Journal of Nonlinear and Convex Analysis
Volume 15, Number 4, 2014, 711–725

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

DUALITY FOR NONSMOOTH MINIMAX FRACTIONAL PROGRAMMING WITH EXPONENTIAL (p, r)-INVEXITY

SHUN-CHIN HO AND HANG-CHIN LAI*

Abstract. In this paper, we focus on a nonsmooth minimax fractional programming problem with exponential (p, r)-invexity. We establish a nonparametric necessary and sufficient optimality conditions. The nonparametric necessary and sufficient optimality conditions deduce to two parameter-free type dual models: Mond-Weir type dual and Wolfe type dual problems. On these duality types, we establish the duality theorems under exponential (p, r)-invexities including weak duality, strong duality, and strict converse duality theorems. Consequently, such duality types are no duality gap with respect to the primary problem in the framwork.

1. Introduction

In this paper, we consider the following minimax fractional programming problem

$$(P) \qquad \min_{x\in X} \max_{y\in Y} \frac{f(x,y)}{g(x,y)}$$
$$\text{subject to} \quad X = \left\{x \in \mathrm{R}^n \;\middle|\; h(x) \in -\mathrm{R}^p_+\right\}$$
$$\text{and } Y \text{ is a compact subset of } \mathrm{R}^m,$$

where $f(\cdot,\cdot)$ and $g(\cdot,\cdot) : \mathrm{R}^n \times \mathrm{R}^m \mapsto \mathrm{R}$ are continuous functions and for each $y \in Y$, $f(\cdot, y)$, $g(\cdot, y)$, and $h(\cdot) : \mathrm{R}^n \mapsto \mathrm{R}^p$ are locally Lipschitz functions. Without loss of generality, we may assume that $g(x, y) > 0$ and $f(x, y)$ is nonnegative for all $(x, y) \in X \times Y$. Then for $x \in X$, we denote

$$J(x) = \{j \in J \mid h_j(x) = 0\} \text{ where } J = \{1, 2, \dots, p\}, \text{ and}$$
$$Y(x) = \left\{y \in Y \;\middle|\; \frac{f(x,y)}{g(x,y)} = \sup_{z\in Y} \frac{f(x,z)}{g(x,z)}\right\}.$$

By the compactness of Y, the continuous function has finite points attended to its maximum points say s. Thus we can set

$$K(x) = \left\{(s,t,y) \in \mathrm{N} \times \mathrm{R}^s_+ \times \mathrm{R}^{ms} \;\middle|\; \begin{array}{l} t = (t_1, t_2, \dots, t_s) \in \mathrm{R}^s_+, \sum_{i=1}^{s} t_i = 1 \text{ and} \\ y = (y_1, y_2, \dots, y_s),\ y_i \in Y(x), i = 1, 2, \dots, s \end{array}\right\}.$$

Problems of this type are known in the area of the mathematical programming as general minimax programming problems, and have been the subject of immense interests in the past few years. The importance of minimax models and methods

2010 *Mathematics Subject Classification.* 90C46, 90C32, 90C30.
Key words and phrases. Minimax fractional programming, exponential type (p, r)-invex function, duality problems, duality theorems.
*Corresponding author.

is well-known in a great variety of optimal decision making situations. Recently, optimality conditions and various duality results have been obtained for minimax fractional programming problems involving the optimization of several ratios in the objective function. In 1977, Schmitendorf [17] first established the necessary and sufficient conditions for minimax programming problem. Later many authors proved duality theorems involving in convexity, generalized convexity (cf. [1], [4], [5], [8], [11], [12], [13], [16], [17], [18]) invexity, and generalized invexity (cf. [3], [2], [7], [9], [10], [14], [15]) for mathematical programming problem.

Recently, Lai and Liu [13] considered nonsmooth minimax fractional programming problem. They reduced minimax fractional programming problem to an equivalent nonfractional parametric problem under continuous functions without convexity. They established necessary and sufficient optimality conditions for nonsmooth minimax fractional programming problem and proved duality theorems for parametric duality. Ho and Lai [7] introduced exponential (Exp for brevity) (p, r)-invexity for Lipschiz function as well as the definition of differentiable (p, r)-invex function given by Antczak [3]. They established necessary optimality conditions and sufficient optimality conditions involving in Exp (p, r)-invexity. Furthermore, Ho and Lai [7] proved duality theorems of parametric duality for nonsmooth minimax fractional programming problem with Exp (p, r)-invexity.

In this paper, we deal with minimax fractional programming problem involving Exp (p, r)-invex function. We apply the optimality conditions to perform parameter-free duality models for the minimax fractional programming problem (P). For convenience, we recall the Exp (p, r)-invex function in section 2 and optimality conditions in section 3. Furthermore, we treat with duality theorems for Mond-Weir type and Wolfe type duality models which the functions occurring belong to Exp (p, r)-invexity.

2. Exponential (p, r)-invex function

Throughout the paper, R^n is the n-dimensional Euclidean space and R^n_+ is its nonnegative orthant. Let S be an open subset of R^n.

A function $f : S \longrightarrow \mathrm{R}$ is said to be **locally Lipschitz** at $x \in S$ if there exist a positive constant $K \in \mathrm{R}$ and a neighborhood Γ of $x \in S$ such that

$$|f(y) - f(z)| \leqq K\|y - z\| \qquad \text{for all} \quad z, y \in \Gamma.$$

where $\|\cdot\|$ stands for any norm of R^n.

For any vector ν in R^n, the **generalized directional derivative** of f at x in the direction $\nu \in \mathrm{R}^n$ in Clarke sense (see [6]) is given by

$$f^\circ(x;\nu) = \limsup_{\substack{y \longrightarrow x \\ \lambda \longrightarrow 0^+}} \frac{f(y + \lambda\nu) - f(y)}{\lambda}.$$

The **generalized subdifferential** of f at $x \in S$ is defined by the set

$$\partial^c f(x) = \{\xi \in \mathrm{R}^n : f^\circ(x;\nu) \geq \langle \xi, \nu\rangle \quad \text{for all} \quad \nu \in \mathrm{R}^n\}$$

where $\langle \xi; \nu\rangle$ denotes the inner product in R^n.

In order to establish the nonparametric sufficient conditions that the duality theorems hold for parameter-free type duality models, we will use the following definition in our paper.

Definition 2.1 (cf. [7]). Let p, r be arbitrary real numbers. A locally Lipschitz function $f : S \subseteq \mathrm{R}^n \to \mathrm{R}$ is said to be **exponential $(\mathbf{p}, \mathbf{r})$-invexity (strictly)** at $u \in S$ if there exists a function $\eta : S \times S \longrightarrow \mathrm{R}^n$ with property $\eta(x, u) = 0$ only if $u = x$ in S such that for each $x \in S$, the following inequality holds for $\xi \in \partial^c f(u)$

$$\frac{1}{r} e^{rf(x)} \geq \frac{1}{r} e^{rf(u)} \left[1 + \frac{r}{p} \left\langle \xi \text{ , } (e^{p\eta(x,u)} - \mathbf{1}) \right\rangle \right] \ (> \text{ if } x \neq u) \text{ for } p \neq 0,\ r \neq 0. \tag{2.1}$$

If p or r is zero, then (2.1) can give some modification by using the limit of $p \to 0$ or $r \to 0$.

(i) If $r \neq 0$, $p \to 0$ in (2.1), then

$$e^{rf(x)} - e^{rf(u)} \geq re^{rf(u)} \langle \xi \text{ , } \eta(x, u) \rangle \quad (> \text{ if } x \neq u) \text{ for } r \neq 0,\ p = 0,$$

(ii) If $p \neq 0$, $r \to 0$, then (2.1) becomes

$$f(x) - f(u) \geq \frac{1}{p} \left\langle \xi \text{ , } (e^{p\eta(x,u)} - \mathbf{1}) \right\rangle \quad (> \text{ if } x \neq u) \text{ for } p \neq 0,\ r = 0, \tag{2.2}$$

(iii) If $r = 0$, $p \to 0$, then (2.2) becomes

$$f(x) - f(u) \geq \langle \xi \text{ , } \eta(x, u) \rangle \quad (> \text{ if } x \neq u) \text{ for } p = 0,\ r = 0,$$

where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathrm{R}^n$, $(e^{p\eta(x,u)} - \mathbf{1})$ stands for the n-vector $(e^{p\eta_1(x,u)} - 1, e^{p\eta_2(x,u)} - 1, \ldots, e^{p\eta_n(x,u)} - 1)$, and $\langle \cdot \text{ , } \cdot \rangle$ stands for the inner product in R^n throughout this paper.

Remark 2.2. *All theorems in our work will be described only in the case of* $p \neq 0$ *and* $r \neq 0$. *We omit the proof of other cases like in (i), (ii), and (iii).*

Now, we say that a constraint qualification holds at $x^* \in X$ if there exists an $x \in X$ such that $h_j(x) < 0$ for $j \in J(x^*)$.

Next section, we state the necessary optimality conditions and sufficient optimality conditions.

3. Optimality conditions

In [7], Ho and Lai derived the following necessary optimality conditions and sufficient optimality conditions for problem (P).

Theorem 3.1 (Necessary Optimality Conditions (cf. [7, Theorem 3.1])). *Let x^* be a (P)-optimal solution and the constraint qualification hold at x^*. Then there exist $(s^*, t^*, y^*) \in K(x^*)$, $\lambda^* \in \mathrm{R}_+$, and a p-vector Lagrange multiplier $\mu^* \in \mathrm{R}^p_+$ such that*

$$0 \in \sum_{i=1}^{s^*} t_i^* \left\{ \partial^c f(x^*, y_i^*) + \lambda^* \partial^c (-g(x^*, y_i^*)) \right\} + \langle \mu^* \text{ , } \partial^c h(x^*) \rangle_p, \tag{3.1}$$

$$f(x^*, y_i^*) - \lambda^* g(x^*, y_i^*) = 0, \qquad i = 1, 2, \ldots, s^*, \tag{3.2}$$

$$\mu_j^* h_j(x^*) = 0, \qquad j = 1, 2, \ldots, p, \tag{3.3}$$

$$\mu^* \in \mathrm{R}^p_+,\ t^* \in I,\ y_i^* \in Y(x^*), \quad i = 1, 2, \ldots, s^*, \tag{3.4}$$

where $I = \{t^* \in \mathrm{R}^{s^*}_+ : t^* = (t_1^*, t_2^*, \ldots, t_{s^*}^*)$ *with* $\sum_{i=1}^{s^*} t_i^* = 1\}$ *and* $\langle \mu^* , \partial^c h(x^*)\rangle_p \equiv \sum_{j=1}^{p} \mu_j^* \partial^c h_j(x^*)$.

We state the sufficient optimality by using the converse of the necessary optimality condition with extra assumptions for Exp (p, r)-invexities to establish the following theorem.

Theorem 3.2 (Sufficient Optimality Conditions (cf. [7, Theorem 4.1])**.** *Let* $\widetilde{x}$ *be a feasible solution of* (P) *satisfying the necessity conditions* (3.1) $\sim$ (3.4). *Suppose further that any one of the following* (*a*) *or* (*b*) *holds*

(a) $A_1(\cdot) = \sum_{i=1}^{\widetilde{s}} \widetilde{t}_i(f(\cdot, \widetilde{y}_i) - \widetilde{\lambda} g(\cdot, \widetilde{y}_i))$ *and* $A_2(\cdot) = \sum_{j=1}^{p} \widetilde{\mu}_j h_j(\cdot)$ *are Exp* (p, r)-*invexities with respect to (w.r.t., for brevity) the same function* η *at* $\widetilde{x}$ *in* X,

(b) $A_3(\cdot) = \sum_{i=1}^{\widetilde{s}} \widetilde{t}_i(f(\cdot, \widetilde{y}_i) - \widetilde{\lambda} g(\cdot, \widetilde{y}_i)) + \sum_{j=1}^{p} \widetilde{\mu}_j h_j(\cdot)$ *is an Exp* (p, r)-*invexity w.r.t.* η *at* $\widetilde{x}$ *in* X.

Then $\widetilde{x}$ *is an optimal solution of* (P).

In order to discuss Mond-Weir type dual model for the considered generalized fractional minimax programming problem (P), we restate Theorem 3.1 by using the parameter λ^* instead of $\dfrac{f(x^*, y_i^*)}{g(x^*, y_i^*)}$ in (3.1) and rewriting the multiplier functions associated with constraints as the following (cf. [4], [18]).

Theorem 3.3 (Nonparametric Necessary Optimality Conditions)**.** *Let* x^* *be a* (P)-*optimal solution and the constraint qualification hold at* x^*. *Then there exist* $(s^*, t^*, y^*) \in K(x^*)$, *and a p-vector Lagrange multiplier* $\mu^* \in \mathrm{R}^p_+$ *such that*

$$0 \in \sum_{i=1}^{s^*} t_i^* \left\{ g(x^*, y_i^*)\partial^c f(x^*, y_i^*) + f(x^*, y_i^*)\partial^c(-g(x^*, y_i^*)) \right\} + \langle \mu^* , \partial^c h(x^*)\rangle_p, \tag{3.5}$$

$$\mu_j^* h_j(x^*) = 0, \quad j = 1, 2, \ldots, p, \tag{3.6}$$

$$\mu^* \in \mathrm{R}^p_+,\ t^* \in I,\ y_i^* \in Y(x^*), \quad i = 1, 2, \ldots, s^*, \tag{3.7}$$

Actually the expression (3.5) is the same as

$$\begin{aligned} 0 \in \sum_{i=1}^{s^*} t_i^* g(x^*, y_i^*) &\left[\sum_{i=1}^{s^*} t_i^* \partial^c f(x^*, y_i^*) + \langle \mu^* , \partial^c h(x^*)\rangle_p \right] \\ &+ \left[\sum_{i=1}^{s^*} t_i^* f(x^*, y_i^*) + \langle \mu^* , h(x^*)\rangle_p \right] \sum_{i=1}^{s^*} t_i^* \partial^c(-g(x^*, y_i^*)), \end{aligned} \tag{3.8}$$

The nonparametric sufficient optimality conditions follow from the inverse of nonparametric necessary optimality conditions with extra assumptions. Thus the sufficient optimality theorem varies depending on the extra assumptions and the duality model is various. Now, we employ the necessary optimality conditions and Exp (p, r)-invexity to establish sufficient optimality conditions.

Theorem 3.4 (Sufficient Optimality Conditions)**.** *If $\widetilde{x}$ is a feasible solution of (P) and the necessity conditions* (3.5) $\sim$ (3.7) *hold. Denote a function $A_4 : X \to \mathrm{R}$ by*

$$A_4(\cdot) = \sum_{i=1}^{s} t_i \left\{ g(\widetilde{x}, y_i) f(\cdot, y_i) - f(\widetilde{x}, y_i) g(\cdot, y_i) \right\} + \langle \mu \ , \ h(\cdot) \rangle_p$$

with $A_4(\widetilde{x}) = 0$. Assume that A_4 is an Exp (p, r)-invexity w.r.t. η at $\widetilde{x}$ in X. Then $\widetilde{x}$ is an optimal solution of (P).

Proof. Suppose that $\widetilde{x}$ is not an optimal solution of (P). Then there exists a (P)-feasible solution x, such that

$$\frac{f(\widetilde{x}, y_i)}{g(\widetilde{x}, y_i)} > \max_{y \in Y} \frac{f(x, y)}{g(x, y)} \qquad \text{for all} \quad i = 1, 2, \ldots, s.$$

For all $i = 1, 2, \ldots, s$, the above inequality reduces

$$f(x, y) g(\widetilde{x}, y_i) - g(x, y) f(\widetilde{x}, y_i) < 0 \qquad \text{for all} \quad y \in Y. \tag{3.9}$$

By relations (3.7) and (3.9), we have

$$\sum_{i=1}^{s} t_i (f(x, y_i) g(\widetilde{x}, y_i) - g(x, y_i) f(\widetilde{x}, y_i)) < 0. \tag{3.10}$$

On the other hand, from $h_j(x) \leq 0, \ j \in J$ and $\mu \in \mathrm{R}^p_+$, we obtain

$$\langle \mu \ , \ h(x) \rangle_p \leq 0, \tag{3.11}$$

thus from the inequalities (3.10) and (3.11), it yields

$$\begin{aligned} A_4(x) \ &= \sum_{i=1}^{s} t_i (f(x, y_i) g(\widetilde{x}, y_i) - g(x, y_i) f(\widetilde{x}, y_i)) + \langle \mu \ , \ h(x) \rangle_p \\ &< 0 = A_4(\widetilde{x}). \end{aligned} \tag{3.12}$$

By the relation (3.5), there exist $\xi_i \in \partial^c f(\widetilde{x}, y_i)$, $\varsigma_i \in \partial^c(-g(\widetilde{x}, y_i))$ for all $i = 1, 2, \ldots, s$, and $\rho_j \in \partial^c h_j(\widetilde{x})$ for all $j \in J$, such that

$$\langle \ a_4 \ \rangle \equiv \sum_{i=1}^{s} t_i \left\{ g(\widetilde{x}, y_i) \xi_i + f(\widetilde{x}, y_i) \varsigma_i \right\} + \langle \mu \ , \ \rho \rangle_p = 0,$$

that is, $\langle \ a_4 \ \rangle$ is a zero vector, where $\langle \mu \ , \ \rho \rangle_p = \sum_{j=1}^{p} \mu_j \rho_j$ and $\rho = (\rho_1, \rho_2, \ldots, \rho_p)$.

It follows the inner product

$$\frac{1}{p} \left\langle \ \langle \ a_4 \ \rangle \ , \ (e^{p\eta(x, \widetilde{x})} - \mathbf{1}) \ \right\rangle = 0. \tag{3.13}$$

Since A_4 is an Exp (p,r)-invex function w.r.t. η at $\widetilde{x}$ in X, we have

$$\frac{1}{r}e^{rA_4(x)} \geq \frac{1}{r}e^{rA_4(\widetilde{x})}\left[1+\frac{r}{p}\left\langle\ \langle\ a_4\ \rangle\ ,\ (e^{p\eta(x,\widetilde{x})}-\mathbf{1})\ \right\rangle\right].$$

This inequality together with equality (3.13) yields

$$\frac{1}{r}e^{rA_4(x)} \geq \frac{1}{r}e^{rA_4(\widetilde{x})}. \tag{3.14}$$

Using fundamental property of the exponential function and inequality (3.14), we get

$$A_4(x) \geq A_4(\widetilde{x})$$

which contradicts the inequality (3.12). Hence, the proof is complete. □

Similar to the proof of Theorem 3.4, we establish Theorem 3.5 which is simply stated.

Theorem 3.5 (Sufficient Optimality Conditions)**.** *If $\widetilde{x}$ is a feasible solution of (P) and the necessary conditions (3.6) $\sim$ (3.8) hold. Denote a function $A_5 : X \to \mathrm{R}$ by*

$$A_5(\cdot) = \sum_{i=1}^{s} t_i g(\widetilde{x}, y_i)\left[\sum_{i=1}^{s} t_i f(\cdot, y_i) + \langle\mu\ ,\ h(\cdot)\rangle_p\right] + \left[\sum_{i=1}^{s} t_i f(\widetilde{x}, y_i) + \langle\mu\ ,\ h(\widetilde{x})\rangle_p\right]\sum_{i=1}^{s} t_i(-g(\cdot, y_i))$$

with $A_5(\widetilde{x}) = 0$. Assume that A_5 is an Exp (p,r)-invexity w.r.t. η at $\widetilde{x}$ in X. Then $\widetilde{x}$ is an optimal solution of (P).

4. Mond-Weir type duality model

The Mond-Weir type duality contains no constraint of problem (P) in the objective fractional functional of (MWD), as the following form

$$(MWD) \qquad \max_{u} \quad \max_{(\mu,s,t,y)\in K_1(u)} \quad \frac{f(u,y)}{g(u,y)}$$

subject to

$$0 \in \sum_{i=1}^{s} t_i\left\{g(u,y_i)\partial^c f(u,y_i) + f(u,y_i)\partial^c(-g(u,y_i))\right\} + \langle\mu\ ,\ \partial^c h(u)\rangle_p, \tag{4.1}$$

$$\sum_{j\in J} \mu_j h_j(u) = 0, \tag{4.2}$$

$$\mu \in \mathrm{R}^p_+,\ t \in I,\ y_i \in Y(u) \quad \text{for} \quad i = 1, 2, \ldots, s, \tag{4.3}$$

where $u \in X$ is the (P)-feasible solution and the $K_1(u)$ stand in (MWD) is represented by the set

$$K_1(u) = \left\{(\mu, s, t, y) \in \mathrm{R}^p_+ \times \mathrm{N} \times \mathrm{R}^s_+ \times \mathrm{R}^{ms}\ \middle|\ \begin{array}{l} t \in I\ ,\ y = (y_1, y_2, \ldots, y_s)\ , \\ y_i \in Y(u)\ ,\ i = 1, 2, \ldots, s \end{array}\right\},$$

which means that elements in $K_1(u)$ satisfy the expressions (3.5) and (3.6) in the necessary optimality conditions of (P) in Theorem 3.3.

In order to establish (MWD) is the dual problem w.r.t. the primal problem (P), we denote Γ_1 the constraint set of (MWD). Moreover, denote by the elements satisfying the necessary optimality conditions of (P) which is defined by a projective-like set as the feasible solutions of problem (P) to be

$$pr_X\Gamma_1 = \{u \in X \subseteq \mathrm{R}^n |\ (u; \mu, s, t, y) \in \Gamma_1\}.$$

At first, we show the weak duality theorem related to problems (P) and (MWD), as following.

Theorem 4.1 (Weak Duality). *Let x and $(u; \mu, s, t, y)$ be (P)-feasible and (MWD)-feasible, respectively. Denote a function $A_6 : X \to \mathrm{R}$ by*

$$A_6(\cdot) = \sum_{i=1}^{s} t_i \{g(u, y_i)f(\cdot, y_i) - f(u, y_i)g(\cdot, y_i)\} + \langle \mu\ ,\ h(\cdot)\rangle_p$$

with $A_6(u) = 0$ if $u \in \Gamma_1$. Suppose that $A_6(\cdot)$ is an Exp (p, r)-invex function w.r.t. η at $u \in pr_X\Gamma_1$.
Then $\max_{y\in Y} \dfrac{f(x, y)}{g(x, y)} \geq \dfrac{f(u, y)}{g(u, y)}$

Proof. Suppose on the contrary. It would have $\max_{y\in Y} \dfrac{f(x, y)}{g(x, y)} < \dfrac{f(u, y)}{g(u, y)}$ and then there is a feasible solution $x \in X$ such that

$$\max_{y\in Y} \frac{f(x, y)}{g(x, y)} < \frac{f(u, y)}{g(u, y)} \text{ for any } (u; \mu, s, t, y) \in \Gamma_1. \tag{4.4}$$

Thus the inequality (4.4) would reduce

$$\frac{f(x, y)}{g(x, y)} < \frac{f(u, y)}{g(u, y)} \quad \text{for all} \quad y \in Y.$$

This is equivalent to

$$f(x, y)g(u, y) - f(u, y)g(x, y) < 0 \quad \text{for all} \quad y \in Y.$$

Multiplying the above expression respectively by $t_i \geq 0$, $i = 1, 2, \ldots, s$ and then summing up, it would yield

$$\sum_{i=1}^{s} t_i(f(x, y_i)g(u, y_i) - f(u, y_i)g(x, y_i)) < 0. \tag{4.5}$$

On the other hand, by the feasible solution x in (P) together with $\mu \in \mathrm{R}^p_+$, we have

$$\langle \mu\ ,\ h(x)\rangle_p \leq 0. \tag{4.6}$$

Then from (4.5) and (4.6), we obtain

$$A_6(x) = \sum_{i=1}^{s} t_i(f(x, y_i)g(u, y_i) - f(u, y_i)g(x, y_i)) + \langle \mu\ ,\ h(x)\rangle_p < 0 = A_6(u). \tag{4.7}$$

By expression (4.1), there exist $\xi_i \in \partial^c f(u, y_i)$, $\varsigma_i \in \partial^c(-g(u, y_i))$ for all $i = 1, 2, \ldots, s$, and $\rho_j \in \partial^c h_j(x)$ for all $j \in J$, such that

$$\langle\ a_6\ \rangle \equiv \sum_{i=1}^{s} t_i \{g(x, y_i)\xi_i + f(x, y_i)\varsigma_i\} + \langle \mu\ ,\ \rho \rangle_p = 0,$$

that is, the vector $\langle\ a_6\ \rangle$ is a zero vector, where $\langle \mu\ ,\ \rho \rangle_p \equiv \sum_{j=1}^{p} \mu_j \rho_j$ with $\rho = (\rho_1, \rho_2, \ldots, \rho_p)$.
Then it yields the next inner product

$$\frac{1}{p} e^{rA_6(u)} \Big\langle\ \langle\ a_6\ \rangle\ ,\ (e^{p\eta(x,u)} - \mathbf{1}) \Big\rangle = 0. \tag{4.8}$$

By assumption, $A_6(\cdot)$ is an Exp (p, r)-invex function w.r.t. η at $u \in pr_X\Gamma_1$. By Definiton 2.1 and relation (4.7), it reduces to

$$\frac{1}{p} e^{rA_6(u)} \Big\langle \langle\ a_6\ \rangle\ ,\ (e^{p\eta(x,u)} - \mathbf{1}) \Big\rangle < 0.$$

This contradicts inequality (4.8). Hence the proof is complete. □

Theorem 4.2 (Strong Duality). *Let $\overline{x}$ be the efficient solution of problem (P) satisfying the constraint qualification at $\overline{x}$. Then there exists $(\mu^*, s^*, t^*, y^*) \in K_1(\overline{x})$, such that $(\overline{x}; \mu^*, s^*, t^*, y^*)$ is a feasible point for (MWD). If the hypotheses of Theorem 4.1 are fulfilled, then $(\overline{x}; \mu^*, s^*, t^*, y^*)$ is an efficient solution to problem (MWD), and the two problems (P) and (MWD) have the same optimal values.*

Proof. By assumption, $\overline{x}$ is an efficient point of (P) and the constraint qualification holds at $\overline{x}$. Then, by conditions (3.5) $\sim$ (3.7), we conclude that $(\overline{x}; \mu^*, s^*, t^*, y^*)$ is a feasible for (MWD). Since

$$\frac{f(\overline{x}, y_i^*)}{g(\overline{x}, y_i^*)} = \max_{y \in Y} \frac{f(\overline{x}, y)}{g(\overline{x}, y)},$$

then, using the weak duality theorem (Theorem 4.1), we conclude that $(\overline{x}; \mu^*, s^*, t^*, y^*)$ is an efficient solution of problem (MWD). Consequently, the two problems (P) and (MWD) have the same optimal values. □

Theorem 4.3 (Strict Converse Duality). *Let $\overline{x}$ and $(u^*; \mu^*, s^*, t^*, y^*)$ be the efficient solutions to (P) and (MWD), respectively, and the constraint qualification be satisfied at $\overline{x}$. Let a function $A_7 : X \to \mathrm{R}$ be*

$$A_7(\cdot) = \sum_{i=1}^{s^*} t_i^* \{g(u^*, y_i^*) f(\cdot, y_i^*) - f(u^*, y_i^*) g(\cdot, y_i^*)\} + \langle \mu^*\ ,\ h(\cdot) \rangle_p.$$

Then $A_7(u^*) = 0$. Assume that $A_7(\cdot)$ is a strictly Exp (p, r)-invex function w.r.t. η at $u^* \in pr_X\Gamma_1$ for all optimal vectors $\overline{x}$ for (P), and $(u^*; \mu^*, s^*, t^*, y^*)$ for (MWD), respectively. Then, $\overline{x} = u^*$, and the optimal values of problems (P) and (MWD) are equal.

Proof. Suppose on the contrary that $\overline{x} \neq u^*$. Then by the expression (4.1), there exist $\xi_i \in \partial^c f(u^*, y_i^*)$, $\varsigma_i \in \partial^c(-g(u^*, y_i^*))$ for all $i = 1, 2, \ldots, s^*$, and $\rho_j \in \partial^c h_j(u^*)$ for all $j \in J$, such that

$$\langle\ a_7\ \rangle \equiv \sum_{i=1}^{s^*} t_i^* \left\{g(u^*, y_i^*)\xi_i + f(u^*, y_i^*)\varsigma_i\right\} + \langle \mu^*\ ,\ \rho\rangle_p = 0,$$

that is, the vector $\langle\ a_7\ \rangle$ is a zero vector, where $\langle \mu^*\ ,\ \rho\rangle_p \equiv \sum_{j\in J} \mu_j^* \rho_j$, and $\rho = (\rho_1, \rho_2, \ldots, \rho_p)$. This implies that

$$\frac{1}{p} e^{rA_7(u^*)} \Big\langle\ \langle\ a_7\ \rangle\ ,\ (e^{p\eta(\overline{x}, u^*)} - \mathbf{1})\Big\rangle = 0. \tag{4.9}$$

By assumption, A_7 is a strictly Exp (p, r)-invex function w.r.t. η at $u^* \in pr_X\Gamma_1$. Then, by the definition of strictly Exp (p, r)-invexity and equality (4.9), it follows that

$$\frac{1}{r} e^{rA_7(\overline{x})} - \frac{1}{r} e^{rA_7(u^*)} > 0, \tag{4.10}$$

and so

$$A_7(\overline{x}) > A_7(u^*) \quad \text{for} \quad r \neq 0. \tag{4.11}$$

Since $\overline{x}$ is an optimal solution of (P), we have

$$\langle \mu^*\ ,\ h(\overline{x})\rangle_p \leq 0. \tag{4.12}$$

Following Theorem 4.2, we see that

$$\max_{y\in Y} \frac{f(\overline{x}, y)}{g(\overline{x}, y)} = \frac{f(u^*, y_i^*)}{g(u^*, y_i^*)},$$

and so

$$\frac{f(\overline{x}, y)}{g(\overline{x}, y)} \leq \frac{f(u^*, y_i^*)}{g(u^*, y_i^*)} \quad \text{for all} \quad y \in Y.$$

This implies that

$$f(\overline{x}, y)g(u^*, y_i^*) - g(\overline{x}, y)f(u^*, y_i^*) \leq 0 \quad \text{for all} \quad y \in Y.$$

As $t^* \in I$ and $y_i^* \in Y(u^*)$, $i = 1, 2, \ldots, s^*$, the above expression becomes

$$\sum_{i=1}^{s^*} t_i^* (f(\overline{x}, y_i^*)g(u^*, y_i^*) - g(\overline{x}, y_i^*)f(u^*, y_i^*)) \leq 0.$$

The above inequality together with the inequality (4.12), it would yield

$$A_7(\overline{x}) = \sum_{i=1}^{s^*} t_i^* (f(\overline{x}, y_i^*)g(u^*, y_i^*) - g(\overline{x}, y_i^*)f(u^*, y_i^*)) + \langle \mu^*\ ,\ h(\overline{x})\rangle_p \leq 0 = A_7(u^*).$$

This contradicts the inequality (4.11). Hence the proof is complete. □

5. Wolfe Type duality model

The Wolfe type duality in fractional programming problem can be considered by the objective of fractional functional of (P) by adding the constraint scalarization with a multiplier μ into the numerator of the fractional functional in (P), precisely the Wolfe type dual is stated as follows

$$(WD) \qquad \max_{u\in X} \quad \max_{(\mu,s,t,y)\in K_2(u)} \quad \frac{\sum_{i=1}^{s} t_i f(u,y_i) + \langle \mu \ , \ h(u)\rangle_p}{\sum_{i=1}^{s} t_i g(u,y_i)}$$

subject to

$$(5.1) \qquad \begin{aligned} 0 \in \sum_{i=1}^{s} t_i g(u,y_i) &\left[\sum_{i=1}^{s} t_i \partial^c f(u,y_i) + \langle \mu \ , \ \partial^c h(u)\rangle_p \right] \\ &+ \left[\sum_{i=1}^{s} t_i f(u,y_i) + \langle \mu \ , \ h(u)\rangle_p \right] \sum_{i=1}^{s} t_i \partial^c(-g(u,y_i)), \end{aligned}$$

and

$$(5.2) \qquad \mu \in \mathrm{R}_+^p, \ t \in I, \ y_i \in Y(u) \quad \text{for} \quad i = 1,2,\ldots,s.$$

Here $u \in X$ is a (P)-feasible solution, and denote the set

$$K_2(u) = \left\{ (\mu,s,t,y) \in \mathrm{R}_+^p \times \mathrm{N} \times \mathrm{R}_+^s \times \mathrm{R}^{ms} \ \middle| \ \begin{array}{l} t \in I \ , \ y = (y_1,y_2,\ldots,y_s) \ , \\ y_i \in Y(u) \ , \ i = 1,2,\ldots,s \end{array} \right\},$$

as elements in $K_2(u)$ satisfying the expressions (3.6) and (3.8) which is equivalent to $K_1(u)$ for the necessary optimality conditions of (P) in Theorem 3.3 taken as the constraint given in the problem (MWD).

In order to show the problem (WD) being surely a dual problem w.r.t. the primal problem (P), we denote Γ_2 as the constraint set of (WD). Actually Γ_2 is also defined by projective-like as the feasible solutions of problem (P)

$$pr_X\Gamma_2 = \{u \in X \subseteq \mathrm{R}^n | \ (u;\mu,s,t,y) \in \Gamma_2\}.$$

To the aim we proceed to show the following theorems related to problems (P) and (WD):

Theorem 5.1 (Weak Duality). *Let x and $(u;\mu,s,t,y)$ be (P)-feasible and (WD)-feasible, respectively. Denote a function $A_8 : X \to \mathrm{R}$ by*

$$A_8(\cdot) = \sum_{i=1}^{s} t_i g(u,y_i)[f(\cdot,y_i) + \langle \mu, h(\cdot)\rangle_p] - \sum_{i=1}^{s} t_i g(\cdot,y_i)[f(u,y_i) + \langle \mu, h(u)\rangle_p]$$

with $A_8(u) = 0$. Suppose that $A_8(\cdot)$ is an Exp (p,r)-invex function w.r.t. η at $u \in pr_X\Gamma_2$. Then

$$\max_{y\in Y} \frac{f(x,y)}{g(x,y)} \geq \frac{\sum_{i=1}^{s} t_i f(u,y_i) + \langle \mu \ , \ h(u)\rangle_p}{\sum_{i=1}^{s} t_i g(u,y_i)}.$$

Proof. Suppose that $\max_{y\in Y}\dfrac{f(x,y)}{g(x,y)} \geq \dfrac{\sum_{i=1}^{s} t_i f(u,y_i) + \langle \mu \ , \ h(u)\rangle_p}{\sum_{i=1}^{s} t_i g(u,y_i)}$ were not true.

Then there is a feasible solution $x \in X$ such that

$$\max_{y\in Y}\frac{f(x,y)}{g(x,y)} < \frac{\sum_{i=1}^{s} t_i f(u,y_i) + \langle \mu \ , \ h(u)\rangle_p}{\sum_{i=1}^{s} t_i g(u,y_i)}$$

for any $(u;\mu,s,t,y) \in \Gamma_2$.
This implies that

$$\frac{f(x,y)}{g(x,y)} < \frac{\sum_{i=1}^{s} t_i f(u,y_i) + \langle \mu \ , \ h(u)\rangle_p}{\sum_{i=1}^{s} t_i g(u,y_i)} < 0 \quad \text{for all} \quad y \in Y,$$

or equivalently,

$$f(x,y)\sum_{i=1}^{s} t_i g(u,y_i) - g(x,y)\sum_{i=1}^{s} t_i f(u,y_i) + \langle \mu \ , \ h(u)\rangle_p < 0 \quad \text{for all} \quad y \in Y.$$

Furthermore, multiplying the above expression respectively by $t_i \geq 0$, $i = 1,2,\ldots,s$ and summing up, it yields

$$\sum_{i=1}^{s} t_i f(x,y_i)\sum_{i=1}^{s} t_i g(u,y_i) - \sum_{i=1}^{s} t_i g(x,y_i)[\sum_{i=1}^{s} t_i f(u,y_i) + \langle \mu \ , \ h(u)\rangle_p] < 0.$$

It reduces to

$$(5.3) \quad \sum_{i=1}^{s} t_i g(u,y_i)\Big[\sum_{i=1}^{s} t_i f(x,y_i) + \langle \mu \ , \ h(x)\rangle_p\Big] - \sum_{i=1}^{s} t_i g(x,y_i)\Big[\sum_{i=1}^{s} t_i f(u,y_i) + \langle \mu \ , \ h(u)\rangle_p\Big] = A_8(x) < \sum_{i=1}^{s} t_i g(u,y_i)\langle \mu \ , \ h(x)\rangle_p.$$

Since the relations (5.2), $g_i(u,y_i) > 0,\ i = 1,2,\ldots,s$, and $h_j(x) \leq 0$, we get

$$\sum_{i=1}^{s} t_i g(u,y_i)\langle \mu \ , \ h(x)\rangle_p \leq 0.$$

Therefore, from (5.3), we obtain

$$(5.4) \qquad A_8(x) < 0 = A_8(u).$$

Let x and $(u;\mu,s,t,y)$ be (P)-feasible and (WD)-feasible, respectively. According to the expression (5.1), there exist $\xi_i \in \partial^c f(u,y_i)$, $\zeta_i \in \partial^c(-g)(u,y_i)$, $i = 1,2,\ldots,s$, and $\rho_j \in \partial^c h_j(u)$, $j \in J$ such that the vector

$$\langle \ a_8 \ \rangle \equiv \sum_{i=1}^{s} t_i g(u,y_i)\Big[\langle t \ , \ \xi\rangle_s + \langle \mu \ , \ \rho\rangle_p\Big] + \left[\sum_{i=1}^{s} t_i f(u,y_i) + \langle \mu \ , \ h(u)\rangle_p\right]\langle t \ , \ \zeta\rangle_s = 0,$$

that is, the vector $\langle\ a_8\ \rangle$ is a zero vector, where

$$\langle t\ ,\ \xi\rangle_s \equiv \sum_{i=1}^{s} t_i\xi_i,\ \langle t\ ,\ \zeta\rangle_s \equiv \sum_{i=1}^{s} t_i\zeta_i,\ \langle \mu\ ,\ \rho\rangle_p \equiv \sum_{j=1}^{p} \mu_j\rho_j,$$

$$\xi = (\xi_1, \xi_2, \dots, \xi_s),\ \zeta = (\zeta_1, \zeta_2, \dots, \zeta_s),\ \text{and } \rho = (\rho_1, \rho_2, \dots, \rho_p).$$

This implies that

$$\frac{1}{p}\langle\langle\ a_8\ \rangle\ ,\ (e^{p\eta(x,u)} - \mathbf{1})\rangle = 0. \tag{5.5}$$

If A_8 is an Exp (p,r)-invexity w.r.t η at u in $pr_X\Gamma_2$, we get

$$\frac{1}{r}e^{rA_8(x)} - \frac{1}{r}e^{rA_8(u)} \geq \frac{1}{p}e^{rA_8(u)}\langle\langle\ a_8\ \rangle\ ,\ (e^{p\eta(x,u)} - \mathbf{1})\rangle.$$

According to the above relation and (5.5), we have $\frac{1}{r}e^{rA_8(x)} - \frac{1}{r}e^{rA_8(u)} \geq 0$.
By the exponential function, we obtain

$$A_8(x) \geq A_8(u) = 0 \qquad \text{for} \qquad r \neq 0,$$

which contradicts the inequality (5.4). Hence the inequality $\max_{y\in Y} \frac{f(x,y)}{g(x,y)} \geq \frac{\sum_{i=1}^{s} t_i f(u,y_i) + \langle \mu\ ,\ h(u)\rangle_p}{\sum_{i=1}^{s} t_i g(u,y_i)}$ is true, and the theorem is proved. □

Theorem 5.2 (Strong Duality). *Let $\overline{x}$ be the efficient solution of problem (P) satisfying the constraint qualification at $\overline{x}$. Then there exists $(\mu^*, s^*, t^*, y^*) \in K_2(\overline{x})$, such that $(\overline{x}; \mu^*, s^*, t^*, y^*)$ is feasible for (WD). If the hypotheses of Theorem 5.1 are fulfilled, then $(\overline{x}; \mu^*, s^*, t^*, y^*)$ is an efficient solution to problem (WD) and the two problems (P) and (WD) have the same optimal values.*

Proof. Let $\overline{x}$ be an efficient point of (P) and satisfy the constraint qualification at $\overline{x}$. By the necessary conditions (3.6) $\sim$ (3.8), we conclude that $(\overline{x}; \mu^*, s^*, t^*, y^*)$ is a feasible solution of (WD). It follows that

$$\frac{\sum_{i=1}^{s^*} t_i^* f(\overline{x}, y_i^*) + \langle \mu^*\ ,\ h(\overline{x})\rangle_p}{\sum_{i=1}^{s^*} t_i^* g(\overline{x}, y_i^*)} = \max_{y\in Y} \frac{f(\overline{x}, y)}{g(\overline{x}, y)}.$$

Employing the weak duality theorem (Theorem 5.1), it yields that $(\overline{x}; \mu^*, s^*, t^*, y^*)$ is the efficient point for problem (WD). Therefore, the two problems (P) and (WD) have the same optimal values. □

Theorem 5.3 (Strict Converse Duality). *Let $\overline{x}$ and $(u^*; \mu^*, s^*, t^*, y^*)$ be the efficient solutions to (P) and (WD), respectively, and satisfying the constraint qualification at $\overline{x}$. Denote a function $A_9 : X \to \mathrm{R}$ by*

$$A_9(\cdot) = \sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*)[f(\cdot, y_i^*) + \langle \mu^*, h(\cdot)\rangle_p] - \sum_{i=1}^{s^*} t_i^* g(\cdot, y_i^*)[f(u^*, y_i^*) + \langle \mu, h(u^*)\rangle_p]$$

with $A_9(u^*) = 0$. Assume that $A_9(\cdot)$ is a strictly Exp (p, r)-invex function w.r.t. η at $u^* \in pr_X\Gamma_2$ for all optimal vectors $\overline{x}$ for (P) and $(u^*; s^*, t^*, y^*)$ for (WD), respectively. Then $\overline{x} = u^*$, and the efficient values of (P) and (WD) are equal.

Proof. We want to prove $\overline{x} = u^*$.
Suppose on contrary that $x^* \neq u^*$. It will deduce to a contradiction. According to relation (5.1), there exist $\xi_i \in \partial^c f(u^*, y_i^*)$, $\zeta_i \in \partial^c g(u^*, y_i^*)$ for all $i = 1, 2, \ldots, s^*$, and $\rho_j \in \partial^c h_j(u^*)$ for all $j \in J$, such that the vector

$$\begin{aligned}\langle\ a_9\ \rangle \equiv\ & \sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*)\,[\langle t^*\ ,\ \xi\rangle_{s^*} + \langle \mu^*\ ,\ \rho\rangle_p] \\ & + \left[\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \langle \mu^*\ ,\ h(u^*)\rangle_p\right] \langle t^*\ ,\ \zeta\rangle_{s^*} = 0,\end{aligned} \tag{5.6}$$

that is, $\langle\ a_9\ \rangle$ is a zero vector, where

$$\langle t^*\ ,\ \xi\rangle_{s^*} \equiv \sum_{i=1}^{s^*} t_i^* \xi_i,\ \langle t^*\ ,\ \zeta\rangle_{s^*} \equiv \sum_{i=1}^{s^*} t_i^* \zeta_i, \langle \mu^*\ ,\ \rho\rangle_p \equiv \sum_{j=1}^{p} \mu_j^* \rho_j,$$
$$\xi = (\xi_1, \xi_2, \ldots, \xi_{s^*}),\ \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{s^*}),\ \text{and } \rho = (\rho_1, \rho_2, \ldots, \rho_p).$$

This implies that

$$\frac{1}{p}\langle\langle\ a_9\ \rangle\ ,\ (e^{p\eta(\overline{x}, u^*)} - \mathbf{1})\rangle = 0. \tag{5.7}$$

From the strictly Exp (p, r)-invexity w.r.t. η at u^* of A_9 and equality (5.7), we get

$$A_9(\overline{x}) > A_9(u^*). \tag{5.8}$$

From Theorem 5.2, we see that

$$\max_{y\in Y} \frac{f(\overline{x}, y)}{g(\overline{x}, y)} = \frac{\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \langle \mu^*\ ,\ h(u^*)\rangle_p}{\sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*)},$$

and so

$$\frac{f(\overline{x}, y)}{g(\overline{x}, y)} \leq \frac{\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \langle \mu^*\ ,\ h(u^*)\rangle_p}{\sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*)} \quad \text{for all} \quad y \in Y.$$

This implies that

$$f(\overline{x}, y) \sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*) - g(\overline{x}, y) \left[\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \langle \mu^*\ ,\ h(u^*)\rangle_p\right] \leq 0 \quad \text{for all} \quad y \in Y.$$

By $t^* \in I$ and $y_i^* \in Y(u^*)$, $i = 1, 2, \ldots, s^*$, we obtain
(5.9)

$$\sum_{i=1}^{s^*} t_i^* f(\overline{x}, y_i^*) \sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*) - \sum_{i=1}^{s^*} t_i^* g(\overline{x}, y_i^*) \left[\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \langle \mu^*\ ,\ h(u^*)\rangle_p\right] \leq 0.$$

This implies that

$$\begin{aligned}\text{(5.10)}\quad & \sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*) \left[\sum_{i=1}^{s^*} t_i^* f(\overline{x}, y_i^*) + \langle \mu^*\ ,\ h(\overline{x})\rangle_p\right] \\ & - \sum_{i=1}^{s^*} t_i^* g(\overline{x}, y_i^*) \left[\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \langle \mu^*\ ,\ h(u^*)\rangle_p\right]\end{aligned}$$

$$= A_9(\overline{x}) \leq \sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*)\langle \mu^* \ , \ h(\overline{x})\rangle_p.$$

By the expression (5.2), $y_i^* \in Y(u^*)$ and $g(u^*, y_i^*) > 0$, $i = 1, 2, \ldots, s^*$, but $h(\overline{x}) \in -\mathrm{R}_+^p$, it yields

$$\sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*)\langle \mu^* \ , \ h(\overline{x})\rangle_p \leq 0.$$

Therefore, from (5.10), we get

$$A_9(\overline{x}) \leq 0 = A_9(u^*). \tag{5.11}$$

Consequently, the expression (5.9) contradicts the inequality (5.8). Hence u^* is an optimal solution to (P), and $A_9(\overline{x}) = A_9(u^*)$ deduces $u^* = \overline{x}$. Therefore

$$\max_{y \in Y} \frac{f(u^*, y)}{g(u^*, y)} = \frac{\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \langle \mu^* \ , \ h(u^*)\rangle_p}{\sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*)}.$$

This proves the optimal values of the dual problem (WD) and the primal problem (P) are equal. □

References

[1] I. Ahmad and Z. Husain, *Duality in nondifferentiable minimax fractional programming with generalized convexity*, Appl. Math. Comp. **176** (2006), 545–551.

[2] T. Antczak, *(p, r)-invex sets and functions*, J. Math. Anal. Appl. **263** (2001), 355–379.

[3] T. Antczak, *Minimax programming under (p, r)-invexity*, Eur. J. Oper. Res. **158** (2004), 1–19.

[4] C. R. Bector, S. Chandra and M. K. Bector, *Generalized fractional programming duality: A parametric approch*, J. Math. Anal. Appl. **60** (1989), 243–260.

[5] S. Chandra and V. Kumar, *Duality in fractional minimax programming*, J. Austr. Math. Soc. Ser. A. **58** (1995), 376–386.

[6] F. H. Clarke, *Optimization and Non-smooth Analysis*, Wiley-Interscience, New York, 1983.

[7] S. C. Ho and H. C. Lai, *Optimality and duality for nonsmooth minimax fractional programming problem with exponential (p,r)-invexity*, J. Nonlinear and Convex Anal. **13** (2012), 433–447.

[8] H. C. Lai and H. M. Chen, *Duality on a nondifferentiable minimax fractional programming*, J. Glob. Optim. **54** (2012), 295–306.

[9] H. C. Lai and S. C. Ho, *Optimality and duality for nonsmooth multiobjective fractional programmings involving exponential V-r-invexity*, Nonlinear Anal. **75** (2012), 3157–3166.

[10] H. C. Lai and T. Y. Huang, *Minimax fractional programming for n-set functions and mixed-type duality under generalized invexity*, J. Optim. Theory Appl. **139** (2008), 295–313.

[11] H. C. Lai and T. Y. Huang, *Nondifferentiable minimax fractional programming in complex with parametric duality*, J. Glob. Optim. **53** (2012), 243–254.

[12] H. C. Lai, J. C. Lee and S. C. Ho, *Parametric duality on minimax programming involving generalized convexity in complex space*, J. Math. Anal. Appl. **323** (2006), 1104–1115.

[13] H. C. Lai and J. C. Liu, *A new characterization on optimality and duality for nondifferentiable minimax fractional programming problems*, J. Nonlinear and Convex Anal. **12** (2011), 69–80.

[14] J. C. Lee and H. C. Lai, *Parameter-free dual moddels for fractional programming with generalized invexity*, Annals Oper. Res. **133** (2005), 47–61.

[15] B. Mond, S. Chandra and I. Husain, *Duality for variational problems with invexity*, J. Math. Anal. Appl. **134** (1988), 322–328.

[16] B. Mond and T. Weir, *Generalized concavity and duality*, in Generalized Concavity in Optimization and Economics, S. Schaible, W. T. Ziemba (eds.) , Academic Press, New York, 1981, pp. 263–279.

[17] W. E. Schmitendorf, *Necessary conditions and sufficient conditions for static minimax problems*, J. Math. Anal. and Appl. **57** (1977), 683–693.
[18] G. J. Zalmai, *Optimality conditions and duality models for generalized fractional programming problems containing locally subdifferentiable and ρ-convex functions*, Optimization **32** (1995), 95–124.

Manuscript received December 3, 2012
revised February 28, 2013

Shun-Chin Ho
Chung-Jen Junior College of Nursing, Health Sciences and Management, Chia-Yi, Taiwan
E-mail address: scho31@gmail.com

Hang-Chin Lai
Department of Mathematics, National Tsing Hua University, Hsing-Chu, Taiwan
E-mail address: laihc@mx.nthu.edu.tw