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DYNAMIC STRING-AVERAGING PROJECTION METHODS FOR CONVEX FEASIBILITY PROBLEMS IN THE PRESENCE OF COMPUTATIONAL ERRORS

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Abstract. In the present paper we study convergence of dynamic string-averaging projection methods for solving convex feasibility problems in a Hilbert space. Our goal is to obtain an approximate solution of the problem in the presence of computational errors. We show that our dynamic string-averaging projection algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant.

1. Introduction

In this paper we study the convergence behavior of a class of projection methods which are an important tool for solving convex feasibility problems [1-19, 21-30]. Our goal is to obtain an ϵ-approximate solution of the problem in the presence of computational errors, where ϵ is a small positive number. We apply a dynamic string-averaging projection methods for solving convex feasibility problems and show that this algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant. Clearly, in practice it is sufficient to find a good approximate solution instead of constructing a minimizing sequence. On the other hand, in practice computations induce numerical errors and if one uses methods in order to solve minimization problems or feasibility problems these methods usually provide only approximate solutions of the problems.

Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ which induces a complete norm $\|\cdot\|$.

For each $x \in X$ and each nonempty set $E \subset X$ put

$$d(x, E) = \inf\{\|x - y\| : \ y \in E\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \le r\}.$$

It is well-known that the following proposition holds.

Proposition 1.1. *Let D be a nonempty, closed and convex subset of X. Then for each $x \in X$ there is a unique point $P_D(x) \in D$ satisfying*

$$\|x - P_D(x)\| = \inf\{\|x - y\| : \ y \in D\}.$$

Moreover,

$$\|P_D(x) - P_D(y)\| \le \|x - y\| \text{ for all } x, y \in X$$

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and for each $x \in X$ *and each* $z \in D$,

$$\langle z - P_D(x), x - P_D(x)\rangle \le 0.$$

Thus the mapping P_D is nonexpansive [20].

Corollary 1.2. *Assume that* D *is a nonempty, convex and closed subset of* X*. Then for each* $x \in X$ *and each* $z \in D$,

$$\|z - P_D(x)\|^2 + \|x - P_D(x)\|^2 \le \|z - x\|^2.$$

Suppose that $C_1, \dots, C_m$ are nonempty, closed and convex subsets of X where m is a natural number. Set

$$C := \cap_{i=1}^m C_i. \tag{1.1}$$

We suppose that

$$C \neq \emptyset. \tag{1.2}$$

For $i = 1, \dots, m$ set

$$P_i = P_{C_i}. \tag{1.3}$$

A point $x \in C$ is called a solution of our convex feasibility problem. For a given $\epsilon > 0$ a point $x \in X$ is called an ϵ-approximate solution of the feasibility problem if

$$d(x, C_i) \le \epsilon,\ i = 1, \dots, m.$$

It should be mentioned that there are cases when a point $x \in X$ is an ϵ-approximate solution of the feasibility problem with a small constant ϵ, but x is rather far from the intersection of the sets C_i, $i = 1, \dots, m$. Nevertheless such situations do not happen if the sets $C_1, \dots, C_m$ possess the (bounded) regularity property (see Section 5 of [4]).

In the present paper we apply a dynamic string-averaging projection (DSAP) method with variable strings and weights [6, 8-12] in order to obtain a good approximative solution of the feasibility problem.

Next we describe the dynamic string-averaging projection (DSAP) method with variable strings and weights.

By an index vector, we a mean a vector $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, p$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q,\ P[t] = P_{t_q} \cdots P_{t_1}. \tag{1.4}$$

It is easy to see that for each index vector t

$$\|P[t](x) - P[t](y)\| \le \|x - y\| \text{ for all } x, y \in X, \tag{1.5}$$

$$P[t](x) = x \text{ for all } x \in C. \tag{1.6}$$

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ be such that } \sum_{t \in \Omega} w(t) = 1. \tag{1.7}$$

A pair $(\Omega, w) \in \mathcal{M}$ is called an amalgamator and w is called a fit weight function [6].

Let $(\Omega, w) \in \mathcal{M}$. Define

$$P_{\Omega,w}(x) = \sum_{t \in \Omega} w(t)P[t](x),\ x \in X. \tag{1.8}$$

It is easy to see that

$$\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\| \le \|x - y\| \text{ for all } x, y \in X, \tag{1.9}$$

$$P_{\Omega,w}(x) = x \text{ for all } x \in C. \tag{1.10}$$

The dynamic string-averaging projection (DSAP) method with variable strings and variable weights can now be described by the following algorithm [6, 8-12].

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector x_k pick a pair

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate the next iteration vector x_{k+1} by

$$x_{k+1} = P_{\Omega_{k+1}, w_{k+1}}(x_k).$$

The convergence properties and the, so called, perturbation resilience of this DSAP method were analyzed in [6, 8-12].

Fix a number

$$\Delta \in (0, m^{-1}] \tag{1.11}$$

and an integer

$$\bar{q} \ge m. \tag{1.12}$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(t) \le \bar{q} \text{ for all } t \in \Omega, \tag{1.13}$$

$$w(t) \ge \Delta \text{ for all } t \in \Omega. \tag{1.14}$$

Fix a natural number $\bar{N}$.

In the studies of the convex feasibility problem the goal is to find a point $x \in C$. In order to meet this goal we apply an algorithm generated by

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

such that for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}).$$

This algorithm generates, for any starting point $x_0 \in X$, a sequence $\{x_k\}_{k=0}^{\infty} \subset X$, where

$$x_{k+1} = P_{\Omega_{k+1}, w_{k+1}}(x_k).$$

According to the results known in the literature, this sequence should converge to an element of C [6, 8-12]. In this paper, we study the behavior of the sequences generated by $\{(\Omega_i, w_i)\}_{i=1}^{\infty}$ taking into account computational errors which always present in practice. These computational errors are bounded from above by a small constant depending only on our computer system which is denoted in the paper by δ. This computational error δ presents in all calculations which we do using our computer system. For example, if $x \in X$ and $i \in \{1, \dots, m\}$ and we need to calculate $P_i(x)$, then using our computer system we obtain a point $y \in X$ satisfying

$$\|y - P_i(x)\| \le \delta.$$

If k is a natural number, $y_i \in X$, $i = 1, \dots, k$, $\alpha_i > 0$, $i = 1, \dots, k$ satisfying $\sum_{i=1}^k \alpha_i = 1$ and if need to calculate $\sum_{i=1}^k \alpha_i y_i$, then by using our computer system we obtain a point $y \in X$ satisfying

$$\left\| y - \sum_{i=1}^k \alpha_i y_i \right\| \le \delta.$$

Surely, in this situation one cannot expect that the sequence of iterates generated by our algorithm converges to the set C. The goal of our paper is to understand what approximate solutions of the feasibility problem can be obtained.

We will prove the following result (Theorem 1.3), which shows that in the presence of computational errors bounded from above by a constant δ, an ϵ-approximate solution can be obtained after $(n_0 - 1)\bar{N}$ iterations of the algorithm, where ϵ and n_0 are constants depending on δ (see (1.19) and (1.26)).

In order to state our main result we need the following definitions.

Let $\delta \ge 0$, $x \in X$ and let $t = (t_1, \dots, t_{p(t)})$ be an index vector. Define

$$\begin{aligned} A_0(x, t, \delta) = \{(y, \lambda) \in X \times R^1 : \ & \text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that} \\ & y_0 = x \text{ and for all } i = 1, \dots, p(t), \\ & \|y_i - P_{t_i}(y_{i-1})\| \le \delta, \\ & y = y_{p(t)}, \\ & \lambda = \max\{\|y_i - y_{i-1}\| : \ i = 1, \dots, p(t)\}\}. \end{aligned} \tag{1.15}$$

Let $\delta \ge 0$, $x \in X$ and let $(\Omega, w) \in \mathcal{M}$. Define

$$\begin{aligned} A(x, (\Omega, w), \delta) = \Big\{ & (y, \lambda) \in X \times R^1 : \ \text{there exist} \\ & (y_t, \lambda_t) \in A_0(x, t, \delta), \ t \in \Omega \text{ such that} \\ & \left\| y - \sum_{t \in \Omega} w(t) y_t \right\| \le \delta, \ \lambda = \max\{\lambda_t : \ t \in \Omega_t\} \Big\}. \end{aligned} \tag{1.16}$$

Denote by $\mathrm{Card}(A)$ the cardinality of a set A. Suppose that the sum over empty set is zero.

Theorem 1.3. *Let $M > 0$ satisfy*

$$B(0, M) \cap C \ne \emptyset, \tag{1.17}$$

$\delta > 0$ satisfy

$$\delta \le (2\bar{q}\bar{N})^{-1} \tag{1.18}$$

and let a natural number n_0 satisfy

$$n_0 \ge 1 + 4M^2 \delta^{-1} (\bar{q} + 1)^{-1} (2M + 4)^{-1} (4\bar{N})^{-1}. \tag{1.19}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{1.20}$$

satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\}), \tag{1.21}$$

$$x_0 \in B(0, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X, \ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{1.22}$$

satisfy for each natural number i*,*

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta). \tag{1.23}$$

Then there exists an integer $q \in [0, n_0 - 1]$ *such that*

$$\|x_i\| \leq 3M + 1, \ i = 0, \dots, q\bar{N}, \tag{1.24}$$

$$\lambda_i \leq (64\Delta^{-1}\delta(\bar{q}+1)(2M+4)4\bar{N})^{1/2}, \tag{1.25}$$

$$i = q\bar{N} + 1, \dots, (q+1)\bar{N}.$$

Moreover, if an integer $q \in [0, n_0 - 1]$ *satisfies (1.25), then for each* $s = 1, \dots, m$ *and each* $i = q\bar{N}, \dots, (q+1)\bar{N}$*,*

$$d(x_i, C_s) \leq (\bar{q}+1)(\bar{N}+1)(64\Delta^{-1}\delta(\bar{q}+1)(2M+4)4\bar{N})^{1/2}, \tag{1.26}$$

and

$$\|x_i - x_j\| \leq (\bar{q}+1)\bar{N}(64\Delta^{-1}\delta(\bar{q}+1)(2M+4)4\bar{N})^{1/2} \tag{1.27}$$

for each $i, j \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$.

Theorem 1.3 is proved in Section 2. It provides the estimations for the constants ϵ and n_0, which follow from (1.26) and (1.19). According to (1.26),

$$\epsilon = (\bar{q}+1)(\bar{N}+1)(64\Delta^{-1}\delta(\bar{q}+1)(2M+4)4\bar{N})^{1/2}. \tag{1.28}$$

Note that $\epsilon = c_1\delta^{1/2}$ and $n_0 = [c_2\delta^{-1}] + 1$, where c_1 and c_2 are positive constants depending on M and $[u]$ denotes the integer part of u.

Let $\delta > 0$ satisfy (1.18) and a natural number n_0 satisfy (1.19). Assume that we apply an algorithm associated with

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

which satisfies (1.21) for each natural number j, under the presence of computational errors bounded from above by a constant δ and that our goal is to find an ϵ-approximate solution with ϵ defined by (1.28). Theorem 1.3 also answers an important question: how we can find an iteration number k for which x_k is an ϵ-approximate solution of the feasibility problem. By Theorem 1.3 we need just to find the smallest integer $q \in [0, \dots, n_0 - 1]$ such that (1.24) and (1.25) hold.

2. Proof of Theorem 1.3

By (1.17) there exists

$$z \in B(0,M) \cap C. \tag{2.1}$$

Fix a positive number

$$\epsilon_0 = (64\Delta^{-1}\delta(\bar{q}+1)(2M+4)4\bar{N})^{1/2}. \tag{2.2}$$

Assume that a nonnegative integer s is such that for each integer $k \in [0,s]$,

$$\max\{\lambda_i : \ i = k\bar{N}+1,\dots,(k+1)\bar{N}\} > \epsilon_0. \tag{2.3}$$

By (2.1) and (1.22),

$$\|x_0 - z\| \le 2M. \tag{2.4}$$

Assume that an integer $k \in [0,s]$ satisfies

$$\|x_{k\bar{N}} - z\| \le 2M. \tag{2.5}$$

We prove the following auxiliary result.

Lemma 2.1. *Assume that an integer*

$$i \in [0, \bar{N}-1] \tag{2.6}$$

satisfies

$$\|x_{k\bar{N}+i} - z\| \le 2M + i\delta(\bar{q}+1). \tag{2.7}$$

Then

$$\|x_{k\bar{N}+i+1} - z\| \le \delta(\bar{q}+1) + \|x_{k\bar{N}+i} - z\| \tag{2.8}$$

and

$$\|x_{k\bar{N}+i+1} - z\|^2 \le \|x_{k\bar{N}+i} - z\|^2 + \delta(\bar{q}+1)(4M+3). \tag{2.9}$$

If $\lambda_{k\bar{N}+i+1} > \epsilon_0$*, then*

$$\|x_{k\bar{N}+i+1} - z\|^2 - \|x_{k\bar{N}+i} - z\|^2 \le -32^{-1}\Delta\epsilon_0^2. \tag{2.10}$$

Proof By (1.23),

$$(x_{k\bar{N}+i+1}, \lambda_{k\bar{N}+i+1}) \in A(x_{k\bar{N}+i}, (\Omega_{k\bar{N}+i+1}, w_{k\bar{N}+i+1}), \delta). \tag{2.11}$$

In view of (2.11) and (1.16) there exists

$$(y_t, \alpha_t) \in A_0(x_{k\bar{N}+i}, t, \delta), \ t \in \Omega_{k\bar{N}+i+1} \tag{2.12}$$

such that

$$\Big\| x_{k\bar{N}+i+1} - \sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) y_t \Big\| \le \delta, \tag{2.13}$$

$$\lambda_{k\bar{N}+i+1} = \max\{\alpha_t : \ t \in \Omega_{k\bar{N}+i+1}\}. \tag{2.14}$$

By (2.12) and (1.15) for each $t = (t_1,\dots,t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$ there exists a finite sequence $\{y_i^{(t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_{k\bar{N}+i}, \ y_{p(t)}^{(t)} = y_t, \tag{2.15}$$

$$\|y_j^{(t)} - P_{t_j}(y_{j-1}^{(t)})\| \le \delta \text{ for each integer } j = 1, \dots, p(t), \tag{2.16}$$

$$\alpha_t = \max\{\|y_{i+1}^{(t)} - y_i^{(t)}\| : \ i = 0, \dots, p(t) - 1\}. \tag{2.17}$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}.$$

Let

$$j \in \{1, \dots, p(t)\}. \tag{2.18}$$

It follows from Corollary 1.2, (1.1), (1.3) and (2.1) that

$$\|z - P_{t_j}(y_{j-1}^{(t)})\|^2 + \|P_{t_j}(y_{j-1}^{(t)}) - y_{j-1}^{(t)}\|^2 \le \|z - y_{j-1}^{(t)}\|^2. \tag{2.19}$$

By (1.1), (1.3), (2.1), (2.16), (2.18) and Proposition 1.1,

$$\begin{aligned} \|z - y_j^{(t)}\|^2 =& \|z - P_{t_j}(y_{j-1}^{(t)}) + P_{t_j}(y_{j-1}^{(t)}) - y_j^{(t)}\|^2 \\ \le& \|z - P_{t_j}(y_{j-1}^{(t)})\|^2 + \|P_{t_j}(y_{j-1}^{(t)}) - y_j^{(t)}\|^2 \\ &+ 2\|z - P_{t_j}(y_{j-1}^{(t)})\|\|P_{t_j}(y_{j-1}^{(t)}) - y_j^{(t)}\| \\ \le& \|z - P_{t_j}(y_{j-1}^{(t)})\|^2 + \delta^2 + 2\delta\|z - P_{t_j}(y_{j-1}^{(t)})\| \\ \le& \|z - P_{t_j}(y_{j-1}^{(t)})\|^2 + \delta^2 + 2\delta\|z - y_{j-1}^{(t)}\|. \end{aligned} \tag{2.20}$$

In view of (2.19) and (2.20),

$$\begin{aligned} \|z - y_j^{(t)}\|^2 \le& \|z - y_{j-1}^{(t)}\|^2 - \|P_{t_j}(y_{j-1}^{(t)}) - y_{j-1}^{(t)}\|^2 \\ &+ \delta^2 + 2\delta\|z - y_{j-1}^{(t)}\|. \end{aligned} \tag{2.21}$$

By (2.21),

$$\|z - y_j^{(t)}\| \le \|z - y_{j-1}^{(t)}\| + \delta. \tag{2.22}$$

Thus we have shown that the following property holds:

(P1) for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$ and each $j \in \{1, \dots, p(t)\}$ (2.21) and (2.22) hold.

It follows from (P1), (2.22), (2.15) and (1.13) that for each $t \in \Omega_{k\bar{N}+i+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$\begin{aligned} \|z - y_j^{(t)}\| \le& \|z - y_0^{(t)}\| + \delta j = \|z - x_{k\bar{N}+i}\| + \delta j \\ \le& \|z - x_{k\bar{N}+i}\| + \delta\bar{q}. \end{aligned} \tag{2.23}$$

In view of (2.23) and (2.7) for each $t \in \Omega_{k\bar{N}+i+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$\|z - y_j^{(t)}\| \le 2M + (1 + \bar{q})i\delta + \delta\bar{q} \le 2M + \delta(\bar{q}(i+1) + i). \tag{2.24}$$

By (1.18), (2.24), (2.6), (2.7) and (2.15) the following property holds:

(P2) for each $t \in \Omega_{k\bar{N}+i+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$\|z - y_j^{(t)}\| \le 2M + 2\delta\bar{q}\bar{N} \le 2M + 1. \tag{2.25}$$

By (2.15) and (2.23) for each $t \in \Omega_{k\bar{N}+i+1}$,

$$\|z - y_t\| = \|z - y^{(t)}_{p(t)}\| \le \|z - x_{k\bar{N}+i}\| + \delta\bar{q}. \tag{2.26}$$

It follows from (2.13), (2.26) and (1.7) that

$$\begin{aligned}\|x_{k\bar{N}+i+1} - z\| &\le \Big\|x_{k\bar{N}+i+1} - \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t\Big\| \\ &\quad + \Big\|\sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t - z\Big\| \\ &\le \delta + \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)\|y_t - z\| \le \delta + \|x_{k\bar{N}+i} - z\| + \delta\bar{q},\end{aligned}$$

$$\|x_{k\bar{N}+i+1} - z\| \le \delta(\bar{q}+1) + \|x_{k\bar{N}+i} - z\|$$

and (2.8) is true.

By (2.8), (2.7), (2.6) and (1.18)

$$\begin{aligned}\|x_{k\bar{N}+i+1} - z\|^2 &\le \|x_{k\bar{N}+i} - z\|^2 + \delta^2(\bar{q}+1)^2 + 2\delta(\bar{q}+1)\|x_{k\bar{N}+i} - z\| \\ &\le \|x_{k\bar{N}+i} - z\|^2 + \delta^2(\bar{q}+1)^2 + 2\delta(\bar{q}+1)(2M+1) \\ &\le \|x_{k\bar{N}+i} - z\|^2 + \delta(\bar{q}+1)(4M+3).\end{aligned}$$

Thus (2.9) holds.

Assume that

$$\lambda_{k\bar{N}+i+1} > \epsilon_0. \tag{2.27}$$

By (2.14) there is

$$s = (s_1, \dots, s_{p(s)}) \in \Omega_{k\bar{N}+i+1} \tag{2.28}$$

such that

$$\alpha_s = \lambda_{k\bar{N}+i+1} > \epsilon_0. \tag{2.29}$$

In view of (2.29), (2.28) and (2.17), there is

$$j_0 \in \{1, \dots, p(s)\} \tag{2.30}$$

such that

$$\|y^{(s)}_{j_0} - y^{(s)}_{j_0-1}\| = \alpha_s > \epsilon_0. \tag{2.31}$$

By (P1), (P2) and (2.21) applied with $t = s$, $j = j_0$

$$\begin{aligned}\|z - y^{(s)}_{j_0}\|^2 &\le \|z - y^{(s)}_{j_0-1}\|^2 - \|P_{s_{j_0}}(y^{(s)}_{j_0-1}) - y^{(s)}_{j_0-1}\|^2 \\ &\quad + \delta^2 + 2\delta(2M+1) \\ &\le \|z - y^{(s)}_{j_0-1}\|^2 - \|P_{s_{j_0}}(y^{(s)}_{j_0-1}) - y^{(s)}_{j_0-1}\|^2 + 2\delta(2M+2). \qquad (2.32)\end{aligned}$$

By (2.31) and (2.16),

$$\|P_{s_{j_0}}(y^{(s)}_{j_0-1}) - y^{(s)}_{j_0-1}\| \ge \|y^{(s)}_{j_0} - y^{(s)}_{j_0-1}\| - \|y^{(s)}_{j_0} - P_{s_{j_0}}(y^{(s)}_{j_0-1})\| > \epsilon_0 - \delta. \tag{2.33}$$

Relations (2.32) and (2.33) imply that

$$\|z - y^{(s)}_{j_0}\|^2 \le \|z - y^{(s)}_{j_0-1}\|^2 - (\epsilon_0 - \delta)^2 + 2\delta(2M+2). \tag{2.34}$$

By (P2) applied with $t = s$ for all $j \in \{0, 1, \dots, p(s)\}$,

$$\|z - y_j^{(s)}\| \le 2M + 1. \tag{2.35}$$

By (P1), (2.22) with $t = s$ and (2.35), for all $j \in \{1, \dots, p(s)\}$,

$$\begin{aligned} \|z - y_j^{(s)}\|^2 &\le \|z - y_{j-1}^{(s)}\|^2 + \delta^2 + 2\delta\|z - y_{j-1}^{(s)}\| \\ &\le \|z - y_{j-1}^{(s)}\|^2 + 2\delta(2M + 2). \end{aligned} \tag{2.36}$$

In view of (2.15), (2.30), (2.34), (2.36) and (1.13),

$$\begin{aligned} \|z - x_{k\bar{N}+i}\|^2 - \|z - y_s\|^2 &= \sum_{i=1}^{p(s)} [\|z - y_{i-1}^{(s)}\|^2 - \|z - y_i^{(s)}\|^2] \\ &\ge (\epsilon_0 - \delta)^2 - 2\delta(2M + 2) - 2\delta(2M + 2)\bar{q} \\ &\ge (\epsilon_0 - \delta)^2 - 2\delta(2M + 2)(\bar{q} + 1). \end{aligned} \tag{2.37}$$

By (P1), (P2) and (2.22), for all $t \in \Omega_{k\bar{N}+i+1}$ and all $j \in \{1, \dots, p(t)\}$,

$$\begin{aligned} \|z - y_j^{(t)}\|^2 &\le \|z - y_{j-1}^{(t)}\|^2 + \delta^2 + 2\delta\|z - y_{j-1}^{(t)}\| \\ &\le \|z - y_{j-1}^{(t)}\|^2 + 2\delta(2M + 2), \end{aligned}$$

$$\|z - y_{j-1}^{(t)}\|^2 - \|z - y_j^{(t)}\|^2 \ge -2\delta(2M + 2). \tag{2.38}$$

By (2.15), (2.38) and (1.13), for all $t \in \Omega_{k\bar{N}+i+1}$,

$$\begin{aligned} \|z - x_{k\bar{N}+i}\|^2 - \|z - y_t\|^2 &= \sum_{i=1}^{p(t)} [\|z - y_{i-1}^{(t)}\|^2 - \|z - y_i^{(t)}\|^2] \\ &\ge -2\bar{q}\delta(2M + 2). \end{aligned} \tag{2.39}$$

Since the function $u \to \|u - z\|^2$, $u \in X$ is convex it follows from (1.7) and (2.28) that

$$\begin{aligned} \Big\| \sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) y_t - z \Big\|^2 &\le \sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) \|y_t - z\|^2 \\ &= \|z - x_{k\bar{N}+i}\|^2 + \sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)[\|y_t - z\|^2 - \|z - x_{k\bar{N}+i}\|^2] \\ &\le \|z - x_{k\bar{N}+i}\|^2 + w_{k\bar{N}+i+1}(s)[\|y_s - z\|^2 - \|z - x_{k\bar{N}+i}\|^2] \\ &\quad + \sum \{w_{k\bar{N}+i+1}(t)[\|y_t - z\|^2 - \|z - x_{k\bar{N}+i}\|^2] : \ t \in \Omega_{k\bar{N}+i+1} \setminus \{s\}\}. \end{aligned} \tag{2.40}$$

By (2.40), (2.37), (2.39), (1.7), (2.2), (1.14)

$$\begin{aligned} \Big\| \sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) y_t - z \Big\|^2 &\le w_{k\bar{N}+i+1}(s)[-(\epsilon_0 - \delta)^2 + 2\delta(2M + 2)(\bar{q} + 1)] \\ &\quad + 2\bar{q}\delta(2M + 2) + \|z - x_{k\bar{N}+i}\|^2 \\ &\le \|z - x_{k\bar{N}+i}\|^2 + 2\bar{q}\delta(2M + 2) \\ &\quad - w_{k\bar{N}+i+1}(s)[4^{-1}\epsilon_0^2 - 2\delta(2M + 2)(\bar{q} + 1)] \\ &\le \|z - x_{k\bar{N}+i}\|^2 + 2\delta\bar{q}(2M + 2) \\ &\quad - \Delta(4^{-1}\epsilon_0^2 - 2\delta(M + 2)(\bar{q} + 1)). \end{aligned} \tag{2.41}$$

By (2.41) and (2.2)

$$\Big\|\sum_{t\in\Omega_{k\bar N+i+1}} w_{k\bar N+i+1}(t)y_t - z\Big\|^2 - \|x_{k\bar N+i}-z\|^2 \le 2\delta(2M+2)\bar q - 8^{-1}\Delta\epsilon_0^2$$

$$\ge\ -16^{-1}\Delta\epsilon_0^2. \tag{2.42}$$

It follow from (2.13), (2.42) and (1.7) that

$$\begin{aligned}
&\|x_{k\bar N+i+1}-z\|^2\\
&= \Big\|x_{k\bar N+i+1} - \sum_{t\in\Omega_{k\bar N+i+1}} w_{k\bar N+i+1}(t)y_t + \sum_{t\in\Omega_{k\bar N+i+1}} w_{k\bar N+i+1}(t)y_t - z\Big\|^2\\
&\le \Big\|x_{k\bar N+i+1} - \sum_{t\in\Omega_{k\bar N+i+1}} w_{k\bar N+i+1}(t)y_t\Big\|^2\\
&\quad + \Big\|\sum_{t\in\Omega_{k\bar N+i+1}} w_{k\bar N+i+1}(t)y_t - z\Big\|^2\\
&\quad + 2\Big\|x_{k\bar N+i+1} - \sum_{t\in\Omega_{k\bar N+i+1}} w_{k\bar N+i+1}(t)y_t\Big\|\Big\|\sum_{t\in\Omega_{k\bar N+i+1}} w_{k\bar N+i+1}(t)y_t - z\Big\|\\
&\le \delta^2 - 16^{-1}\Delta\epsilon_0^2 + \|x_{k\bar N+i}-z\|^2\\
&\quad + 2\delta\Big\|\sum_{t\in\Omega_{k\bar N+i+1}} w_{k\bar N+i+1}(t)y_t - z\Big\|\\
&\le \delta^2 - 16^{-1}\Delta\epsilon_0^2 + \|x_{k\bar N+i}-z\|^2\\
&\quad + 2\delta\max\{\|y_t-z\|:\ t\in\Omega_{k\bar N+i+1}\}.
\end{aligned} \tag{2.43}$$

By (2.43), (P2), (2.15) and (2.2),

$$\begin{aligned}
\|x_{k\bar N+i+1}-z\|^2 - \|x_{k\bar N+i}-z\|^2 &\le \delta^2 - 16^{-1}\Delta\epsilon_0^2 + 2\delta(2M+1)\\
&\le -16\Delta\epsilon_0^2 + 2\delta(2M+2) \le -32^{-1}\Delta\epsilon_0^2.
\end{aligned}$$

Lemma 2.1 is proved.

By (2.5), Lemma 2.1 applied by induction and (1.18) for all $i=0,\dots,\bar N-1$,

$$\|x_{k\bar N+i+1}-z\| \le \|x_{k\bar N+i}-z\| + \delta(\bar q+1),$$

$$\|x_{k\bar N+i+1}-z\| \le 2M + \delta(\bar q+1)(i+1) \le 2M + \delta(\bar q+1)\bar N \le 2M+1, \tag{2.44}$$

$$\|x_{k\bar N+i}-z\| \le 2M+1,\ i=0,\dots,\bar N. \tag{2.45}$$

It follows from (2.5), (2.44), (2.3), (2.2) and Lemma 2.1 that

$$\begin{aligned}
\|x_{(k+1)\bar N}-z\|^2 - \|x_{k\bar N}-z\|^2 &= \sum_{i=0}^{\bar N-1}[\|x_{k\bar N+i+1}-z\|^2 - \|x_{k\bar N+i}-z\|^2]\\
&\le -32^{-1}\Delta\epsilon_0^2 + \bar N\delta(\bar q+1)(4M+3) \le -64^{-1}\Delta\epsilon_0^2.
\end{aligned}$$

Thus we have shown that the following property holds:

(P3) if an integer $k \in [0, s]$ satisfies $\|x_{k\bar{N}} - z\| \le 2M$, then

$$\|x_j - z\| \le 2M + 1,\ j = k\bar{N}, \dots, (k+1)\bar{N},$$

$$\|x_{(k+1)\bar{N}} - z\|^2 - \|x_{k\bar{N}} - z\|^2 \le -64^{-1}\Delta\epsilon_0^2. \tag{2.46}$$

By (2.4) and property (P3),

$$\|x_j - z\| \le 2M + 1,\ j = 0, \dots, (s+1)\bar{N} \tag{2.47}$$

and (2.46) holds for all $k = 0, \dots, s$.

In view of (2.46) and (2.4),

$$\begin{aligned}64^{-1}\Delta\epsilon_0^2(s+1) &\le \sum_{k=0}^{s}[\|x_{k\bar{N}} - z\|^2 - \|x_{(k+1)\bar{N}} - z\|^2]\\ &= \|x_0 - z\|^2 - \|x_{(s+1)\bar{N}} - z\|^2 \le \|x_0 - z\|^2 \le 4M^2,\end{aligned}$$

$$s + 1 \le 256M^2\Delta^{-1}\epsilon_0^{-2}.$$

Thus we have shown that the following property holds:

(P4) If an integer $s \ge 0$ and for each integer $k \in [0, s]$ (2.3) holds, then

$$s \le 256M^2\Delta^{-1}\epsilon_0^{-2} - 1,$$
$$\|x_j - z\| \le 2M + 1,\ j = 0, \dots, (s+1)\bar{N},$$
$$\|x_{k\bar{N}} - z\| \le 2M,\ k = 0, \dots, s+1.$$

(P4), (1.19) and (2.2) imply that there exists an integer $q \in [0, n_0 - 1]$ such that for each integer k satisfying $0 \le k < q$,

$$\max\{\lambda_i :\ i = k\bar{N} + 1, \dots, (k+1)\bar{N}\} > \epsilon_0,$$
$$\max\{\lambda_i :\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}\} \le \epsilon_0.$$

By (2.4), the choice of q, (P4) and (2.1),

$$\|x_{q\bar{N}} - z\| \le 2M,$$
$$\|x_j - z\| \le 2M + 1,\ j = 0, \dots, q\bar{N},$$
$$\|x_j\| \le 3M + 1,\ j = 0, \dots, q\bar{N}.$$

Assume that an integer $q \in [0, n_0 - 1]$ satisfies

$$\lambda_i \le \epsilon_0,\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}. \tag{2.48}$$

Let

$$j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}. \tag{2.49}$$

Then by (1.23) and (1.49),

$$(x_{j+1}, \lambda_{j+1}) \in A(x_j, (\Omega_{j+1}, w_{j+1}), \delta). \tag{2.50}$$

It follows from (2.50), (1.16) and (2.48) that there exist

$$(y_t^{(j)}, \alpha_t^{(j)}) \in A_0(x_j, t, \delta),\ t \in \Omega_{j+1} \tag{2.51}$$

such that

$$\Big\|x_{j+1} - \sum_{t \in \Omega_{j+1}} w_{j+1}(t)y_t^{(j)}\Big\| \le \delta, \tag{2.52}$$

$$\max\{\alpha_t^{(j)} : t \in \Omega_{j+1}\} \le \epsilon_0. \tag{2.53}$$

By (2.51), (1.15) and (2.53) for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{j+1}$ there exists a finite sequence $\{y_i^{(t,j)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(t,j)} = x_j, \tag{2.54}$$

for each integer $i = 1, \ldots, p(t)$,

$$\|y_i^{(t,j)} - P_{t_i}(y_{i-1}^{(t,j)})\| \le \delta, \tag{2.55}$$

$$y_{p(t)}^{(t,j)} = y_t^{(j)}, \tag{2.56}$$

$$\epsilon_0 \ge \alpha_t^{(j)} = \max\{\|y_i^{(t,j)} - y_{i-1}^{(t,j)}\| : i = 1, \ldots, p(t)\}. \tag{2.57}$$

By (2.54), (2.56), (2.57) and (1.13), for each $t \in \Omega_{j+1}$ and each integer $i = 1, \ldots, p(t)$,

$$\|x_j - y_i^{(t,j)}\| \le i\epsilon_0 \le \epsilon_0 \bar{q}, \tag{2.58}$$

$$\|x_j - y_t^{(j)}\| \le \epsilon_0 \bar{q}. \tag{2.59}$$

By (2.56) and (2.55) for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{j+1}$ and each $i = 1, \ldots, p(t)$,

$$\begin{aligned} d(x_j, C_{t_i}) &\le \|x_j - P_{t_i}(y_{i-1}^{(t,j)})\| \\ &\le \|x_j - y_i^{(t,j)}\| + \|y_i^{(t,j)} - P_{t_i}(y_{i-1}^{(t,j)})\| \\ &\le \epsilon_0 \bar{q} + \delta. \end{aligned} \tag{2.60}$$

In view of (2.52), (2.59) and (1.7),

$$\begin{aligned} \|x_{j+1} - x_j\| &\le \Big\|x_{j+1} - \sum_{t \in \Omega_{j+1}} w_{j+1}(t) y_t^{(j)}\Big\| + \Big\|\sum_{t \in \Omega_{j+1}} w_{j+1}(t) y_t^{(j)} - x_j\Big\| \\ &\le \delta + \sum_{t \in \Omega_{j+1}} w_{j+1}(t) \|y_t^{(j)} - x_j\| \\ &\le \delta + \epsilon_0 \bar{q}. \end{aligned}$$

Together with (2.2) this implies that

$$\|x_{j+1} - x_j\| \le \epsilon_0(\bar{q} + 1). \tag{2.61}$$

By (2.60) and (2.2),

$$d(x_j, C_{t_i}) \le \epsilon_0(\bar{q} + 1),\ t \in \Omega_{j+1},\ i = 1, \ldots, p(t). \tag{2.62}$$

Clearly, (2.61) and (2.62) hold for all $j = q\bar{N}, \ldots, (q+1)\bar{N} - 1$.

By (2.61) for each $j_1, j_2 \in \{q\bar{N}, \ldots, (q+1)\bar{N}\}$,

$$\|x_{j_1} - x_{j_2}\| \le \epsilon_0(\bar{q} + 1)\bar{N}. \tag{2.63}$$

Let $s \in \{1, \ldots, m\}$. By (1.21) there exist $j \in \{q\bar{N}, \ldots, (q+1)\bar{N} - 1\}$ and $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{j+1}$ such that

$$s \in \{t_1, \ldots, t_{p(t)}\}.$$

Together with (2.62) this implies that

$$d(x_j, C_s) \le \epsilon_0(\bar{q}+1). \tag{2.64}$$

In view of (2.63) and (2.64) for each $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$d(x_i, C_s) \le \|x_i - x_j\| + d(x_j, C_s) \le \epsilon_0(\bar{q}+1)\bar{N} + \epsilon_0(\bar{q}+1) \le \epsilon_0(\bar{q}+1)(\bar{N}+1). \tag{2.65}$$

It follows from (2.63) and (2.2) that for all $j_1, j_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$\|x_{j_1} - x_{j_2}\| \le (\bar{q}+1)\bar{N}(64\Delta^{-1}\delta(\bar{q}+1)(2M+4)4\bar{N})^{1/2}.$$

By (2.2) and (2.65), for each $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$ and each $s \in \{1, \dots, m\}$,

$$d(x_i, C_s) \le (\bar{q}+1)(\bar{N}+1)(64\Delta^{-1}\delta(\bar{q}+1)(2M+4)4\bar{N})^{1/2}.$$

Theorem 1.3 is proved.

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