)$ then from (2.9), we get $d(x_0, y_0) \le cd(x_0, y_0)$ and so $x_0 = y_0$ (note that $c < 1$).

Therefore from (2.8) we deduce that there exists $\overline{x} \in X$ such that $\lim_{n\to\infty} T^n x = \overline{x}$ for each $x \in X$. To prove that $T\overline{x} = \overline{x}$ notice first that since T^m is continuous, we have

$$\overline{x} = \lim_{n\to\infty} T^{m+n} x = \lim_{n\to\infty} T^m(T^n x) = T^m(\overline{x}).$$

Then $\overline{x}$ is a fixed point of T^m. Now we show that T^m has a unique fixed point. To prove the claim let us suppose that $\overline{y}$ is another fixed point of T^m, that is, $T^m(\overline{y}) = \overline{y}$. Then for each $n \in \mathbb{N}$, $T^{nm}(\overline{y}) = \overline{y}$ and so

$$\overline{x} = \lim_{n\to\infty} T^{nm}(\overline{y}) = \overline{y}.$$

Thus $\overline{x}$ is the unique fixed point of T^m. Since $\overline{x} = T^m(\overline{x})$, then

$$T(\overline{x}) = T(T^m(\overline{x})) = T^m(T(\overline{x})),$$

and so $T(\overline{x})$ is also a fixed point of T^m. Thus by the uniqueness, we get $T(\overline{x}) = \overline{x}$. To show that $\overline{x}$ is the unique fixed point of T, let us suppose that $\overline{y}$ is another fixed point of T. Then $\overline{x}$ and $\overline{y}$ are fixed points of T^m and so by the above $\overline{x} = \overline{y}$. □

Now, we give the following example to support our main result.

Example 2.3. *Let $X = \mathbb{R}$, $d(x, y) = |x - y|$ for each $x, y \in \mathbb{R}$ and let Q denotes the set of rational numbers. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by*

$$T(x) = \begin{cases} \sqrt{2}, & \text{if } x \in Q \\ \sqrt{3}. & \text{otherwise} \end{cases}$$

Then $T^2(x) = \sqrt{3}$ for each $x \in \mathbb{R}$ and so T is a 2-quasi contraction map such that T is discontinuous, and T^2 is continuous. Thus all of the conditions of Theorem 2.2 are satisfied and $\sqrt{3}$ is the unique fixed point of T. But we show that T is not a quasi-contraction map and so we cannot invoke the above mentioned Theorem 1.1 of Ćirić to show that the mapping T has a fixed point in X. To show the claim, let $x = \frac{3}{2}$ and $y = \frac{\sqrt{2}+\sqrt{3}}{2}$. Then for each $0 \le c < 1$, we have

$$\begin{aligned} c \max\{&d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\ &\le \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\ &= \sqrt{3} - \frac{3}{2} \\ &< \sqrt{3} - \sqrt{2} = d(Tx, Ty). \end{aligned}$$

Merryfield and Stein Jr. [3] and Arvanitakis [1] independently proved the following interesting generalization of the Banach contraction principle.

Theorem 2.4 (Generalized Banach Contraction Principle(GBCP))**.** *Let $T : X \to X$ be a map of a complete metric space (X, d), and let $0 \le c < 1$. Let J be a positive integer. Assume that for each $x, y \in X$,*

$$\min\{d(T^k x, T^k y) : 1 \le k \le J\} \le c d(x, y).$$

Then T has a unique fixed point.

Now, we pose the following conjecture:

Generalized Quasi-Contraction Conjecture (GQCC). Let $T : X \to X$ be a continuous map of a complete metric space (X, d), and let $0 \le c < 1$. Let $p \le q$ are positive integers. Assume that for each $x, y \in X$,

$\min\{d(T^k x, T^k y) : p \le k \le q\}$

$$\leq c \max\{d(T^i u, T^j v) : u, v \in \{x, y\}, 0 \leq i, j \leq p \text{ and } i + j < 2p\}.$$

Then T has a unique fixed point.

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