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FIXED POINTS OF FUNCTIONALLY LIPSCHITZIAN MAPS

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Abstract. We introduce new classes of selfmaps on a normed space equipped with the weak topology, and investigate the existence of their fixed points as well as their weakly virtual stability in order to obtain an explicit retraction from the weak convergence sets onto the fixed point sets.

1. Introduction

In metric fixed point theory, the fixed point set of a quasi-nonexpansive selfmap on a convex subset of a strictly convex Banach space can be directly shown to be convex (hence connected and contractible). Such a result also holds if we replace "a strictly convex Banach space" in the previous statement by "a CAT(0) space" (see [5]) or "a uniformly convex metric space" (see [6]). On the other hand, the celebrated Bruck's result in [2] states that if X is a weakly compact convex subset of a Banach space and $T : X \to X$ is nonexpansive selfmap satisfying the conditional fixed point property, then the fixed point set of T is a nonexpansive retract of X. Later in [1], Bruck's result is improved to include an even larger class of (weakly) asymptotically nonexpansive selfmaps. An advantage of these Bruck-type results is a connection (through a nonexpasive retraction, even though not explicitly defined) between the fixed point set and the domain of the map that indirectly results in (since X is convex) some interesting topological structures (such as connectedness and contractibility) of the fixed point set. Motivated by such a connection, the first author introduces in [3] the notion of virtually nonexpansive selfmaps generalising various nonexpansive-type selfmaps, and shows that the fixed point set of a virtually nonexpansive selfmap on a metric space is naturally a retract of a certain subset, called the convergence set, of the domain of the map. Not only this work can be considered as a more general Bruck-type result (in the sense that the convergence set is not neccessary the domain of the map), it also gives an explicit retraction onto the fixed point set (instead of nonexpansive retractions defined implicitly using Zorn's lemma in [1] and [2]). Moreover, the first author also introduces in [4] the notion of virtually stable maps that defines on general Hausdorff spaces (rather than metric spaces) and yet captures the essense of virtually nonexpansive maps at the same time. This certainly allows us to obtain a Bruck-type result in a non-metrizable setting, especially in a normed space equipped with the weak topology. However, the continuity as well as the virtual stability with respect to the weak topology cannot be generally duduced from the usual nonexpansive-type conditions, and surprisingly, the discussion about suitable conditions that guarantee those properties seems to

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be missing from the liturature. Therefore, in this paper, we will introduce some new classes of selfmaps that are naturally continuous and virtually stable with respect to the weak topology, as well as formulate new fixed point theorems for those maps.

In the first section, we recall some backgrounds in functional analysis, topology and fixed point theory as well as results from [4]. Then, in section 2, we introduce the notions of a functionally lipschitzian selfmap and a functionally uniformly lipschitzian selfmap on a normed space, and show that they are weakly continuous and weakly virtually stable, respectively. Together with Theorem 2.6 in [4], it is immediate that the fixed point set of a functionally uniformly lipschitzian selfmap is always a retract of its weak convergence set. In section 3, we prove some new fixed point theorems for functionally uniformly lipschitzian selfmaps with respect to certain sequences. These maps somehow resemble contractions in metric fixed point theory. Finally, in section 4, we give some criteria for a map to be functionally uniformly lipschitzian in an infinite dimensional Banach space having a Shauder basis. As a result, we obtain some explicit examples of functionally uniformly lipschitzian selfmaps including the one in Example 5.7 that is not nonexpansive and hence falls outside the framework of [2].

2. Preliminaries

Let $(E, \|.\|)$ be a normed space, $X \subseteq E$ and E^* denote its dual space (the space of all continuous linear functionals on E). The weak topology on E is the coarsest topology making each $f \in E^*$ continuous. We will denote X equipped with the subspace topology of the weak topology on E by X^w. As usual, each open set in X^w is regarded as a weakly open subset of X, and by a weak neighborhood of $x \in X$, we mean a weakly open subset of X containing x. It is well known (see [7]) that E^w is always regular (but it may not be metrizable), and hence so is X^w. A family $\mathcal{F}$ of continuous functionals on X is called weakly equicontinuous at $x \in X$ if for each $\epsilon > 0$, there exists a weak neighborhood U (in X) of x such that $f(U) \subseteq (f(x) - \epsilon, f(x) + \epsilon)$ for all $f \in \mathcal{F}$. To each selfmap $T : X \to X$, we will associate its weak counterpart $T^w : X^w \to X^w$ given by $T^w(x) = T(x)$. Clearly, $F(T^w) = F(T)$ and we will say that T is weakly continuous if T^w is continuous. Notice also that the continuity of T does not always imply the weak continuity :

Example 2.1. Consider $T : \ell_2 \to \ell_2$ given by $T(x) = (\|x\|, 0, 0, \dots)$ for any $x \in \ell_2$. Let $e_n = (0, \dots, 0, \overbrace{1}^{n-th}, 0, \dots)$. Clearly, T is continuous (being nonexpansive). However, since the sequence (e_n) weakly converges to 0 while the sequence $(T(e_n)) = (e_1)$ does not converge to $T(0) = 0$, the map T is not weakly continuous.

Also recall that a selfmap $T : X \to X$ is said to be lipschitzian if there is $L \geq 0$ such that $\|T(x) - T(y)\| \leq L\|x - y\|$ for all $x, y \in X$. The smallest such L of a lipschitzian map T will be called the Lipschitz constant of T, and denoted by $L(h)$.

When E is finite dimensional, a finite sequence $e_1, \dots, e_N$ is a basis for E if each $x \in E$ can be uniquely written as $x = \sum_{n=1}^{N} \alpha_n e_n$ where $\alpha_1, \dots, \alpha_N$ are real numbers. Similarly, in an infinite dimensional (real) Banach space E, a sequence (e_n) is a Shauder basis for E if each $x \in E$ can be uniquely written as $x = \sum_{n\in\mathbb{N}} \alpha_n e_n$

where (α_n) is a sequence of real numbers. In both situations, each coordinate functional $e_m^* : E \to \mathbb{R}$ given by $e_m^*(\sum_{n\in\mathbb{N}} \alpha_n e_n) = \alpha_m$ is continuous and we can always assume that the basis is normalised; i.e., $\|e_n\| = 1$ for all n (see [7] for more details).

Following [4], we will define the *weak convergence set* of a selfmap $T : X \to X$ to be $C(T^w)$ and simply denote it by $C^w(T)$. Since $C^w(T)$ is generally larger than $C(T)$, it tends to be topologically simpler than $C(T)$. For instance, $C^w(T)$ has a better chance than $C(T)$ to be the whole domain X. Recall also that a selfmap $T : X \to X$ is virtually stable if it is continuous, $F(T) \neq \emptyset$, and for each $x \in F(T)$ and each neighborhood U of x, there exist a neighborhood V of x and an increasing sequence (k_n) of positive integers such that $T^{k_n}(V) \subseteq U$ for all $n \in \mathbb{N}$. When the sequence (k_n) is independent of both x and U, we will call T uniformly virtually stable. Thus, we will say that T is *weakly (uniformly) virtually stable* if T^w is (uniformly) virtually stable. With these terminologies, Theorem 2.6 [4] immediately gives us the following connection between the fixed point set of T and its weak convergence set :

Theorem 2.2. *If $T : X \to X$ is weakly virtually stable, then $F(T)$ is a retract of $C^w(T)$.*

3. Functionally lipschitzian selfmaps

As seen in Example 2.1, nonexpansive maps do not behave well with respect to the weak continuity. In this section, we will explore the weak continuity and the weak virtual stability for new classes of maps whose definitions are motivated by those of lipschitzian maps. As usual, we let X be a nonempty subset of a normed space E and $T : X \to X$ a selfmap on X with $F(T) \neq \emptyset$. Also, for each $f \in E^*$, let $\|.\|_f$ denote the seminorm defined by $\|x\|_f = |f(x)|$ for each $x \in E$.

Definition 3.1. T is called

- *functionally lipschitzian* if for each $f \in E^*$, there exist $g_1, g_2, \ldots, g_n \in E^*$ such that
$$\|T(x) - T(y)\|_f \leq \sum_{i=1}^{n} \|x - y\|_{g_i}$$
for any $x, y \in X$.
- *functionally uniformly lipschitzian* if for each $f \in E^*$, there exist $g_1, g_2, \ldots,$ $g_n \in E^*$ such that
$$\|T^k(x) - T^k(y)\|_f \leq \sum_{i=1}^{n} \|x - y\|_{g_i}$$
for any $x, y \in X$ and $k \in \mathbb{N}$.
- *functionally uniformly quasi-lipschitzian* if $F(T) \neq \emptyset$ and for each $f \in E^*$, there exist $g_1, g_2, \ldots, g_n \in E^*$ such that
$$\|T^k(x) - p\|_f \leq \sum_{i=1}^{n} \|x - p\|_{g_i}$$
for any $x \in X$, $p \in F(T)$ and $k \in \mathbb{N}$.

It is clear from the above definition that a functionally uniformly lipschitzian selfmap is both functionally lipschitzian and functionally uniformly quasi-lipschitzian. Although one can readily extend the above definitions to a nonself map $T : X \to E$, we will restrict our attention only to a selfmap for the purpose of this paper.

Example 3.2. A continuous affine map on a Banach space E is always functionally lipschitzian. For if $T : E \to E$ is a continuous affine map, says $T(x) = L(x) + c$ for some linear map $L : E \to E$ and some $c \in E$, we have for each $f \in E^*$, $\|T(x) - T(y)\|_f = |f \circ L(x) - f \circ L(y)|$. Hence, by letting $g = f \circ L \in E^*$, we clearly obtain $\|T(x) - T(y)\|_f \le \|x - y\|_g$ for any $x, y \in X$.

Theorem 3.3. *Suppose E is finite dimensional. Then T is functionally lipschitzian if and only if T is lipschitzian.*

Proof. Let $e_1, e_2, \dots, e_N$ be a normalised basis for E. So, for each $x \in X$, we write $x = \sum_{n=1}^N e_n^*(x)e_n$ and $T(x) = \sum_{n=1}^N (e_n^* \circ T(x))e_n$, where $e_n^* : E \to \mathbb{R}$ denotes the usual n-th coordinate functional.

($\Rightarrow$) Since T is functionally lipschitzian, for each $n = 1, \dots, N$, there are $g_{n,1}, \dots, g_{n,k_n} \in E^*$ such that for any $x, y \in X$,

$$|e_n^*(T(x) - T(y))| = \|T(x) - T(y)\|_{e_n^*} \le \sum_{i=1}^{k_n} \|x - y\|_{g_{n,i}} \le \sum_{i=1}^{k_n} \|g_{n,i}\| \|x - y\|$$

Then, for any $x, y \in X$, we have

$$\begin{aligned} \|T(x) - T(y)\| &= \Big\| \sum_{n=1}^{N} (e_n^*(T(x) - T(y)))e_n \Big\| \\ &\le \sum_{n=1}^{N} |e_n^*(T(x) - T(y))| \\ &\le \Big(\sum_{n=1}^{N} \sum_{i=1}^{k_n} \|g_{n,i}\| \Big) \|x - y\|. \end{aligned}$$

($\Leftarrow$) Suppose $\|T(x) - T(y)\| \le L\|x - y\|$ for some $L \ge 0$ and for all $x, y \in X$. Without loss of generality, we may assume that $L > 0$.

Then for any $f \in E^*$ and $x, y \in X$, we have

$$\begin{aligned} \|T(x) - T(y)\|_f &= \|f(T(x) - T(y))\| \\ &\le \|f\| \|T(x) - T(y)\| \\ &\le L\|f\| \|x - y\| \\ &\le L\|f\| \Big\| \sum_{n=1}^{N} e_n^*(x - y)e_n \Big\| \\ &\le \sum_{n=1}^{N} \|L\|f\| e_n^*(x - y)\| \end{aligned}$$

$$= \sum_{n=1}^{N} \|x-y\|_{L\|f\|e_n^*}.$$

□

Following the proof of the previous theorem, we also have

Theorem 3.4. *Suppose E is finite dimensional. Then T is functionally uniformly lipschitzian if and only if T is uniformly lipschitzian.*

When E is infinite diemensional, the notions of functionally (uniformly) lipschitzian maps and (uniformly) lipschitzian maps are not equivalent. In fact, we will see later on that the map in Example 2.1 (which is a lipschitzian) is not functionally lipschitzian. In the next section, we will furthur explore functionally lipschitzian maps and functionally uniformly lipschitzian maps on infinite dimensional Banach spaces, but for now, we will show that functionally (uniformly quasi-) lipschitzian maps behave very well with respect to the weak continuity (weak virtual stability). First, we need the following lemmas :

Lemma 3.5. *A map $T : X \to X$ is weakly continuous if and only if, for each $f \in E^*$, $f \circ T$ is weakly continuous.*

Proof. See Corollary 2.4.5 in [7]. □

Theorem 3.6. *If $T : X \to X$ is functionally lipschitzian, then it is weakly continuous.*

Proof. Let $f \in E^*$, $x \in X$ and $\epsilon > 0$. Then, there exist $g_1, g_2, \ldots, g_n \in E^*$ such that

$$\|T(x) - T(y)\|_f \le \sum_{i=1}^{n} \|x-y\|_{g_i},$$

for all $x, y \in X$. Clearly, $U := X \cap \bigcap_{i=1}^{n} g_i{}^{-1}\Big(g_i(x) - \frac{\epsilon}{n}, g_i(x) + \frac{\epsilon}{n}\Big)$ is a weak neighborhood (in X) of x such that for each $y \in U$, we have

$$|f \circ T(x) - f \circ T(y)| = \|T(x) - T(y)\|_f \le \sum_{i=1}^{n} \|x-y\|_{g_i} = \sum_{i=1}^{n} |g_i(x) - g_i(y)| < \epsilon;$$

i.e., $T(y) \in f^{-1}\Big(f \circ T(x) - \epsilon, f \circ T(x) + \epsilon\Big)$. It follows that

$$U \subseteq T^{-1}\Big(f^{-1}\Big(f \circ T(x) - \epsilon, f \circ T(x) + \epsilon\Big)\Big) = (f \circ T)^{-1}\Big(f \circ T(x) - \epsilon, f \circ T(x) + \epsilon\Big),$$

and hence $f \circ T$ is weakly continuous at x. Then by Lemma 3.5, T is weakly continuous. □

Example 3.7. The map $T : \ell_2 \to \ell_2$ as in Example 2.1 is not functionally lipschitzian because it is not weakly continuous.

Lemma 3.8. *A weakly continuous map $T : X \to X$ is weakly uniformly virtually stable with respect to a sequence (k_n) if and only if, for each $f \in E^*$, the family $\{f \circ T^{k_n}\}$ is weakly equicontinuous at each fixed point of T.*

Proof. ($\Rightarrow$) Let $f \in E^*$, $\epsilon > 0$ and $p \in F(T)$. Since $U := X \cap f^{-1}(f(p) - \epsilon, f(p) + \epsilon)$ is a weak neighborhood (in X) of p, then, by weak virtual stability of T, there exists a weak neighborhood V (in X) of p such that $T^{k_n}(V) \subseteq U$ for all $n \in \mathbb{N}$. It follows that $f \circ T^{k_n}(V) \subseteq (f(p) - \epsilon, f(p) + \epsilon)$ for all $n \in \mathbb{N}$, and hence family $\{f \circ T^{k_n}\}$ is weakly equicontinuous at x.

($\Leftarrow$) Let $p \in F(T)$ and U a weak neighborhood (in X) of p. Then, there exist $f_1, \dots, f_n \in E^*$ and $\epsilon > 0$ such that $X \cap \bigcap_{i=1}^{m} f_i^{-1}(f_i(p) - \epsilon, f_i(p) + \epsilon) \subseteq U$. By weak equicontinuity of $\{f \circ T^{k_n}\}$ at p, for each $i = 1, \dots, m$, there is a weak neighborhood V_i (in X) of p such that

$$f_i \circ T^{k_n}(V_i) \subseteq (f_i \circ T^{k_n}(p) - \epsilon, f_i \circ T^{k_n}(p) + \epsilon) = (f_i(p) - \epsilon, f_i(p) + \epsilon),$$

for all $n \in \mathbb{N}$. By letting $V = \bigcap_{i=1}^{m} V_i$, we obtain a weak neighborhood V (in X) of p such that for all $n \in \mathbb{N}$,

$$T^{k_n}(V) = T^{k_n}(\bigcap_{i=1}^{m} V_i) \subseteq \bigcap_{i=1}^{m} T^{k_n}(V_i) \subseteq X \cap \bigcap_{i=1}^{m} f_i^{-1}(f_i(p) - \epsilon, f_i(p) + \epsilon) \subseteq U.$$

Therefore, T is weakly uniformly virtually stable with respect to (k_n). □

Theorem 3.9. *If $T : X \to X$ is weakly continuous and functionally uniformly quasi-lipschitzian, then it is weakly uniformly virtually stable, and hence $F(T)$ is a retract of $C^w(T)$.*

Proof. Let $f \in E^*$, $p \in F(T)$ and $\epsilon > 0$. Then, there exists $g_1, g_2, \dots, g_n \in E^*$ such that

$$\|T^k(x) - p\|_f \le \sum_{i=1}^{n} \|x - p\|_{g_i},$$

for all $x \in X$ and $k \in \mathbb{N}$. Clearly, $U := X \cap \bigcap_{i=1}^{n} {g_i}^{-1}\Big(g_i(p) - \frac{\epsilon}{n}, g_i(p) + \frac{\epsilon}{n}\Big)$ is a weak neighborhood (in X) of p such that for each $x \in U$ and $k \in \mathbb{N}$, we have

$$|f \circ T^k(x) - f(p)| = \|T^k(x) - p\|_f \le \sum_{i=1}^{n} \|x - p\|_{g_i} = \sum_{i=1}^{n} |g_i(x) - g_i(p)| < \epsilon;$$

i.e., $T^k(x) \in f^{-1}(f(p) - \epsilon, f(p) + \epsilon)$. It follows that $f \circ T^k(U) \subseteq (f(p) - \epsilon, f(p) + \epsilon)$ for all $k \in \mathbb{N}$, and hence $\{f \circ T^k\}$ is weakly equicontinuous at p. Since T is also assumed to be weakly continuous, then by Lemma 3.8, T is weakly uniformly virtually stable with respect to the sequence of all natural numbers, and by Theorem 2.2, $F(T)$ is a retract of $C^w(T)$. □

The previous two theorems immediately imply :

Corollary 3.10. *If $T : X \to X$ is functionally uniformly lipschitzian, then it is weakly uniformly virtually stable, and hence $F(T)$ is a retract of $C^w(T)$.*

In particular, we obtain the following criterion for contractibility of fixed point sets :

Corollary 3.11. *If $T : X \to X$ is functionally uniformly lipschitzian and $C^w(T)$ is connected [contractible], then $F(T)$ is is connected [contractible].*

Remark 3.12. We may tempt to combine Corollary 3.11 with Opial's type results (for example, see [8]) to obtain some explicit contractibility criteria for the fixed point set of a nonexpansive functionally uniformly lipschitzian map. However, such results are not new since in those settings, X is always convex in a strictly convex Banach space and hence $F(T)$ is automatically convex, connected and even contractible. Therefore, in order that Corollary 3.11 will have some significance in application, some new results on connectedness or contractibility of $C^w(T)$ in a more general setting need to be established.

4. Fixed Point Theorems

To prove some fixed point thorems, we need to introduce a variant of functionally uniformly lipschitzian selfmaps. Let X be a nonempty subset of a normed space E and $T : X \to X$ a selfmap as in the previous section, but now we will not assume that T has a fixed point.

Definition 4.1. Let (z_k) be a sequence of positive real numbers. The map $T : X \to X$ is called *functionally uniformly lipschitzian with respect to* (z_k) if for each $f \in E^*$, there exist $g_1, g_2, \ldots, g_n \in E^*$ such that

$$\|T^k(x) - T^k(y)\|_f \leq z_k \sum_{i=1}^{n} \|x - y\|_{g_i}$$

for any $x, y \in X$ and $k \in \mathbb{N}$.

Remark 4.2. A functionally uniformly lipschitzian selfmap is always functionally uniformly lipschitzian with respect to the constant sequence (1), while a functionally uniformly lipschitzian selfmap with respect to a bounded sequence is always functionally uniformly lipschitzian.

Theorem 4.3. *Suppose $T : X \to X$ is functionally uniformly lipschitzian selfmap with respect to a sequence (z_k) converging to 0. If there exists $x_0 \in X$ such that the sequence $(T^k(x_0))$ has a weakly convergent subsequence, then T has a unique fixed point in X.*

Proof. Let $(T^{j_k}(x_0))$ be a subsequence of $(T^k(x_0))$ such that $(T^{j_k}(x_0))$ converges weakly to $p \in X$. Then we have

$$\begin{aligned}\|T^{j_k+1}(x_0) - p\|_f &\leq \|T^{j_k}(T(x_0)) - T^{j_k}(x_0)\|_f + \|T^{j_k}(x_0) - p\|_f \\ &\leq z_{j_k} \sum_{i=1}^{n} \|T(x_0) - x_0\|_{g_i} + \|T^{j_k}(x_0) - p\|_f\end{aligned}$$

Since (z_k) converges to 0, the sequence $(T^{j_k+1}(x_0))$ converges weakly to p. Also, since (z_k) is bounded, T is weakly continuous (by Theorem 3.6) and hence the sequence $(T^{j_k+1}(x_0))$ converges weakly to $T(p)$. Since the weak topology is Hausdorff, it follows that $T(p) = p$.

For uniqueness, suppose x and y are fixed points of T. Then for each $f \in E^*$ and $k \in \mathbb{N}$,

$$\|x - y\|_f = \|T^k(x) - T^k(y)\|_f \leq z_k \sum_{i=1}^{n} \|x - y\|_{g_i}.$$

Since the sequence (z_k) converges to 0, we must have $x = y$. □

Corollary 4.4. *If X is closed and bounded subset of a reflexive Banach space, and $T : X \to X$ is functionally uniformly lipschitzian selfmap with respect to a sequence (z_k) converging to 0, then T has a unique fixed point in X.*

Proof. By the assumption, every sequence in X has a weakly convergent subsequence, and hence, T has a unique fixed point in X by the previous theorem. □

Theorem 4.5. *Suppose X is weakly sequentially complete and $T : X \to X$ is functionally uniformly lipschitzian selfmap with respect to a sequence (z_k). If the sequence (z_k) is summable, then T has a unique fixed point in X and and $C^w(T) = X$.*

Proof. Let $x \in X$. Then for any $f \in E^*$ and $m \le l$, we have

$$\begin{aligned}\|T^m(x) - T^l(x)\|_f &\le \sum_{m\le j<l} \|T^j(x) - T^{j+1}(x)\|_f \\ &\le \sum_{m\le j<l} \Big(z_j \sum_{i=1}^{n} \|x - T(x)\|_{g_i} \Big) \\ &= \Big(\sum_{m\le j<l} z_j \Big) \sum_{i=1}^{n} \|x - T(x)\|_{g_i}.\end{aligned}$$

Since (z_k) is summable, $(T^k(x))$ is a weakly Cauchy sequence in X. By the weak sequential completeness of X, there exists $p \in X$ such that $(T^k(x))$ converges weakly to p. Then $(T^k(x))$ also converges weakly to $T(p)$ because of the weak continuity of T. Since the weak topology is Hausdorff, it follows that $T(p) = p$. The uniqueness is obtained in the similar manner to Theorem 4.3. □

Example 4.6. Let $0 \ne c = (c_1, c_2, \dots) \in \ell_2$ and $T : \ell_2 \to \ell_2$ be defined by

$$T(x_1, x_2, \dots) = \Big(\frac{1}{2}\sin(g(x)\Big), \frac{1}{8}\sin^2(g(x)), \dots, \frac{1}{n2^n}\sin^n(g(x)), \dots),$$

where $g(x) = \frac{1}{\|c\|_2+1}\sum_{n=1}^{\infty} c_n x_n$ for all $x = (x_1, x_2, \dots) \in \ell_2$. We will see in the next section (Theorem 5.3 and Example 5.4) that T satisfies every conditions in previous theorem. So, T has a unique fixed point and $C^w(T) = X$. Notice that $(0, 0, \dots)$ is clearly a fixed point of T, but it is far from trivial to show directly that T has no other fixed point.

5. Examples in Infinite Dimensional Banach Spaces

We will now give some criteria for a map to be functionally uniformly lipschitzian on an infinite dimensional Banach space. Suppose E an infinite dimensional Banach space having a normalised Schauder basis (e_n). As usual, X denotes a nonempty subset of E, and $T : X \to X$ denotes a selfmap whose $F(T) \ne \emptyset$. Moreover, for a lipschitzian selfmap $h : \mathbb{R} \to \mathbb{R}$, we will use $L(h)$ to denote the Lipschitz constant of h.

Proposition 5.1. *If T is functionally lipschitzian, then $e_n^* \circ T$ is a lipschitzian functional for each $n \in \mathbb{N}$.*

Proof. Since T is functionally lipschitzian, for each $n \in \mathbb{N}$, there are $g_1, g_2, \ldots, g_m \in E^*$ such that for any $x, y \in X$,

$$\begin{aligned} |e_n^* \circ T(x) - e_n^* \circ T(y)| &= \|T(x) - T(y)\|_{e_n^*} \\ &\le \sum_{i=1}^{m} \|(x-y)\|_{g_i} \\ &= \sum_{i=1}^{m} |g_i(x-y)| \\ &\le \Big(\sum_{i=1}^{m} \|g_i\|\Big)\|x-y\|, \end{aligned}$$

which implies that $e_n^* \circ T$ is lipschitzian. □

Lemma 5.2. *Let $g_1, g_2, \ldots, g_N \in E^*$, and for any $k \in \mathbb{N}$, (c_n^k) sequences of non-negative numbers with $\sum_{n=1}^{\infty} c_n^k < \infty$. If for each $n, k \in \mathbb{N}$ and $x, y \in X$,*

$$|e_n^*(T^k(x) - T^k(y))| \le c_n^k \sum_{i=1}^{N} |g_i(x-y)|,$$

then T is functionally uniformly lipschitzian with respect to the sequence (z_k), where $z_k = \sum_{n=1}^{\infty} c_n^k$.

Proof. Let $z_k = \sum_{n=1}^{\infty} c_n^k$.
For each $f \in E^*$ and $x, y \in X$, we have

$$\begin{aligned} \|T^k(x) - T^k(y)\|_f &= \Big|f\Big(\sum_{n=1}^{\infty} e_n^*(T^k(x) - T^k(y))e_n\Big)\Big| \\ &\le \|f\| \sum_{n=1}^{\infty} |e_n^*(T^k(x) - T^k(y))| \\ &\le \|f\| \sum_{n=1}^{\infty} \Big(c_n^k \sum_{i=1}^{N} |g_i(x-y)|\Big) \\ &= \|f\| z_k \sum_{i=1}^{N} |g_i(x-y)| \\ &= z_k \sum_{i=1}^{N} \|x-y\|_{\|f\|g_i}. \end{aligned}$$

□

Theorem 5.3. *Let $g_1, g_2, \ldots, g_N \in E^*$ and $\{h_{n,i} : n \in \mathbb{N}; i = 1, \ldots, N\}$ a collection on lipschitzian selfmaps on $\mathbb{R}$ satisfying $\sum_{n=1}^{\infty} \max\{L(h_{n,i}) : i = 1, \ldots, N\} < \infty$. If for each $n \in \mathbb{N}$*

$$e_n^* \circ T = \sum_{i=1}^{N} h_{n,i} \circ g_i|_X,$$

then T is functionally uniformly lipschitzian with respect to the sequence (z_k), where

$$z_k = \Big[\Big(\sum_{n=1}^{\infty} \max\{L(h_{n,i}) : i = 1, \dots, N\}\Big)\Big(\sum_{i=1}^{N} \|g_i\|\Big)\Big]^{k-1}.$$

Consequently,

(1) *if* $\Big(\sum_{n=1}^{\infty} \max\{L(h_{n,i}) : i = 1, \dots, N\}\Big)\Big(\sum_{i=1}^{N} \|g_i\|\Big) \leq 1$, *then T is functionally uniformly lipschitzian.*
(2) *if* $\Big(\sum_{n=1}^{\infty} \max\{L(h_{n,i}) : i = 1, \dots, N\}\Big)\Big(\sum_{i=1}^{N} \|g_i\|\Big) < 1$, *then T has a unique fixed point and $C^w(T) = X$.*

Proof. Let $B = \sum_{i=1}^{N} \|g_i\|$, $C_n = \max\{L(h_{n,i}) : i = 1, \dots, N\}$ and $C = \sum_{n=1}^{\infty} C_n$.
First notice that, for any $n \in \mathbb{N}$ and $x, y \in X$, we have

$$\begin{aligned}
|e_n^*(T(x) - T(y))| &= \Big|\sum_{i=1}^{N} \Big(h_{n,i} \circ g_i(x) - h_{n,i} \circ g_i(y)\Big)\Big| \\
&\leq \sum_{i=1}^{N} |h_{n,i}(g_i(x)) - h_{n,i}(g_i(y))| \\
&\leq \sum_{i=1}^{N} L(h_{n,i})|g_i(x - y)| \\
&\leq C_n \sum_{i=1}^{N} |g_i(x - y)|.
\end{aligned}$$

Next, we claim that for each $n, k \in \mathbb{N}$ and $x, y \in X$,

$$|e_n^*(T^k(x) - T^k(y))| \leq C_n(BC)^{k-1} \sum_{i=1}^{N} |g_i(x - y)|.$$

When $k = 1$, the statement immediately holds from the previous paragraph. Suppose it is also true for some $k \in \mathbb{N}$, we then have

$$\begin{aligned}
|e_n^*(T^{k+1}(x) - T^{k+1}(y))| &\leq C_n(BC)^{k-1} \sum_{i=1}^{N} |g_i(T(x) - T(y))| \\
&\leq C_n(BC)^{k-1} \Big(\sum_{i=1}^{N} \|g_i\|\Big) \|T(x) - T(y)\| \\
&\leq C_n(BC)^{k-1} B \sum_{n=1}^{\infty} |e_n^*(T(x) - T(y))| \\
&\leq C_n(BC)^{k-1} B \sum_{n=1}^{\infty} \Big(C_n \sum_{i=1}^{N} |g_i(x - y)|\Big) \\
&= C_n(BC)^{k-1} B \Big(\sum_{n=1}^{\infty} C_n\Big) \sum_{i=1}^{N} |g_i(x - y)|
\end{aligned}$$

$$= C_n(BC)^{k-1}BC\sum_{i=1}^{N}|g_i(x-y)|$$
$$= C_n(BC)^{k}\sum_{i=1}^{N}|g_i(x-y)|,$$

which proves the claim.
By the previous Lemma, we have

$$\|T^k(x)-T^k(y)\|_f \le (BC)^{k-1}\sum_{i=1}^{N}\|x-y\|_{C\|f\|g_i}$$

and hence T is functionally uniformly lipschitzian with respect to the sequence (z_k), where $z_k = (BC)^{k-1}$. Therefore, if $BC \le 1$, then (z_k) is bounded and T is functionally uniformly lipschitzian by Remark 4.2. Also, if $BC < 1$, the sequence (z_k) is summable, and hence by Theorem 4.5, T has a unique fixed point and $C^w(T) = X$. □

Example 5.4. Consider the map T given in Example 4.6. By letting $N = 1$, $h_n = \frac{1}{n2^n}\sin^n$, we have $C = \sum_{n=1}^{\infty} L(h_n) \le 1$ and $B = \|g\| < 1$. The previous theorem then implies that T has a unique fixed point and $C^w(T) = X$.

In the final part of this work, we will give another criterion for a map to be functionally uniformly lipschitzian. It will help us, for the sake of completeness, construct a simple selfmap, in the last example, whose fixed point set is not convex and hence guaranteed be a retract of its weak convergence set only by our results.

Lemma 5.5. *Let $l \in \mathbb{N}$, $\{g_{n,i} : n = 1, \ldots, l; i = 1, \ldots, m_n\} \subseteq E^*$ and $a \in \mathbb{R}$. Suppose that $e_n^*(T(x)-T(y)) = ae_n^*(x-y)$ for all $n > l$ and $x, y \in X$.*
If for each $n \le l$, $k \in \mathbb{N}$ and $x, y \in X$, there is $c_k \ge 0$

$$|e_n^*(T^k(x)-T^k(y))| \le c_k \textstyle\sum_{i=1}^{m_n}|g_{n,i}(x-y)|,$$

then T is functionally uniformly lipschitzian with respect to the sequence (z_k), where $z_k = \max\{c_k, |a|^k\}$.

Proof. Let $f \in E^*$, $k \in \mathbb{N}$ and $x, y \in X$. Then

$$\begin{aligned}
\|T^k(x)-T^k(y)\|_f &= |f(T^k(x)-T^k(y))| \\
&= \Big|f\Big(\sum_{n=1}^{\infty} e_n^*(T^k(x)-T^k(y))e_n\Big)\Big| \\
&= \Big|f\Big(\sum_{n\le l} e_n^*(T^k(x)-T^k(y))e_n\Big) + f\Big(\sum_{n>l} e_n^*(T^k(x)-T^k(y))e_n\Big)\Big| \\
&\le \Big|f\Big(\sum_{n\le l} e_n^*(T^k(x)-T^k(y))e_n\Big)\Big| + \Big|f\Big(\sum_{n>l} e_n^*(T^k(x)-T^k(y))e_n\Big)\Big| \\
&\le c_k\|f\|\sum_{n\le l}\sum_{i=1}^{m_n}|g_{n,i}(x-y)| + \Big|f\Big(\sum_{n>l} a^k e_n^*(x-y)e_n\Big)\Big| \qquad (**)
\end{aligned}$$

$$= c_k\|f\| \sum_{n\le l}\sum_{i=1}^{m_n} |g_{n,i}(x-y)| + \Big|f\Big(a^k(x-y) - \sum_{n\le l} a^k e_n^*(x-y)e_n\Big)\Big|$$

$$\le c_k\|f\| \sum_{n\le l}\sum_{i=1}^{m_n} |g_{n,i}(x-y)| + |a^k f(x-y)| + \Big|f\Big(\sum_{n\le l} a^k e_n^*(x-y)e_n\Big)\Big|$$

$$\le c_k\|f\| \sum_{n\le l}\sum_{i=1}^{m_n} |g_{n,i}(x-y)| + |a^k f(x-y)| + \sum_{n\le l} |a^k f(e_n)e_n^*(x-y)|$$

$$\le \max\{c_k, |a|^k\}\Big(\sum_{n\le l}\sum_{i=1}^{m_n} \|x-y\|_{\|f\|g_{n,i}} + \|x-y\|_f + \sum_{n\le l}\|x-y\|_{|f(e_n)|e_n^*}\Big).$$

Therefore, T is functionally uniformly lipschitzian with respect to the sequence (z_k), where $z_k = \max\{c_k, |a|^k\}$. □

Theorem 5.6. *Let $l \in \mathbb{N}$; $g_1, \dots, g_l \in E^*$; $h_1, \dots, h_l$ lipschitzian selfmaps on $\mathbb{R}$ and $a, b_{l+1}, b_{l+2}, \dots \in \mathbb{R}$ satisfying $|a| \le 1$ and $\sum_{n=1}^{l} L(h_n)\|g_n\| < 1$.*

If for each $n \in \mathbb{N}$,

$$e_n^* \circ T = \begin{cases} h_n \circ g_n|_X & ; n \le l, \\ ae_n^* + b_n & ; n > l, \end{cases}$$

then T is functionally uniformly lipschitzian and hence weakly uniformly virtually stable.

Proof. Let $R = \sum_{n=1}^{l} L(h_n)\|g_n\|$, $c_1 = 1$, $c_k = \max\{R^{k-2}, |a|^{k-1}, \sum_{j=1}^{k-2}|a|^{k-j-1}R^{j-1}\}$ for any $k \ge 2$ and $x, y \in X$. Also, to save some spaces, we will simply write L_n instead of $L(h_n)$. From (**) and $k = 1$ in the previous lemma, we have for any $f \in E^*$,

$$\|T(x) - T(y)\|_f \le \|f\| \sum_{n\le l} L_n|g_n(x-y)| + \Big|f\Big(\sum_{n>l} ae_n^*(x-y)e_n\Big)\Big|,$$

and, in particular, for each $n = 1, \dots, l$,

$$|g_n(T(x) - T(y))| \le \|g_n\| \sum_{i\le l} L_i|g_i(x-y)| + \Big|g_n\Big(\sum_{i>l} ae_i^*(x-y)e_i\Big)\Big|.$$

For each $n = 1, \dots, l$, we claim that for all $k \ge 2$,

$$\begin{aligned}|e_n^*(T^k(x) - T^k(y))| \le{}& L_n\|g_n\|R^{k-2}\sum_{i\le l} L_i|g_i(x-y)| \\ &+ |a|^{k-1}L_n\Big|g_n\Big(\sum_{i>l} e_i^*(x-y)e_i\Big)\Big| \\ &+ L_n\|g_n\|\Big(\sum_{j=1}^{k-2}|a|^{k-j-1}R^{j-1}\Big)\sum_{i\le l} L_i\Big|g_i\Big(\sum_{j>l} e_j^*(x-y)e_j\Big)\Big|.\end{aligned}$$

Once the claim is proved, we will have for each $k \ge 2$

$$|e_n^*(T^k(x) - T^k(y))| \le c_k\Big\{L_n\|g_n\|\sum_{i\le l} L_i|g_i(x-y)| + L_n\Big|g_n\Big(\sum_{i>l} e_i^*(x-y)e_i\Big)\Big|$$

$$
\begin{aligned}
&+ L_n\|g_n\|\sum_{i\le l} L_i\Big|g_i\Big(\sum_{j>l} e_j^*(x-y)e_j\Big)\Big|\Big\}\\
&= c_k\Big\{L_n\|g_n\|\sum_{i\le l} L_i|g_i(x-y)| + L_n\Big|g_n\Big((x-y)-\sum_{i\le l} e_i^*(x-y)e_i\Big)\Big|\\
&+ L_n\|g_n\|\sum_{i\le l} L_i\Big|g_i\Big((x-y)-\sum_{j\le l} e_j^*(x-y)e_j\Big)\Big|\Big\}\\
&\le c_k\Big\{L_n\|g_n\|\sum_{i\le l} L_i|g_i(x-y)| + L_n|g_n(x-y)| + L_n\sum_{i\le l}|g_n(e_i)e_i^*(x-y)|\\
&+ L_n\|g_n\|\sum_{i\le l} L_i|g_i(x-y)| + L_n\|g_n\|\sum_{i\le l}\sum_{j\le l} L_i|g_n(e_j)e_j^*(x-y)|\Big\}.
\end{aligned}
$$

Also, since $|e_n^*(T(x)-T(y))| \le c_1 L_n|g_n(x-y)|$, the above inequality holds for all $k \in \mathbb{N}$. Therefore, T is functionally uniformly lipschitzian with respect to the sequence (z_k) where $z_k = \max\{c_k, |a|^k\}$. If $|a| \le 1$ and $R < 1$, then (c_k) is bounded, so is (z_k). Therefore T is functionally uniformly lipschitzian by the previous lemma.

To prove the claim, it is clear that

$$
\begin{aligned}
|e_n^*(T^2(x)-T^2(y))| &\le L_n|g_n(T(x)-T(y))|\\
&\le L_n\Big(\|g_n\|\sum_{i\le l} L_i|g_i(x-y)| + \Big|g_n\Big(\sum_{i>l} ae_i^*(x-y)e_i\Big)\Big|\Big)\\
&= L_n\|g_n\|\sum_{i\le l} L_i|g_i(x-y)| + |a|L_n\Big|g_n\Big(\sum_{i>l} e_i^*(x-y)e_i\Big)\Big|.
\end{aligned}
$$

Now, assume that the claim holds for some $k \ge 2$. Then

$$
\begin{aligned}
|e_n^*(T^{k+1}(x)-T^{k+1}(y))| &\le L_n\|g_n\|R^{k-2}\sum_{i\le l} L_i|g_i(T(x)-T(y))|\\
&+ |a|^{k-1}L_n\Big|g_n\Big(\sum_{i>l} e_i^*(T(x)-T(y))e_i\Big)\Big|\\
&+ L_n\|g_n\|\Big(\sum_{j=1}^{k-2}|a|^{k-j-1}R^{j-1}\Big)\sum_{i\le l} L_i\Big|g_i(\sum_{j>l} e_j^*(T(x)-T(y))e_j)\Big|\\
&\le L_n\|g_n\|R^{k-2}\sum_{i\le l} L_i\Big[\|g_i\|\sum_{j\le l} L_j|g_j(x-y)| + \Big|g_i\Big(\sum_{j>l} ae_j^*(x-y)e_j\Big)\Big|\Big]\\
&+ |a|^k L_n\Big|g_n\Big(\sum_{i>l} e_i^*(x-y)e_i\Big)\Big|\\
&+ L_n\|g_n\|\Big(\sum_{j=1}^{k-2}|a|^{k-j}R^{j-1}\Big)\sum_{i\le l} L_i\Big|g_i(\sum_{j>l} e_j^*(x-y)e_j)\Big|\\
&\le L_n\|g_n\|R^{k-2}\Big(\sum_{i\le l} L_i\|g_i\|\Big)\sum_{j\le l} L_j|g_j(x-y)|
\end{aligned}
$$

$$
\begin{aligned}
&+ L_n\|g_n\||a|R^{k-2}\sum_{i\le l} L_i\Big|g_i\Big(\sum_{j>l} e_j^*(x-y)e_j\Big)\Big| \\
&+ |a|^k L_n\Big|g_n\Big(\sum_{i>l} e_i^*(x-y)e_i\Big)\Big| \\
&+ L_n\|g_n\|\Big(\sum_{j=1}^{k-2}|a|^{k-j}R^{j-1}\Big)\sum_{i\le l} L_i\Big|g_i(\sum_{j>l} e_j^*(x-y)e_j)\Big| \\
= {}& L_n\|g_n\|R^{k-2}R\sum_{i\le l} L_i|g_i(x-y)| + |a|^k L_n\Big|g_n\Big(\sum_{i>l} e_i^*(x-y)e_i\Big)\Big| \\
&+ L_n\|g_n\|\Big(|a|R^{k-2} + \sum_{j=1}^{k-2}|a|^{k-j}R^{j-1}\Big)\sum_{i\le l} L_i\Big|g_i(\sum_{j>l} e_j^*(x-y)e_j)\Big| \\
= {}& L_n\|g_n\|R^{k-1}\sum_{i\le l} L_i|g_i(x-y)| + |a|^k L_n\Big|g_n\Big(\sum_{i>l} e_i^*(x-y)e_i\Big)\Big| \\
&+ L_n\|g_n\|\Big(\sum_{j=1}^{k-1}|a|^{k-j}R^{j-1}\Big)\sum_{i\le l} L_i\Big|g_i(\sum_{j>l} e_j^*(x-y)e_j)\Big|,
\end{aligned}
$$

and the claim is proved by induction. □

Example 5.7. Let $X = \Big\{(x_n) \in \ell_2 : |x_1|, |x_2| \le 10 \quad \text{and for any } i \ge 3, \quad |x_i| \le \frac{10}{2^{i-2}}\Big\}$ and fix $c = (10, 10, 15, 4, 8, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots) \in \ell_2$. Notice that X is weakly-compact and convex. Consider $T : X \to X$ defined by

$$T(x_1, x_2, \dots) = (\sin(g(x)), \cos(g(x)), x_3, x_4, \dots),$$

where $g(x) = \frac{1}{2\|c\|_2+1}\sum_{n=1}^{\infty} c_n x_n$ for all $x = (x_1, x_2, \dots) \in \ell_2$. By letting $l = 2$, $g_1 = g_2 = g$, $h_1 = \sin$, $h_2 = \cos$, $a = 1$ and $0 = b_3 = b_4 = \dots$, we have $\|g_1\| = \|g_2\| = \|g\| = \frac{\|c\|_2}{2\|c\|_2+1}$ and $L(h_1)\|g_1\| + L(h_2)\|g_2\| < 1$. By the previous theorem, T is weakly uniformly virtually stable and hence $F(T)$ is a retract of $C^w(T)$. Notice also that $x = (0, 1, 0, 0, -\frac{5}{4}, 0, \dots)$, $y = (1, 0, \frac{(2\|c\|_2+1)\pi}{30}, -\frac{5}{2}, 0, \dots) \in F(T)$ but $\frac{1}{2}x + \frac{1}{2}y = (\frac{1}{2}, \frac{1}{2}, \frac{(2\|c\|_2+1)\pi}{60}, -\frac{5}{4}, -\frac{5}{8}, 0, \dots) \notin F(T)$; i.e., $F(T)$ is not convex. Therefore, since ℓ_2 is uniformly convex, T is not nonexpansive.

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