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EXISTENCE AND CONVERGENCE OF FIXED POINTS FOR A STRICT PSEUDO-CONTRACTION VIA AN ITERATIVE SHRINKING PROJECTION TECHNIQUE

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Abstract. In this paper, the existence and convergence results of a strict pseudo-contraction in the terminology of Browder and Petryshyn [5] are obtained by using an iterative shrinking projection technique involving some necessary and sufficient conditions. The method permits us to obtain a strong convergence iteration for finding some fixed points of a strict pseudo-contraction in the framework of real Hilbert spaces. Moreover, the main results in the paper extend and improve some results announced by Aoyama, Kohsaka and Takahashi [K. Aoyama, F. Kohsaka, and W. Takahashi, Shrinking projection methods for firmly nonexpansive mappings, Nonlinear Anal. 71 (2009) e1626-e1632.]. In addition, we also provide certain applications of the main theorems to confirm the existence of the zeros of an inverse strongly monotone operator along with its convergent results.

1. Introduction

There are several attempts to establish an iteration method to find a fixed point of some well-known nonlinear mappings, for instant, nonexpansive mapping. We note that Mann's iterations [13] have only weak convergence even in a Hilbert space (see e.g., [7]). Nakajo and Takahashi [18] modified the Mann iteration method so that strong convergence is guaranteed, later well known as a hybrid projection method. Since then, the hybrid method has received rapid developments. For the details, the readers are referred to papers [3, 4, 8–12, 14–16, 19–21, 23, 25–30] and the references cited therein. In 2008, Takahashi, Takeuchi and Kobota [25] introduced an alternative projection method, subsequently well known as the shrinking projection method, and they showed several strong convergence theorems for a family of nonexpansive mappings; see also [2]. In 2009, Aoyama, Kohsaka and Takahashi [1] applied the hybrid shrinking projection method along with creating some necessary and sufficient conditions to confirm the existence of a fixed point of firmly nonexpansive mapping.

Let H be a real Hilbert space, a mapping T with domain $D(T)$ and range $R(T)$ in H is called firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \quad \forall x, y \in D(T). \tag{1.1}$$

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It is not hard to verify that (1.1) is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in D(T)$$

(See Remark 2.1 in Section 2).

T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in D(T).$$

Throughout this paper, I stands for an identity mapping. The mapping T is said to be a strict pseudo-contraction in the terminology of Browder and Petryshyn [5] if for all $x, y \in D(T)$ there exists $\gamma > 0$ such that

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \gamma\|x - y - (Tx - Ty)\|^2. \tag{1.2}$$

It is obvious that (1.2) is equivalent to

$$\langle x - y, (I - T)x - (I - T)y\rangle \geq \gamma\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in D(T). \tag{1.3}$$

If we set $A := (I - T)$ that satisfies (1.3), then A is said to be *inverse strongly monotone*. For such a case, A may be called γ-inverse strongly monotone (See in Section 4.). By setting $k := 1 - 2\gamma$, it is not hard to verify that (1.3) (and hence (1.2)) is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)\, x - (I - T)\, y\|^2, \quad \forall x, y \in D(T)$$

(See Remark 2.2 in Section 2).

In this case, it is easy to see that the constant $k \in (-\infty, 1)$. Moreover,

$$\begin{aligned}
\boxed{T \text{ is firmly nonexpansive}} \quad &\Leftrightarrow \quad \boxed{\gamma = 1} \\
&\Leftrightarrow \quad \boxed{k = -1} \\
&\Leftrightarrow \quad \boxed{T \text{ is } (-1)\text{−strict pseudo-contraction}}. \\
\boxed{T \text{ is nonexpansive}} \quad &\Leftrightarrow \quad \boxed{\gamma = \frac{1}{2}} \\
&\Leftrightarrow \quad \boxed{k = 0} \\
&\Leftrightarrow \quad \boxed{T \text{ is } 0\text{−strict pseudo-contraction}}.
\end{aligned}$$

We use $F(T)$ to denote the set of fixed points of T (i.e., $F(T) = \{x \in D(T) : Tx = x\}$). T is said to be a k-quasi-strict pseudo-contraction in the terminology of Browder and Petryshyn [5] if the set of fixed points $F(T)$ is nonempty and there exists a constant $k < 1$ such that

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \quad \forall x \in D(T) \text{ and } p \in F(T).$$

The class of strict pseudo-contractions contains the classes of nonexpansive mappings and firmly nonexpansive mappings. It is clear that

$$\boxed{\text{firmly nonexpansive}} \Rightarrow \boxed{\text{nonexpansive}} \Rightarrow \boxed{\text{strict pseudo-contraction}}.$$

However, the following examples show that the converse is not true.

Example 1.1. Let H be a real Hilbert space and $\alpha \in (1, \infty)$. Define $T_\alpha : H \to H$ by

$$T_\alpha x = -\alpha x, \quad \forall x \in H.$$

Then, T_α is a strict pseudo-contraction but not a nonexpansive mapping.

Example 1.2. Take $H \neq \{0\}$ and let $T = -I$, it is not hard to verify that T is nonexpansive but not firmly nonexpansive.

From a practical point of view, strict pseudo-contractions have more powerful applications than nonexpansive mappings do in solving inverse problems (see Scherzer [22]). Therefore, it is important to develop theory of iterative methods for strict pseudo-contractions. Within the past several decades, many authors have been devoted to the studies on the existence and convergence of fixed points for strict pseudo-contractions. In 1967, Browder and Petryshyn [5] introduced a convex combination method to study strict pseudo-contractions in Hilbert spaces. On the other hand, Marino and Xu [14] and Zhou [31] developed some iterative scheme for finding a fixed point of a strict pseudo-contraction mapping.

In 2009, Aoyama, Kohsaka and Takahashi [1] provided the useful and interesting lemma to confirm that the sequence generated by the shrinking projection method is well defined even if the firmly nonexpansive mapping T has no fixed points:

Lemma 1.3 (Aoyama, Kohsaka and Takahashi [1, Lemma 4.2])**.** *Let H be a Hilbert space, C a nonempty closed convex subset of H, $T : C \to C$ a firmly nonexpansive mapping and $x_0 \in H$. Let $\{x_n\}$ be a sequence in C and $\{C_n\}$ a sequence of closed convex subsets of H generated by $C_1 = C$ and*

$$\begin{cases} x_n = P_{C_n}(x_0), \\ C_{n+1} = \{z \in C_n : \langle Tx_n - z, x_n - Tx_n \rangle \geq 0\}, \end{cases}$$

for all $n \in \mathbb{N}$. Then C_n is nonempty for every $n \in \mathbb{N}$, and consequently, $\{x_n\}$ is well defined.

By using the lemma mentioned above, they proved the following theorem:

Theorem 1.4 (Aoyama, Kohsaka and Takahashi [1, Theorem 4.3])**.** *Let H be a Hilbert space, C a nonempty closed convex subset of H, $T : C \to C$ a firmly nonexpansive mapping and $x_0 \in H$. Let $\{x_n\}$ be a sequence in C and $\{C_n\}$ a sequence of closed convex subsets of H generated by $C_1 = C$ and*

$$\begin{cases} x_n = P_{C_n}(x_0), \\ C_{n+1} = \{z \in C_n : \langle Tx_n - z, x_n - Tx_n \rangle \geq 0\}, \end{cases}$$

for all $n \in \mathbb{N}$. Then the following are equivalent:

(1) $\bigcap_{n=1}^{\infty} C_n$ *is nonempty;*
(2) $\{x_n\}$ *is bounded;*
(3) $F(T)$ *is nonempty.*

Motivated and inspired by the results mentioned above, in this paper, we provide some existence theorems of a strict pseudo-contraction by the way of the shrinking projection method involving some necessary and sufficient conditions. Then we prove a strong convergence theorem and apply its applications to confirm the existence of a firmly nonexpansive mapping, a nonexpansive mapping and the existence of the zeros of an inverse strongly monotone operator along with its convergent results, respectively.

Throughout the paper, we will use the notation:

(1) $\to$ for strong convergence and $\rightharpoonup$ for weak convergence,
(2) $\omega_w(x_n) = \{x : \exists x_{n_i} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

2. Preliminaries

In this section, some definitions and remarks are provided and some relevant lemmas which are useful to prove in the next section are collected. Most of them are known others are not hard to find and understand the proof.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a closed convex subset of H. For every point $x \in H$ there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C x\| \leqslant \|x - y\|, \quad \forall y \in C.$$

The mapping P_C is called the *metric projection* of H onto C. It is well known that P_C is a firmly nonexpansive mapping of H onto C, that is

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H.$$

Furthermore, for any $x \in H$ and $z \in C$,

$$z = P_C x \Leftrightarrow \langle x - z, z - y \rangle \geq 0, \quad \forall y \in C.$$

Moreover, $P_C x$ is characterized by the following:

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall y \in C.$$

It is obvious that the following identities hold:

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle, \quad \forall x, y \in H, \tag{2.1}$$

and

$$\|x - y\|^2 = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2, \quad \forall x, y \in H. \tag{2.2}$$

Remark 2.1. Let H be a real Hilbert space and T be a mapping with domain $D(T)$ and range $R(T)$. Then the following are equivalent:

(a) T is firmly nonexpansive (i.e., $\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle$, $\forall x, y \in D(T)$);
(b) $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$, $\forall x, y \in D(T)$.

Proof. For each $x, y \in D(T)$ we notice that

$$\begin{aligned} &\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle \\ &\Leftrightarrow \|Tx - Ty\|^2 \leq \|x - y\|^2 - \Big(\|x - y\|^2 - 2\langle x - y, Tx - Ty \rangle + \|Tx - Ty\|^2\Big) \\ &\Leftrightarrow \|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2. \end{aligned}$$

The proof is complete. $\square$

Remark 2.2. Let H be a real Hilbert space and T be a mapping with domain $D(T)$ and range $R(T)$. Then the following are equivalent:

(1) T is a strict pseudo-contraction in the terminology of Browder and Petryshyn [5] (i.e., there exists $\gamma > 0$ such that $\langle Tx - Ty, x - y\rangle \leq \|x-y\|^2 - \gamma\|x - y - (Tx - Ty)\|^2, \quad \forall x, y \in D(T)$);
(2) $\langle x - y, (I-T)x - (I-T)y\rangle \geq \gamma\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in D(T)$;
(3) $\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in D(T)$ where $k := 1 - 2\gamma$.

Proof. It is easy to see that (i) and (ii) are equivalent. On the other hand, for all $x, y \in D(T)$, it follows from the identity (2.2) that

$$\begin{aligned} \|Tx - Ty\|^2 &= \|x - y - [(I-T)\,x - (I-T)\,y]\|^2 \\ &= \|x-y\|^2 + \|(I-T)\,x - (I-T)\,y\|^2 - 2\,\langle x - y, (I-T)\,x - (I-T)\,y\rangle . \end{aligned} \tag{2.3}$$

It follows from (ii) and joining with (2.3), we get that

$$\begin{aligned} &\langle x - y, (I-T)x - (I-T)y\rangle \geq \gamma\|(I-T)x - (I-T)y\|^2 \\ &\Leftrightarrow -2\,\langle x - y, (I-T)x - (I-T)y\rangle \leq -2\gamma\|(I-T)x - (I-T)y\|^2 \\ &\Leftrightarrow \|Tx - Ty\|^2 \leq \|x-y\|^2 + (1-2\gamma)\,\|(I-T)x - (I-T)y\|^2, \end{aligned}$$

for all $x, y \in D(T)$. This shows that (ii) and (iii) are equivalent. The proof is complete. □

Proposition 2.3 ([14, Proposition 2.1])**.** *Assume C is a closed convex subset of a Hilbert space H let $T : C \to C$ be a self-mapping of C.*

(1) *If T is a k-strict pseudo-contraction, then T satisfies the Lipschitz condition*
$$\|Tx - Ty\| \leqslant \frac{1+k}{1-k}\|x-y\| \quad \forall x, y \in C.$$
(2) *If T is a k-strict pseudo-contraction, then $I - T$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and $(I-T)x_n \to 0$, then $(I-T)z = 0$.*
(3) *If T is a k-quasi-strict pseudo-contraction, then the set of fixed point $F(T)$ is closed convex subset of C.*

Next, we will provide some extensions of Proposition 2.3 in the sense of a k-strict pseudo-contraction in the terminology of Browder and Petryshyn [5] (i.e., there exists $k \in (-\infty, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C).$$

Proposition 2.4. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a k-strict pseudo-contraction in the terminology of Browder and Petryshyn [5] (i.e., $k \in (-\infty, 1)$). Then the following are satisfied:*

(1) *T satisfies the Lipschitz condition with Lipschitz constant $L = \max\left\{\frac{1+k}{1-k}, 1\right\}$. That is*
$$\|Tx - Ty\| \leq \max\left\{\frac{1+k}{1-k}, 1\right\}\|x-y\|, \quad \forall x, y \in C.$$

(2) *$I-T$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and $(I-T)x_n \to 0$, then $(I-T)z = 0$.*
(3) *If T is a k-quasi-strict pseudo-contraction in the terminology of Browder and Petryshyn [5] (i.e., $k \in (-\infty, 1)$), then the set $F(T)$ of fixed points of T is a closed convex subset of C.*

Proof. (i) We will divide the proof into two cases.

Case I. $k \leq 0$.

Notice that $k \leq 0 \Leftrightarrow 2k \leq 0 \Leftrightarrow 1+k \leq 1-k \Leftrightarrow \frac{1+k}{1-k} \leq 1 \Leftrightarrow \max\left\{\frac{1+k}{1-k}, 1\right\} = 1$, and then

$$\begin{aligned}\|Tx-Ty\|^2 &\leq \|x-y\|^2 + k\|(I-T)\,x-(I-T)\,y\|^2 \\ &\leq \|x-y\|^2.\end{aligned}$$

This implies that

$$\|Tx-Ty\| \leq \|x-y\| = \max\left\{\frac{1+k}{1-k}, 1\right\}\|x-y\|, \quad \forall x, y \in C.$$

Case II. $0 \leq k < 1$.

In this case, we have $1-k > 0$ and then

$$k \geq 0 \Leftrightarrow 2k \geq 0 \Leftrightarrow 1+k \geq 1-k \Leftrightarrow \frac{1+k}{1-k} \geq 1 \Leftrightarrow \max\left\{\frac{1+k}{1-k}, 1\right\} = \frac{1+k}{1-k}.$$

On the other hand, we observe that

$$\begin{aligned}&\|Tx-Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)\,x-(I-T)\,y\|^2 \\ &\Leftrightarrow \|Tx-Ty\|^2 \leq \|x-y\|^2 + k\left(\|x-y\|^2 - 2\,\langle x-y, Tx-Ty\rangle + \|Tx-Ty\|^2\right) \\ &\Leftrightarrow (1-k)\,\|Tx-Ty\|^2 \leq (1+k)\,\|x-y\|^2 - 2k\,\langle x-y, Tx-Ty\rangle\,.\end{aligned}$$

This implies by Cauchy-Schwarz inequality that

$$(1-k)\,\|Tx-Ty\|^2 - 2k\,\|x-y\|\,\|Tx-Ty\| - (1+k)\,\|x-y\|^2 \leq 0.$$

Solving this quadratic inequality, we obtain

$$\|Tx-Ty\| \leq \frac{1+k}{1-k}\,\|x-y\| = \max\left\{\frac{1+k}{1-k}, 1\right\}\|x-y\|, \quad \forall x, y \in C.$$

The proofs of (ii) and (iii) are the same as Marino and Xu [14, Proposition 2.1]. □

Lemma 2.5 ([24, Theorem 7.1.8])**.** *Let K be a bounded closed convex subset of a Hilbert space H and $A : K \to H$ a continuous monotone mapping. Then there exists an element $u_0 \in K$ such that $\langle v-u_0, Au_0\rangle \geq 0$ for all $v \in K$.*

3. Main results

In this section, motivated by Aoyama, Kohsaka and Takahashi [1] (see also, Matsushita and Takahashi [17]), we discuss the existence of fixed point of a strict pseudo-contraction in the terminology of Browder and Petryshyn [5] (i.e., $k \in (-\infty, 1)$) by using the shrinking projection technique playing as the tool to guarantee the existence of fixed point of this mapping.

Lemma 3.1. *Let H be a real Hilbert space and T be a k-strict pseudo-contraction in the terminology of Browder and Petryshyn [5] (i.e., $k \in (-\infty, 1)$) with domain $D(T)$ and range $R(T)$. Then for all $x, y \in D(T)$ the following inequalities hold and are equivalent:*

(1) $$\|x - Tx\|^2 + \|y - Ty\|^2 \le \frac{2}{1-k}\langle x - y, (I-T)x\rangle - \frac{2k}{1-k}\langle x - y, y - Ty\rangle - 2\langle Tx - y, y - Ty\rangle;$$

(2) $$\|x - Tx\|^2 + \|y - Ty\|^2 \le \frac{2}{1-k}\langle x - y, (I-T)x\rangle - \frac{2}{1-k}\langle x - y, (I-T)y\rangle + 2\langle x - Tx, y - Ty\rangle;$$

(3) $$\|x - Tx\|^2 + \|y - Ty\|^2 \le \frac{2}{1-k}\langle x - Ty, (I-T)x\rangle - \frac{2}{1-k}\langle Tx - y, (I-T)y\rangle - 2\frac{1+k}{1-k}\langle x - Tx, y - Ty\rangle;$$

(4) $$\|x - Tx\|^2 + \|y - Ty\|^2 \le \langle x - Ty, (I-T)x\rangle - \langle Tx - y, (I-T)y\rangle + \frac{1+k}{2}\|(I-T)x - (I-T)y\|^2.$$

Proof. Firstly, we will show that (2) holds. It follows from the identity (2.2) and the inverse strongly monotone of $(I - T)$(See Remark 2.2) that

$$\begin{aligned}
\|x - Tx\|^2 + \|y - Ty\|^2 &= \|(I-T)\,x - (I-T)\,y\|^2 + 2\langle x - Tx, y - Ty\rangle \\
&\le \frac{2}{1-k}\langle x - y, (I-T)x - (I-T)y\rangle + 2\langle x - Tx, y - Ty\rangle \\
&= \frac{2}{1-k}\langle x - y, (I-T)x\rangle - \frac{2}{1-k}\langle x - y, (I-T)y\rangle \qquad (3.1) \\
&\quad + 2\langle x - Tx, y - Ty\rangle .
\end{aligned}$$

So, we obtain (2). Next, we observe that

$$\begin{aligned}
&-\frac{2}{1-k}\langle x - y, (I-T)y\rangle + 2\langle x - Tx, y - Ty\rangle \\
&\quad = -\frac{2}{1-k}\langle x - y, (I-T)y\rangle + 2\langle x - y, y - Ty\rangle + 2\langle y - Tx, y - Ty\rangle \\
&\quad = \left[2 - \frac{2}{1-k}\right]\langle x - y, y - Ty\rangle - 2\langle Tx - y, y - Ty\rangle \qquad (3.2) \\
&\quad = -\frac{2k}{1-k}\langle x - y, y - Ty\rangle - 2\langle Tx - y, y - Ty\rangle .
\end{aligned}$$

Substituting (3.2) in (3.1), we get (1) and hence (1) and (2) are equivalent. Next, we will show that (4) is true. Let us consider

$$\begin{aligned}
\langle Tx - Ty, (I-T)x - (I-T)y\rangle &= \langle Tx - Ty, (I-T)x\rangle - \langle Tx - Ty, (I-T)y\rangle \\
&= \langle Tx - Ty, (I-T)x\rangle
\end{aligned}$$

$$\begin{aligned} &\quad - \left\langle (Tx - y) + (y - Ty), (I - T)y \right\rangle \\ &= \langle Tx - Ty, (I - T)x \rangle \\ &\quad - \langle Tx - y, (I - T)y \rangle - \|y - Ty\|^2. \end{aligned} \tag{3.3}$$

Adding and subtracting with x for the first term of the last line of (3.3), we have

$$\begin{aligned} \langle Tx - Ty, (I - T)x \rangle &= \langle Tx - x + x - Ty, (I - T)x \rangle \\ &= \langle Tx - x, (I - T)x \rangle + \langle x - Ty, (I - T)x \rangle \\ &= -\|x - Tx\|^2 + \langle x - Ty, (I - T)x \rangle . \end{aligned} \tag{3.4}$$

Replacing (3.4) in (3.3), we obtain

$$\begin{aligned} \langle Tx - Ty, (I - T)x - (I - T)y \rangle = &-\|x - Tx\|^2 + \langle x - Ty, (I - T)x \rangle \\ &- \langle Tx - y, (I - T)y \rangle - \|y - Ty\|^2. \end{aligned} \tag{3.5}$$

On the other hand, we observe that

$$\begin{aligned} &\langle Tx - Ty, (I - T)x - (I - T)y \rangle \\ &= \left\langle x - y - \left[(I - T)x - (I - T)y\right], (I - T)x - (I - T)y \right\rangle \\ &= \langle x - y, (I - T)x - (I - T)y \rangle - \|(I - T)x - (I - T)y\|^2 \\ &\geq \frac{1-k}{2}\|(I - T)x - (I - T)y\|^2 - \|(I - T)x - (I - T)y\|^2 \\ &= \left[\frac{1-k-2}{2}\right]\|(I - T)x - (I - T)y\|^2 \\ &= -\frac{1+k}{2}\|(I - T)x - (I - T)y\|^2. \end{aligned} \tag{3.6}$$

Substituting (3.5) in (3.6), we have that

$$\begin{aligned} -\|x - Tx\|^2 + \langle x - Ty, (I - T)x \rangle - \langle Tx - y, (I - T)y \rangle - \|y - Ty\|^2 \\ \geq -\frac{1+k}{2}\|(I - T)x - (I - T)y\|^2. \end{aligned} \tag{3.7}$$

By simple calculation, (3.7) can be written in the form

$$\begin{aligned} \|x - Tx\|^2 + \|y - Ty\|^2 \leq &\langle x - Ty, (I - T)x \rangle - \langle Tx - y, (I - T)y \rangle \\ &+ \frac{1+k}{2}\|(I - T)x - (I - T)y\|^2. \end{aligned} \tag{3.8}$$

This shows that (4) is true. Next, we will show that (3) is true. Let us look at the last term of (3.8), we get

$$\begin{aligned} &\frac{1+k}{2}\|(I - T)x - (I - T)y\|^2 \\ &\quad = \frac{1+k}{2}\left(\|x - Tx\|^2 - 2\left\langle x - Tx, y - Ty \right\rangle + \|y - Ty\|^2\right) \\ &\quad = \frac{1+k}{2}\|x - Tx\|^2 - (1+k)\left\langle x - Tx, y - Ty \right\rangle + \frac{1+k}{2}\|y - Ty\|^2. \end{aligned} \tag{3.9}$$

Replacing (3.8) with (3.9), we obtain

$$\begin{aligned}\|x-Tx\|^2+\|y-Ty\|^2 \le{}& \langle x-Ty,(I-T)x\rangle-\langle Tx-y,(I-T)y\rangle\\&+\frac{1+k}{2}\|x-Tx\|^2-(1+k)\langle x-Tx,y-Ty\rangle\\&+\frac{1+k}{2}\|y-Ty\|^2\end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned}\left(1-\frac{1+k}{2}\right)\left(\|x-Tx\|^2+\|y-Ty\|^2\right) \le{}& \langle x-Ty,(I-T)x\rangle-\langle Tx-y,(I-T)y\rangle\\&-(1+k)\langle x-Tx,y-Ty\rangle\end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned}\left(\frac{1-k}{2}\right)\left(\|x-Tx\|^2+\|y-Ty\|^2\right) \le{}& \langle x-Ty,(I-T)x\rangle-\langle Tx-y,(I-T)y\rangle\\&-(1+k)\langle x-Tx,y-Ty\rangle\end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned}\|x-Tx\|^2+\|y-Ty\|^2 \le{}& \frac{2}{1-k}\langle x-Ty,(I-T)x\rangle-\frac{2}{1-k}\langle Tx-y,(I-T)y\rangle\\&-2\frac{1+k}{1-k}\langle x-Tx,y-Ty\rangle .\end{aligned}$$

This shows that (3) is true and hence (3) and (4) are equivalent. Next, we will show that (2) and (3) are equivalent. For this fact, let us consider the three terms on the right side of (2)

$$\begin{aligned}&\frac{2}{1-k}\langle x-y,(I-T)\,x\rangle-\frac{2}{1-k}\langle x-y,(I-T)\,y\rangle+2\langle x-Tx,y-Ty\rangle\\&=\frac{2}{1-k}\langle x-Ty,(I-T)\,x\rangle+\frac{2}{1-k}\langle Ty-y,(I-T)\,x\rangle\\&\quad-\frac{2}{1-k}\langle x-Tx,(I-T)\,y\rangle-\frac{2}{1-k}\langle Tx-y,(I-T)\,y\rangle\\&\quad+2\langle x-Tx,y-Ty\rangle\\&=\frac{2}{1-k}\langle x-Ty,(I-T)\,x\rangle-\frac{2}{1-k}\langle Tx-y,(I-T)\,y\rangle\\&\quad+\left[2-\frac{2}{1-k}-\frac{2}{1-k}\right]\langle x-Tx,y-Ty\rangle\\&=\frac{2}{1-k}\langle x-Ty,(I-T)\,x\rangle-\frac{2}{1-k}\langle Tx-y,(I-T)\,y\rangle\\&\quad-2\frac{1+k}{1-k}\langle x-Tx,y-Ty\rangle .\end{aligned}$$

The proof is complete. □

Lemma 3.2. *Let H be a real Hilbert space and T be a k-strict pseudo-contraction in the terminology of Browder and Petryshyn [5] (i.e., $k\in(-\infty,1)$) with domain*

$D(T)$ and range $R(T)$. If there exists $u \in D(T)$ such that $\langle x-u, u-Tu\rangle \geq 0$ and $\langle Tx-u, u-Tu\rangle \geq 0$ for some $x \in D(T)$, then the following inequalities hold:

(3.10)

$$\|x-Tx\|^2 \leq \begin{cases} \frac{2}{1-k}\langle x-u,(I-T)x\rangle, & \begin{cases}\text{if} & k\in[0,1)\,;\\ \text{or if} & \langle x-Tx,u-Tu\rangle\leq 0;\end{cases} \\ \begin{cases}\frac{2}{1-k}\langle x-u,(I-T)x\rangle \\ \frac{2}{1-k}\langle x-Tu,(I-T)x\rangle\end{cases}, & \text{if } \langle x-Tx,u-Tu\rangle\geq 0 \text{ and } k\in[0,1)\,; \\ \frac{2}{1-k}\langle x-Tu,(I-T)x\rangle, & \text{if } \langle x-Tx,u-Tu\rangle\geq 0 \text{ and } k\in[-1,0)\,; \\ \langle x-Tu,(I-T)x\rangle, & \text{if } k\in(-\infty,-1]\,, \end{cases}$$

Proof. If $k \in [0,1)$, then $k<1 \Leftrightarrow 0<1-k$ and note that $0 \leq 2k$, so we have $\frac{2k}{1-k} \geq 0 \Leftrightarrow -\frac{2k}{1-k} \leq 0$, thus by using Lemma 3.1 (1), we obtain

$$\begin{aligned}\|x-Tx\|^2 &\leq \|x-Tx\|^2+\|u-Tu\|^2\\ &\leq \frac{2}{1-k}\langle x-u,(I-T)x\rangle-\frac{2k}{1-k}\langle x-u,u-Tu\rangle\\ &\quad -2\langle Tx-u,u-Tu\rangle\\ &\leq \frac{2}{1-k}\langle x-u,(I-T)x\rangle.\end{aligned} \tag{3.11}$$

If $\langle x-Tx,u-Tu\rangle \leq 0$, then by using Lemma 3.1 (2), we obtain

$$\begin{aligned}\|x-Tx\|^2 &\leq \|x-Tx\|^2+\|u-Tu\|^2\\ &\leq \frac{2}{1-k}\langle x-u,(I-T)x\rangle-\frac{2}{1-k}\langle x-u,(I-T)u\rangle\\ &\quad +2\langle x-Tx,u-Tu\rangle\\ &\leq \frac{2}{1-k}\langle x-u,(I-T)x\rangle.\end{aligned}$$

Before the proof in the next case, let us consider the following.

$$k\in[-1,1)\Leftrightarrow -1\leq k<1\begin{cases}\Leftrightarrow 0\leq 1+k<2.\\ \Leftrightarrow 1\geq -k>-1\Leftrightarrow 2\geq 1-k>0\Leftrightarrow \frac{1}{2}\leq\frac{1}{1-k}.\end{cases}$$

Therefore, we have $2\frac{1+k}{1-k}\geq 1+k\geq 0$ and then

$$-2\frac{1+k}{1-k}\leq 0 \quad \text{whenever } k\in[-1,1)\,. \tag{3.12}$$

If $\langle x-Tx,u-Tu\rangle \geq 0$ and $k\in[0,1)$, then it follows from (3.12) and Lemma 3.1 (3) that

$$\|x-Tx\|^2 \leq \|x-Tx\|^2+\|u-Tu\|^2$$

$$\begin{aligned}
&\leq \frac{2}{1-k}\left\langle x-Tu,(I-T)x\right\rangle - \frac{2}{1-k}\left\langle Tx-u,(I-T)u\right\rangle \\
&\quad - 2\frac{1+k}{1-k}\left\langle x-Tx,u-Tu\right\rangle \\
&\leq \frac{2}{1-k}\left\langle x-Tu,(I-T)x\right\rangle . \qquad (3.13)
\end{aligned}$$

It follows simultaneously that (3.11) holds together, so we can conclude in this case that

$$\|x-Tx\|^2 \leq \begin{cases} \dfrac{2}{1-k}\left\langle x-u,(I-T)x\right\rangle \\ \dfrac{2}{1-k}\left\langle x-Tu,(I-T)x\right\rangle . \end{cases}$$

If $\langle x-Tx,u-Tu\rangle \geq 0$ and $k \in [-1,0)$, then by employing (3.12) and Lemma 3.1 (3), we obtain (3.13).

Finally, if $k \in (-\infty,-1]$, then by the virtue of Lemma 3.1 (4), we obtain

$$\begin{aligned}
\|x-Tx\|^2 &\leq \|x-Tx\|^2 + \|u-Tu\|^2 \\
&\leq \left\langle x-Tu,(I-T)x\right\rangle - \left\langle Tx-u,(I-T)u\right\rangle \\
&\quad + \frac{1+k}{2}\|(I-T)x-(I-T)u\|^2 \\
&\leq \left\langle x-Tu,(I-T)x\right\rangle .
\end{aligned}$$

This completes the proof. □

Every iteration process generated by the shrinking projection method for a k-strict pseudo-contraction T in the terminology of Browder and Petryshyn [5] (i.e., $k \in (-\infty,1)$) is well defined even if T is fixed point free.

Lemma 3.3. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $T : C \to C$ a k-strict pseudo-contraction in the terminology of Browder and Petryshyn [5] (i.e., $k \in (-\infty,1)$) and $x_0 \in H$. Let $\{x_n\}$ be a sequence in C and $\{C_n\}$ a sequence of closed convex subsets of H generated by $C_1 = C$ and*

$$(3.14)\qquad \begin{cases} x_n = P_{C_n}(x_0), \\ C_{n+1} = \left\{ z \in C_n \;\middle|\; \|x_n - Tx_n\|^2 \leq \max\left\{\dfrac{2}{1-k},1\right\}\langle x_n - z, x_n - Tx_n\rangle \right\} \end{cases}$$

for all $n \in \mathbb{N}$. Then C_n is nonempty for every $n \in \mathbb{N}$, and consequently, $\{x_n\}$ is well defined.

Proof. Clearly, C_1 is nonempty. Suppose that C_n is nonempty for some $n \in \mathbb{N}$. Since $C_n \subset C_{n-1} \subset \ldots \subset C_1$, we have $C_1, C_2, \ldots, C_n$ are nonempty and hence $\{x_1, x_2, \ldots, x_n\}$ is well define. Put $r = \max\{\|x_i\|, \|Tx_i\| : i = 1,2,\ldots,n\}$ and $B_r = \{z \in H : \|z\| \leq r\}$. Obviously $C \cap B_r$ is a nonempty bounded closed convex subset of H. Let I denote the identity mapping on C. Since $I - T$ is continuous and monotone, it follows from Lemma 2.5 that there exists $u \in C \cap B_r$ such that

$$\langle y-u,(I-T)u\rangle \geq 0, \quad \forall y \in C \cap B_r.$$

In particular, we have

$$\langle x_i - u, (I-T)u\rangle \geq 0 \quad \text{and} \quad \langle Tx_i - u, (I-T)u\rangle \geq 0, \tag{3.15}$$

for every $i = 1, 2, \ldots, n$.

Case I. $\max\left\{\frac{2}{1-k}, 1\right\} = \frac{2}{1-k}$.

Notice that $\max\left\{\frac{2}{1-k}, 1\right\} = \frac{2}{1-k} \Leftrightarrow 1 \leq \frac{2}{1-k} \Leftrightarrow 1-k \leq 2 \Leftrightarrow -1 \leq k \Leftrightarrow k \in [-1, 1)$, it follows from (3.15) and Lemma 3.2 that

$$\|x_i - Tx_i\|^2 \leq \begin{cases} \dfrac{2}{1-k}\langle x_i - u, (I-T)x_i\rangle, & \begin{cases} \text{if} & k \in [0,1); \\ \text{or if} & \langle x_i - Tx_i, u - Tu\rangle \leq 0; \end{cases} \\ \begin{cases} \dfrac{2}{1-k}\langle x_i - u, (I-T)x_i\rangle \\ \dfrac{2}{1-k}\langle x_i - Tu, (I-T)x_i\rangle \end{cases}, & \text{if } \langle x_i - Tx_i, u - Tu\rangle \geq 0 \text{ and } k \in [0,1); \\ \dfrac{2}{1-k}\langle x_i - Tu, (I-T)x_i\rangle, & \text{if } \langle x_i - Tx_i, u - Tu\rangle \geq 0 \text{ and } k \in [-1,0), \end{cases}$$

for every $i = 1, 2, \ldots, n$. This shows that $u \vee Tu \in C_{n+1}$.

Case II. $\max\left\{\frac{2}{1-k}, 1\right\} = 1$.

Notice that $\max\left\{\frac{2}{1-k}, 1\right\} = 1 \Leftrightarrow \frac{2}{1-k} \leq 1 \Leftrightarrow 2 \leq 1-k \Leftrightarrow k \leq -1 \Leftrightarrow k \in (-\infty, -1]$, it follows from (3.15) and Lemma 3.2 that

$$\|x_i - Tx_i\|^2 \leq \langle x_i - Tu, (I-T)x_i\rangle,$$

for every $i = 1, 2, \ldots, n$. This shows that $Tu \in C_{n+1}$.

By Case I and Case II, we can conclude that $u \vee Tu \in C_{n+1}$. By induction on n, we obtain the desired result. □

The following theorem provides some necessary and sufficient conditions to confirm the existence of a fixed point of a k-strict pseudo-contraction in the terminology of Browder and Petryshyn [5] (i.e., $k \in (-\infty, 1)$) in Hilbert spaces.

Theorem 3.4. *Let all the assumptions be as in Lemma 3.3. Then the following are equivalent:*

(1) $\bigcap_{n=1}^{\infty} C_n$ *is nonempty;*
(2) $\{x_n\}$ *is bounded;*
(3) $F(T)$ *is nonempty.*

Proof. [(1)⇒(2)] Let $u \in \bigcap_{n=1}^{\infty} C_n$, it follows from the nonexpansiveness of P_{C_n} that

$$\|x_n - u\| = \|P_{C_n}x_0 - P_{C_n}u\| \leq \|x_0 - u\|.$$

This shows that $\{x_n\}$ is bounded.

[(2)⇒(3)] Suppose that $\{x_n\}$ is bounded, we observe that

$$\begin{aligned} 0 \le \|x_{n+1} - x_n\|^2 &= \|x_{n+1} - P_{C_n} x_0\|^2 \\ &\le \|x_{n+1} - x_0\|^2 - \|P_{C_n} x_0 - x_0\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2. \end{aligned} \tag{3.16}$$

This shows that $\{\|x_n - x_0\|\}$ is non-decreasing and then with the boundedness of $\{x_n\}$, we have $\lim_{n\to\infty} \|x_n - x_0\|$ exists. By using (3.16), we obtain

$$\|x_{n+1} - x_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Since $x_{n+1} \in C_{n+1}$, we have

$$\|x_n - Tx_n\|^2 \le \max\left\{\frac{2}{1-k}, 1\right\} \langle x_n - x_{n+1}, (I-T)x_n\rangle \to 0 \quad \text{as} \quad n \to \infty. \tag{3.17}$$

Since $\{x_n\}$ is bounded, the reflexivity of H allows a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p \in C$ as $i \to \infty$. By using (3.17) and the demicloseness of $(I - T)$, we obtain $p - Tp = 0$ that is $p \in F(T) \neq \varnothing$.

[(3)⇒(1)] Suppose that $F(T) \neq \varnothing$. We will show that $F(T) \subset C_n$ for every $n \in \mathbb{N}$. If $p \in F(T)$, then we have $(I - T)p = 0$. Let us replace u in the proof of Lemma 3.3 with p, it is not difficult to see that all inequalities are satisfied. This implies that $p \in C_n$ for all $n \in \mathbb{N}$. Therefore $F(T) \subset \bigcap_{n=1}^{\infty} C_n \neq \varnothing$. □

Theorem 3.5. *Let all the assumptions be as in Lemma 3.4. Then, if $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$ ($\Leftrightarrow$ $\{x_n\}$ is bounded $\Leftrightarrow$ $F(T) \neq \varnothing$), then the sequence $\{x_n\}$ generated by* (3.14) *converges strongly to some points of C and its strong limit point is a member of $F(T)$, that is, $\lim_{n\to\infty} x_n = P_{F(T)}x_0 \in F(T)$.*

Proof. If $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$, then Theorem 3.4 ensures that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Thus, there exists $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p \in C$ as $i \to \infty$. By using (3.17) and the demicloseness of $(I - T)$, we obtain $p - Tp = 0$ that is $p \in F(T)$. Since $P_{F(T)}x_0 \in F(T) \subset C_n$, we observe that

$$\|x_n - x_0\| = \|P_{C_n}x_0 - x_0\| \le \left\|P_{F(T)}x_0 - x_0\right\| \tag{3.18}$$

for every $n \in \mathbb{N}$. Since $\|\cdot\|^2$ is weakly lower semicontinuous and $\{\|x_n - x_0\|\}$ is convergent, it follows from (3.18) that

$$\|p - x_0\|^2 \le \liminf_{i\to\infty} \|x_{n_i} - x_0\|^2 = \lim_{n\to\infty} \|x_n - x_0\|^2 \le \left\|P_{F(T)}x_0 - x_0\right\|^2.$$

Taking into account $p \in F(T)$, we obtain $p = P_{F(T)}x_0$. This implies that $x_n \rightharpoonup P_{F(T)}x_0$ and $\|x_n - x_0\| \to \|P_{F(T)}x_0 - x_0\|$. Consequently, from (2.1), we obtain

$$\begin{aligned} \left\|x_n - P_{F(T)}x_0\right\|^2 &= \left\|x_n - x_0 - \left(P_{F(T)}x_0 - x_0\right)\right\|^2 \\ &= \|x_n - x_0\|^2 - \left\|P_{F(T)}x_0 - x_0\right\|^2 \\ &\quad - 2\left\langle x_n - P_{F(T)}x_0, P_{F(T)}x_0 - x_0\right\rangle \to 0. \end{aligned}$$

This completes the proof. □

4. Deduced theorems and applications

In this section, some deduced theorems and applications of the main theorem are provided in order to guarantee the existence of fixed points of a firmly nonexpansive mapping, a nonexpansive mapping and the existence of the zeros of an inverse strongly monotone operator. Moreover, we also have the methods that can be used to find fixed points and zero points of the mappings mentioned above.

In particular case, if we set $k = -1$ in the previous section, then T is reduced to a firmly nonexpansive mapping and $\frac{2}{1-k} = \frac{2}{1-(-1)} = 1$, so we have for any $n \in \mathbb{N}$,

$$\begin{aligned}&\|x_n - Tx_n\|^2 \leq \max\left\{\frac{2}{1-k}, 1\right\}\langle x_n - z, (I-T)x_n\rangle = \langle x_n - z, (I-T)x_n\rangle\\ &\Leftrightarrow \langle Tx_n - z, x_n - Tx_n\rangle \geq 0,\end{aligned}$$

then we obtain the following corollaries:

Every iteration process generated by the shrinking projection method for a firmly nonexpansive mapping T is well defined even if T is fixed point free.

Corollary 4.1 (Aoyama, Kohsaka, Takahashi [1, Lemma 4.2])**.** *Let H be a Hilbert space, C a nonempty closed convex subset of H, $T : C \to C$ a firmly nonexpansive mapping and $x_0 \in H$. Let $\{x_n\}$ be a sequence in C and $\{C_n\}$ a sequence of closed convex subsets of H generated by $C_1 = C$ and*

$$\begin{cases} x_n = P_{C_n}(x_0), \\ C_{n+1} = \{z \in C_n : \langle Tx_n - z, x_n - Tx_n\rangle \geq 0\}, \end{cases}$$

for all $n \in \mathbb{N}$. Then C_n is nonempty for every $n \in \mathbb{N}$, and consequently, $\{x_n\}$ is well defined.

Corollary 4.2 (Aoyama, Kohsaka, Takahashi [1, Theorem 4.3])**.** *Let H be a Hilbert space, C a nonempty closed convex subset of H, $T : C \to C$ a firmly nonexpansive mapping and $x_0 \in H$. Let $\{x_n\}$ be a sequence in C and $\{C_n\}$ a sequence of closed convex subsets of H generated by $C_1 = C$ and*

$$\begin{cases} x_n = P_{C_n}(x_0), \\ C_{n+1} = \{z \in C_n : \langle Tx_n - z, x_n - Tx_n\rangle \geq 0\}, \end{cases}$$

for all $n \in \mathbb{N}$. Then the following are equivalent:

(1) $\bigcap_{n=1}^{\infty} C_n$ *is nonempty;*
(2) $\{x_n\}$ *is bounded;*
(3) $F(T)$ *is nonempty.*

In particular case, if we set $k = 0$ in the previous section, then T is reduced to a nonexpansive mapping and $\frac{2}{1-k} = \frac{2}{1-(0)} = 2$, so we have for any $n \in \mathbb{N}$,

$$\|x_n - Tx_n\|^2 \leq \max\left\{\frac{2}{1-k}, 1\right\}\langle x_n - z, (I-T)x_n\rangle = 2\langle x_n - z, (I-T)x_n\rangle$$

then we obtain the following corollaries:

Every iteration process generated by the shrinking projection method for a nonexpansive mapping T is well defined even if T is fixed point free.

Corollary 4.3. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $T : C \to C$ a nonexpansive mapping and $x_0 \in H$. Let $\{x_n\}$ be a sequence in C and $\{C_n\}$ a sequence of closed convex subsets of H generated by $C_1 = C$ and*

$$\begin{cases} x_n = P_{C_n}(x_0), \\ C_{n+1} = \Big\{ z \in C_n : \|x - Tx\|^2 \le 2 \langle x - z, (I - T)x \rangle \Big\}, \end{cases} \tag{4.1}$$

for all $n \in \mathbb{N}$. Then C_n is nonempty for every $n \in \mathbb{N}$, and consequently, $\{x_n\}$ is well defined.

Corollary 4.4. *Let all the assumptions be as in Lemma 4.3 for all $n \in \mathbb{N}$. Then the following are equivalent:*

(1) $\bigcap\limits_{n=1}^{\infty} C_n$ *is nonempty;*
(2) $\{x_n\}$ *is bounded;*
(3) $F(T)$ *is nonempty.*

Corollary 4.5. *Let all the assumptions be as in Lemma 4.3. Then, if $\bigcap\limits_{n=1}^{\infty} C_n \neq \varnothing$ ($\Leftrightarrow$ $\{x_n\}$ is bounded $\Leftrightarrow$ $F(T) \neq \varnothing$), then the sequence $\{x_n\}$ generated by* (4.1) *converges strongly to some points of C and its strong limit point is a member of $F(T)$, that is $\lim\limits_{n\to\infty} x_n = P_{F(T)} x_0 \in F(T)$.*

Recall that a mapping A is said to be monotone, if $\langle x - y, Ax - Ay \rangle \geqslant 0$ for all $x, y \in H$ and inverse strongly monotone if there exists a real number $\gamma > 0$ such that $\langle x - y, Ax - Ay \rangle \geqslant \gamma \|Ax - Ay\|^2$ for all $x, y \in H$. For the second case, A is said to be γ-inverse strongly monotone. It follows immediately that if A is γ-inverse strongly monotone, then A is monotone and Lipschitz continuous, that is, $\|Ax - Ay\| \leqslant \frac{1}{\gamma}\|x - y\|$. It is well known (see, e.g., [6]) that if A is monotone, then the solutions of the equation $Ax = 0$ correspond to the equilibrium points of some evolution systems. Therefore, it is important to focus on finding the zero point of monotone mappings. The pseudo-contractive mapping and strictly pseudo-contractive mapping are strongly related to the monotone mapping and inverse strongly monotone mapping, respectively. It is well known that

(i) A is γ-inverse strongly monotone
$\Leftrightarrow T := (I - A)$ a k-strict pseudo-contraction in the terminology of Browder and Petryshyn [5] where $k := 1 - 2\gamma$.

Indeed, for (i), we notice that the following equality always holds in a real Hilbert space

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 + \|Ax - Ay\|^2 - 2 \langle x - y, Ax - Ay \rangle, \quad \forall x, y \in H, \tag{4.2}$$

then by the virtue of (4.2) we obtain

$$\langle x - y, Ax - Ay \rangle \geqslant \gamma \|Ax - Ay\|^2 \Leftrightarrow -2 \langle x - y, Ax - Ay \rangle \leqslant -2\gamma \|Ax - Ay\|^2$$

$$\Leftrightarrow \|(I-A)x-(I-A)y\|^2 \leqslant \|x-y\|^2 + (1-2\gamma)\|Ax-Ay\|^2$$
$$\Leftrightarrow \|Tx-Ty\|^2 \leqslant \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2$$
(where $T := (I-A)$ and $k := 1-2\gamma$).

Every iteration process generated by the shrinking projection method for a γ-inverse strongly monotone A is well defined even if A has no zeros.

Corollary 4.6. *Let H be a Hilbert space and $A : H \to H$ be a γ-inverse strongly monotone. Let $x_0 \in H$, $C_1 = C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = P_{C_1}x_0, \\ C_{n+1} = \left\{ z \in C_n \;\middle|\; \|Ax_n\|^2 \leq \max\left\{\dfrac{1}{\gamma}, 1\right\} \langle x_n - z, Ax_n\rangle \right\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \end{cases} \tag{4.3}$$

for all $n \in \mathbb{N}$. Then C_n is nonempty for every $n \in \mathbb{N}$, and consequently, $\{x_n\}$ is well defined.

Proof. Let $T := (I-A)$. Then T is a k-strict pseudo-contraction in the terminology of Browder and Petryshyn [5] and $\frac{1}{\gamma} = \frac{2}{1-(1-2\gamma)} = \frac{2}{1-k}$ where $k := (1-2\gamma) \in (-\infty, 1)$. Hence, applying Theorem 3.3, we have the desired result. □

The following corollary provides some necessary and sufficient conditions to confirm the existence of a zeros of a γ-inverse strongly monotone in Hilbert spaces.

Corollary 4.7. *Let all the assumptions be as in Lemma 4.6 for all $n \in \mathbb{N}$. Then the following are equivalent:*

(1) $\bigcap_{n=1}^{\infty} C_n$ *is nonempty;*
(2) $\{x_n\}$ *is bounded;*
(3) $A^{-1}(0)$ *is nonempty.*

Proof. Let $T := (I-A)$. Then T is a k-strict pseudo-contraction in the terminology of Browder and Petryshyn [5] and $\frac{1}{\gamma} = \frac{2}{1-k}$ where $k := 1-2\gamma$, it is not difficult to show that $F(T) = A^{-1}(0)$. Hence, applying Theorem 3.4, we have the desired result. □

Corollary 4.8. *Let all the assumptions be as in Lemma 4.6. Then, if $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$ ($\Leftrightarrow$ $\{x_n\}$ is bounded $\Leftrightarrow$ $A^{-1}(0) \neq \varnothing$), then the sequence $\{x_n\}$ generated by (4.3) converges strongly to some points of H and its strong limit point is a member of $A^{-1}(0)$, that is $\lim_{n\to\infty} x_n = P_{A^{-1}(0)}x_0 \in A^{-1}(0)$.*

Proof. Let $T := (I-A)$ and by applying Theorem 3.5, we have the desired result. □

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