Journal of Nonlinear and Convex Analysis
Volume 15, Number 4, 2014, 637–645

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

THEOREMS OF DENJOY-WOLFF TYPE FOR FAMILIES OF HOLOMORPHIC RETRACTS

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Abstract. Using the Kobayashi distance k_D, we establish theorems of Denjoy-Wolff type for certain families of holomorphic and k_D-nonexpansive retracts of a bounded and strictly convex domain D in a complex and reflexive Banach space.

1. Introduction

Using the horosphere technique and applying those properties of the Kobayashi distance k_D which are closely connected with convexity, we prove in this paper theorems of Denjoy-Wolff type for certain families of holomorphic and k_D-nonexpansive retracts of a bounded and strictly convex domain D in a complex and reflexive Banach space X (see Theorems 5.1 and 5.4 below). Theorem 5.1, our main result, and its corollaries extend several results in [11], where either $X = \mathbb{C}^k$ or $D = B$, the open unit ball of X.

Our paper is organized as follows. In the next section we recall several properties of the Kobayashi distance k_D, where D is a bounded and convex domain in a complex Banach space. In Section 3 we briefly discuss k_D-nonexpansive mappings and k_D-nonexpansive retracts. In Section 4 we introduce a new kind of horospheres and study their properties. Section 5 is devoted to families of k_D-nonexpansive retracts. In this last section we first use the results of Section 4 to establish Theorem 5.1 and then derive a few of its consequences.

2. The Kobayashi distance and its properties

Let $(X, \|\cdot\|)$ be a complex Banach space, $D \subset X$ be a bounded and convex domain, and let k_D be the Kobayashi distance in D. It is well known that in the case of the open unit disc Δ in the complex plane $\mathbb{C}$, the Kobayashi distance k_Δ coincides with the Poincaré distance on Δ ([26], [27], [28], [35], [16], and also [3], [22] and [36]). It is also known that the Kobayashi distance k_D is locally equivalent to the norm $\|\cdot\|$ in X [21]. We also take note of the following result.

Lemma 2.1 ([24], [32], [34]). *Let D be a bounded and convex domain in a complex Banach space $(X, \|\cdot\|)$.*

2010 *Mathematics Subject Classification.* 32A10, 32F45, 32H02, 46G20, 46T25, 47H09, 47H10, 54E25.

Key words and phrases. Fixed point, holomorphic mapping, holomorphic retract, horosphere, k_D-nonexpansive mapping, k_D-nonexpansive retract, Kobayashi distance.

The first two authors were partially supported by the Polish MNiSW Grant N N201 393737. The third author was partially supported by the Israel Science Foundation (Grant 389/12), the Fund for the Promotion of Research at the Technion, and by the Technion General Research Fund.

(i) *If* $x, y, w, z \in D$ *and* $s \in [0,1]$*, then*

$$k_D\left(sx + (1-s)\,y, sw + (1-s)\,z\right) \le \max\left[k_D\left(x,w\right), k_D\left(y,z\right)\right];$$

(ii) *if* $x, y \in D$ *and* $s, t \in [0,1]$*, then*

$$k_D\left(sx + (1-s)\,y, tx + (1-t)\,y\right) \le k_D\left(x,y\right).$$

In order to recall a characterization of k_D-bounded sets, we need the following notion. Let D be a bounded and convex domain in a complex Banach space $(X, \|\cdot\|)$. A nonempty subset C of D is said to lie *strictly inside* D if

$$\operatorname{dist}_{\|\cdot\|}\left(C, \partial D\right) := \inf\{\|x-y\| : x \in C,\ y \in \partial D\} > 0.$$

It is known [21] that for such a domain D, a nonempty subset C of D is k_D-bounded if and only if C lies strictly inside D.

The following concept of a complex geodesic will play a key role in our considerations.

Definition 2.2 ([15], [38])**.** Let D be a bounded and convex domain in a complex Banach space $(X, \|\cdot\|)$ and let Δ be the open unit disc in the complex plane $\mathbb{C}$. A holomorphic mapping $\phi : \Delta \to D$ is a *complex geodesic* (with respect to k_D) if there exist points $z \neq w$ in Δ such that

$$k_\Delta(w,z) = k_D(\phi(w), \phi(z)).$$

In this case we say that $\phi(w)$ and $\phi(z)$ are joined by a complex geodesic. If, moreover, $w = 0$ and $0 < z \in \mathbb{R}$, we call ϕ a *normalized complex geodesic* joining $\phi(w)$ with $\phi(z)$.

We proceed with the following definition.

Definition 2.3 ([15], [19])**.** We say that a bounded and convex domain D in a complex Banach space $(X, \|\cdot\|)$ is *strictly convex* if for every $x, y \in \overline{D}^{\|\cdot\|}$, the open segment

$$(x,y) = \{z \in X : z = sx + (1-s)\,y \quad \text{for some } 0 < s < 1\}$$

lies in D.

Using strict convexity, one can prove the following very useful theorem and lemma.

Theorem 2.4 ([15])**.** *Let D be a bounded and strictly convex domain in a complex and reflexive Banach space $(X, \|\cdot\|)$. Then any pair of distinct points in D can be joined by a unique normalized k_D-geodesic.*

Lemma 2.5 ([11], [24])**.** *Let D be a bounded and strictly convex domain in a complex Banach space $(X, \|\cdot\|)$. Let $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$ be two nets in D which converge in norm to $\xi \in \partial D$ and to $\eta \in \overline{D}^{\|\cdot\|}$, respectively. If*

$$\sup\left\{k_D\left(x_j, y_j\right) : j \in J\right\} = c < \infty,$$

then $\xi = \eta$.

3. k_D-nonexpansive mappings and k_D-nonexpansive retracts

If D_1 and D_2 are bounded domains in the complex Banach spaces $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$, respectively, and k_{D_1} and k_{D_2} are the Kobayashi distances in D_1 and D_2, respectively, then each holomorphic $f : D_1 \to D_2$ is nonexpansive, that is,

$$k_{D_2}(f(x), f(y)) \leq k_{D_1}(x, y)$$

for all $x, y \in D_1$ [21] (see also [15], [18], [20], [22] and [37]).

In particular, if D is a bounded domain in a complex Banach space $(X, \|\cdot\|)$, then each holomorphic $f : D \to D$ is k_D-nonexpansive (see [20] and [31]).

Definition 3.1. Let D be a bounded and convex domain in a complex Banach space $(X, \|\cdot\|)$. If $\emptyset \neq F \subset D$ and there exists a k_D-nonexpansive (holomorphic) *retraction* of D onto F, then we say that F is a k_D-nonexpansive (holomorphic) *retract* of D.

The following theorem will be applied in the proof of our main theorem.

Theorem 3.2 ([14], [11])**.** *Let D be a bounded and strictly convex domain in a complex reflexive Banach space $(X, \|\cdot\|)$. Let $\mathcal{F}$ be a family of holomorphic (k_D-nonexpansive) retracts of D. If $F = \bigcap_{\tilde{F} \in \mathcal{F}} \tilde{F} \neq \emptyset$, then F is a holomorphic (k_D-nonexpansive) retract of D.*

Recall that, using the Bruck method ([7] and [8]), M. Budzyńska, T. Kuczumow and A. Stachura established the following result (see also [2], [6] and [30]).

Theorem 3.3 ([13])**.** *Let D be a bounded and strictly convex domain in a complex and reflexive Banach space $(X, \|\cdot\|)$. Then, for every family $\mathcal{F}$ of commuting holomorphic (k_D-nonexpansive) self-mappings of D with a nonempty common fixed point set $Fix(\mathcal{F})$, this set $Fix(\mathcal{F})$ is a holomorphic (k_D-nonexpansive) retract of D.*

4. Horospheres

The main tool which we use in the proof of our main result is the newly defined horosphere $\tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma \in \Gamma}\right)$, which is introduced in the following way (for other types of horospheres and their applications see [1], [4], [9], [10], [11], [12], [23], [24], [29], [31] and [32]).

Definition 4.1. Let D be a bounded and convex domain in a complex Banach space $(X, \|\cdot\|)$. Let $\{\gamma\}_{\gamma \in \Gamma}$ be an ultranet, $x \in D$, $\xi \in \partial D$, $R > 0$, $x_\gamma \in D$ for each $\gamma \in \Gamma$, and assume that $\lim_{\gamma \in \Gamma} x_\gamma = \xi$ in $(X, \|\cdot\|)$. The *horosphere* $\tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma \in \Gamma}\right)$ in D is defined as follows:

$$\tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma \in \Gamma}\right) := \left\{ y \in D : \lim_{\gamma \in \Gamma} \left[k_D\left(y, x_\gamma\right) - k_D\left(x, x_\gamma\right)\right] < \frac{1}{2} \log R \right\}.$$

We collect several properties of these horospheres $\tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma \in \Gamma}\right)$ in the following theorem (for the ideas of the proof of this theorem see [1], [4] and [9]).

Theorem 4.2 ([10])**.** *Let D be a bounded and convex domain in a complex and reflexive Banach space $(X, \|\cdot\|)$. Let $\{\gamma\}_{\gamma\in\Gamma}$ be an ultranet, $x \in D$, $\xi \in \partial D$, $R > 0$, $x_\gamma \in D$ for each $\gamma \in \Gamma$, and assume that $\lim_{\gamma\in\Gamma} x_\gamma = \xi$.*

Then the horospheres $\tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right)$ have the following properties:

(i) *if the horosphere $\tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right)$ is nonempty, then it is convex;*

(ii) *for each $0 < R_1 < R_2$, we have*

$$\left[\overline{\tilde{H}\left(x, \xi, R_1, \{x_\gamma\}_{\gamma\in\Gamma}\right)}^{\|\cdot\|} \cap D\right] \subset \tilde{H}\left(x, \xi, R_2, \{x_\gamma\}_{\gamma\in\Gamma}\right);$$

(iii) *if $x, \tilde{x} \in D$ and $\lim_{\gamma\in\Gamma}\left[k_D(\tilde{x}, x_\gamma) - k_B(x, x_\gamma)\right] < \frac{1}{2}\log L$, then*

$$\tilde{H}\left(\tilde{x}, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right) \subset \tilde{H}\left(x, \xi, LR, \{x_\gamma\}_{\gamma\in\Gamma}\right);$$

(iv) *for each $R > 1$, we have $B\left(x, \frac{1}{2}\log R\right) \subset \tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right)$, where $B\left(x, \frac{1}{2}\log R\right)$ is a ball in the metric space (D, k_D);*

(v) *for each $R < 1$, we have $B\left(x, -\frac{1}{2}\log R\right) \cap \tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right) = \emptyset$, where $B\left(x, -\frac{1}{2}\log R\right)$ is a ball in the metric space (D, k_D);*

(vi) $\bigcup_{R>0} \tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right) = D$ *and* $\bigcap_{R>0} \tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right) = \emptyset$*;*

(vii) $\bigcap_{R>0} \overline{\tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right)}^{\|\cdot\|} \subset \partial D$*;*

(viii) *if D is strictly convex and each horosphere is nonempty, then*

$$\bigcap_{R>0} \overline{\tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right)}^{\|\cdot\|} = \{\xi\}.$$

Observe that directly from (iii) it follows that if for some point $\tilde{x} \in D$ all horospheres $\tilde{H}\left(\tilde{x}, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right)$ are nonempty, then the same is true for each point $x \in D$. We do not know, however, whether the horosphere $\tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right)$ is nonempty for each $R > 0$. It is known that this does hold for open unit balls in complex Banach spaces (see the proofs in [24], [29], [31] and [32]; see also [9] for the case of bounded and convex domains in $\mathbb{C}^k$). Now we prove this fact in the special case where the ultranet $\{x_\gamma\}_{\gamma\in\Gamma}$ is connected with a compact k_D-nonexpansive retraction r of D.

Theorem 4.3. *Let D be a bounded and convex domain in a complex reflexive Banach space $(X, \|\cdot\|)$. Let $r : D \to F$ be a compact and k_D-nonexpansive retraction onto F, where $\emptyset \neq F \subset D$. Let $\{\gamma\}_{\gamma\in\Gamma}$ be an ultranet and let a net $\{x_\gamma\}_{\gamma\in\Gamma}$ in D be such that*

$$\lim_{\gamma\in\Gamma} x_\gamma = \xi \in \partial D$$

and

$$r(x_\gamma) = x_\gamma$$

for each $\gamma \in \Gamma$. *Then the horosphere* $\tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right)$ *is nonempty for each* $x \in D$ *and for each* $R > 0$.

Proof. Fix $x \in D$. By Theorem 2.4, for each $w \in D$, where $w \neq x$, and for each $0 < \tilde{\alpha} < k_D(w, x)$, there exists a point $y_{w,x,\tilde{\alpha}}$ in D such that

$$k_D(w, x) = k_D(w, y_{w,x,\tilde{\alpha}}) + k_D(y_{w,x,\tilde{\alpha}}, x)$$

and

$$k_D(y_{w,x,\tilde{\alpha}}, x) = \tilde{\alpha}.$$

Now fix $0 < R < 1$ and let $\alpha > -\frac{1}{2}\log R$. Since $\lim_{\gamma\in\Gamma} x_\gamma = \xi \in \partial D$, we may assume, without loss of generality, that $0 < \alpha < k_D(x_\gamma, \tilde{x})$ for each $\gamma \in \Gamma$. Then for each $y_{x_\gamma,x,\alpha}$, we have

$$k_D(y_{x_\gamma,x,\alpha}, x) = \alpha$$

and

$$k_D(x_\gamma, x) = k_D(x_\gamma, y_{x_\gamma,x,\alpha}) + k_D(y_{x_\gamma,x,\alpha}, x) = k_D(x_\gamma, y_{x_\gamma,x,\alpha}) + \alpha.$$

Set

$$z_{x_\gamma,x,\alpha} = r(y_{x_\gamma,x,\alpha})$$

for each $\gamma \in \Gamma$. Since

$$k_D(z_{x_\gamma,x,\alpha}, r(x)) = k_D(r(y_{x_\gamma,x,\alpha}), r(x)) \leq k_D(y_{x_\gamma,x,\alpha}, x) = \alpha,$$

the ultranet $\{z_{x_\gamma,x,\alpha}\}_{\gamma\in\Gamma}$ lies strictly inside D and in the compact set $\overline{r(D)}^{\|\cdot\|} = \overline{F}^{\|\cdot\|}$. Therefore this ultranet $\{z_{x_\gamma,x,\alpha}\}_{\gamma\in\Gamma}$ is convergent in the norm $\|\cdot\|$ to a limit point $z_{x,\alpha} \in D$. Hence we get

$$\lim_{\gamma\in\Gamma} k_D(z_{x_\gamma,x,\alpha}, z_{x,\alpha}) = 0$$

and

$$\begin{aligned}\lim_{\gamma\in\Gamma}[k_D(z_{x,\alpha}, x_\gamma) - k_D(x, x_\gamma)] &= \lim_{\gamma\in\Gamma}[k_D(z_{x_\gamma,x,\alpha}, x_\gamma) - k_D(x, x_\gamma)] \\ &= \lim_{\gamma\in\Gamma}[k_D(r(y_{x_\gamma,x,\alpha}), r(x_\gamma)) - k_D(x, x_\gamma)] \\ &\leq \lim_{\gamma\in\Gamma} k_D[(y_{x_\gamma,x,\alpha}, x_\gamma) - k_D(x, x_\gamma)] = -\alpha < \frac{1}{2}\log R.\end{aligned}$$

Therefore $z_{x,\alpha} \in \tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right)$ and so the horosphere $\tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right)$ is indeed not empty, as asserted. □

5. Families of k_D-nonexpansive retracts

Before stating our main theorem, we recall that the Hausdorff distance (with respect to the norm in a Banach space $(X, \|\cdot\|)$) between a set A and a point b is given by $\mathrm{Dist}_{\|\cdot\|}(A, b) := \sup\{\|x - b\| : x \in A\}$. Now we are ready to state and prove the main theorem of our paper. It extends Theorems 9.4 and 10.2 in [11], where $X = \mathbb{C}^k$ and $D = B$, the open unit ball of X, respectively.

Theorem 5.1. *Let D be a bounded and strictly convex domain in a complex and reflexive Banach space $(X, \|\cdot\|)$, and let $\{F_i\}_{i\in I}$ be a family of k_D-nonexpansive retracts of D such that $\bigcap_{j\in J} F_j \neq \emptyset$ for each finite set $\emptyset \neq J \subset I$ of indices. If $\bigcap_{i\in I} F_i = \emptyset$ and there exists a nonempty finite set $\emptyset \neq \tilde{J} \subset I$ of indices such that $\overline{\bigcap_{j\in\tilde{J}} F_j}^{\|\cdot\|}$ is compact, then there exists a point $\xi \in \partial D$ such that $\lim_J \mathrm{Dist}_{\|\cdot\|}(\bigcap_{j\in J} F_j, \xi) = 0$, where the family $\{J : \emptyset \neq J \subset I$ and J is a finite set$\}$ of all nonempty finite sets of indices is partially ordered by inclusion.*

Proof. Since the family $\{J : \emptyset \neq J \subset I$ and J is a finite set$\}$ of all nonempty finite sets of indices is partially ordered by inclusion, without any loss of generality we may consider the family $\{J\}_{J\geq\tilde{J}}$ in place of the whole family. Let $\{\gamma\}_{\gamma\in\Gamma}$ be an ultranet such that $\{J_\gamma\}_{\gamma\in\Gamma}$ is a subnet of $\{J\}_{J\geq\tilde{J}}$. The subnet $\{J_\gamma\}_{\gamma\in\Gamma}$ is also an ultranet. These considerations imply that in place of $\{F_i\}_{i\in I}$ we may take $\{\widehat{F}_\gamma\}_{\gamma\in\Gamma} = \{\bigcap_{j\in J_\gamma} F_j\}_{\gamma\in\Gamma}$. Each $\overline{\widehat{F}_\gamma}^{\|\cdot\|}$ is compact and by Theorem 3.2, each $\widehat{F}_\gamma$ is also a k_D-nonexpansive retract of D and $\bigcap_{\gamma\in\Gamma} \widehat{F}_\gamma = \emptyset$. Now we choose $x_\gamma \in \widehat{F}_\gamma$ for each $\gamma \in \Gamma$. Then the ultranet $\{x_\gamma\}_{\gamma\in\Gamma}$ is convergent to ξ and its limit point ξ lies on the boundary ∂D of the domain D. This holds true because $\bigcap_{\gamma\in\Gamma} \widehat{F}_\gamma = \emptyset$, and all retracts are closed in (D, k_D) and relatively compact in $(X, \|\cdot\|)$. Next, we choose $x \in D$ and consider the horosphere $\tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right)$, which is nonempty by Theorem 4.3. For each $\gamma \in \Gamma$, let $\widehat{r}_\gamma$ be a k_D-nonexpansive retraction of D onto $\widehat{F}_\gamma$. Then by the inequality

$$\lim_{\gamma\in\Gamma}\left[k_D\left(\widehat{r}_{\gamma'}(y), x_\gamma\right) - k_D\left(x, x_\gamma\right)\right] \leq \lim_{\gamma\in\Gamma}\left[k_D\left(y, x_\gamma\right) - k_D\left(x, x_\gamma\right)\right] < \frac{1}{2}\log R,$$

which holds for each $y \in \tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right)$ and $\gamma' \in \Gamma$, each horosphere

$$\tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right)$$

is nonempty, convex, and $\widehat{r}_{\gamma'}$-invariant for each $\gamma' \in \Gamma$. Now we fix $z \in D$ and let $\xi(z) \in \partial D$ be the limit point in the norm $\|\cdot\|$ of the ultranet $\{\widehat{r}_\gamma(z)\}_{\gamma\in\Gamma}$. Observe that for each $w \in D$ and $\gamma \in \Gamma$, we have

$$k_D\left(\widehat{r}_\gamma(z), \widehat{r}_\gamma(w)\right) \leq k_D(z, w).$$

Hence, applying Lemma 2.5, we see that the point $\xi(z)$ is independent of the choice of $z \in D$, say $\xi_0 = \xi(z)$ for all $z \in D$. Next, by Theorem 4.2, we obtain

$$\{\xi_0\} \subset \partial D \cap \bigcap_{R>0} \overline{\tilde{H}\left(x, \xi, R, \{x_\gamma\}_{\gamma\in\Gamma}\right)}^{\|\cdot\|} = \{\xi\}.$$

Thus $\xi = \xi_0$ and this means that $\lim_{\gamma\in\Gamma} \widehat{r}_\gamma(w) = \xi$ for each $w \in D$. Finally, observe that each ultranet $\{\tilde{x}_\gamma\}_{\gamma\in\Gamma}$, where $\tilde{x}_\gamma \in \widehat{F}_\gamma$ for $\gamma \in \Gamma$, can play the role of the above ultranet $\{x_\gamma\}_{\gamma\in\Gamma}$. This implies that $\lim_{\gamma\in\Gamma} \mathrm{Dist}_{\|\cdot\|}(\widehat{F}_\gamma, \xi) = 0$ and this leads to our asserted result, namely, $\lim_J \mathrm{Dist}_{\|\cdot\|}(\bigcap_{j\in J} F_j, \xi) = 0$. □

Directly from Theorem 5.1 we deduce the following corollaries.

Corollary 5.2. *Let D be a bounded and strictly convex domain in a complex and reflexive Banach space $(X, \|\cdot\|)$, and let $\{F_i\}_{i\in I}$ be a family of k_D-nonexpansive retracts of D such that $\bigcap_{j\in J} F_j \neq \emptyset$ for each finite set $\emptyset \neq J \subset I$ of indices. If $\bigcap_{i\in I} F_i = \emptyset$ and there exists a nonempty finite set $\emptyset \neq \tilde{J} \subset I$ of indices such that $\overline{\bigcap_{j\in \tilde{J}} F_j}^{\|\cdot\|}$ is compact, then there exists a point $\xi \in \partial D$ such that the net of k_D-nonexpansive retractions $\{r_J\}_J$ converges uniformly on D to the constant map taking the value ξ, where the family $\{\emptyset \neq J \subset I\}$ of all nonempty finite sets J of indices is partially ordered by inclusion and r_J is a k_D-nonexpansive retraction of D onto the intersection $\bigcap_{j\in J} F_j$ for each J.*

Corollary 5.3. *Let D be a bounded and strictly convex domain in a complex and reflexive Banach space $(X, \|\cdot\|)$, and let $\{F_i\}_{i\in I}$ be a family of k_D-nonexpansive retracts of D such that $\bigcap_{j\in J} F_j \neq \emptyset$ for each finite set $\emptyset \neq J \subset I$ of indices. If $\bigcap_{i\in I} F_i = \emptyset$ and there exists a nonempty finite set $\emptyset \neq \tilde{J} \subset I$ of indices such that $\overline{\bigcap_{j\in \tilde{J}} F_j}^{\|\cdot\|}$ is compact, then $\lim_J \operatorname{diam}_{\|\cdot\|}(\bigcap_{j\in J} F_j) = 0$, where the family $\{\emptyset \neq J \subset I\}$ of all nonempty finite sets J of indices is partially ordered by inclusion.*

Finally, applying Theorem 3.3, we obtain our last two results.

Theorem 5.4. *Let D be a bounded and strictly convex domain in a complex and reflexive Banach space $(X, \|\cdot\|)$, and let $\mathcal{F} = \{f_i\}_{i\in I}$ be a commuting family of k_D-nonexpansive self-mappings of D such that $\bigcap_{j\in J} Fix(f_j) \neq \emptyset$ for each finite set $\emptyset \neq J \subset I$ of indices. If the family $\mathcal{F}$ does not have a common fixed point in D and there exists a nonempty finite set $\emptyset \neq \tilde{J} \subset I$ of indices such that $\overline{\bigcap_{j\in \tilde{J}} Fix(f_j)}^{\|\cdot\|}$ is compact, then there exists a point $\xi \in \partial D$ such that $\lim_J \operatorname{Dist}_{\|\cdot\|}(\bigcap_{j\in J} Fix(f_j), \xi) = 0$, where each $\emptyset \neq J \subset I$ is a nonempty finite set of indices and the family of all such sets is partially ordered by inclusion.*

Corollary 5.5. *Let D be a bounded and strictly convex domain in a complex and reflexive Banach space $(X, \|\cdot\|)$, and let $\mathcal{F} = \{f_i\}_{i\in I}$ be a commuting family of k_D-nonexpansive self-mappings of D such that $\bigcap_{j\in J} Fix(f_j) \neq \emptyset$ for each nonempty finite set $\emptyset \neq J \subset I$ of indices. If the family $\mathcal{F}$ does not have a common fixed point in D and there exists a nonempty finite set $\emptyset \neq \tilde{J} \subset I$ of indices such that $\overline{\bigcap_{j\in \tilde{J}} Fix(f_j)}^{\|\cdot\|}$ is compact, then $\lim_J \operatorname{diam}_{\|\cdot\|}(\bigcap_{j\in J} Fix(f_i)) = 0$, where each $\emptyset \neq J \subset I$ is a nonempty finite set of indices and the family of all such sets is partially ordered by inclusion.*

Acknowledgments

Part of this research was carried out when the first two authors were visiting the Technion. They are grateful to their colleagues for their kind hospitality.

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Manuscript received October 2, 2013
revised October 25, 2013

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