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A UNIFIED STUDY OF THE SPLIT FEASIBLE PROBLEMS WITH APPLICATIONS

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Abstract. Split feasibility problem has received a lot of attention due to its diverse applications in signal processing, image reconstruction, with particular progress in intensity-modulated radiation therapy, approximation theory, control theory, biomedical engineering, communications, and geophysics. In this paper, we first prove some properties of firmly nonexpansive mappings. Then we apply these properties to establish a strong convergence theorem with a Regularized-like method to find an element of the solutions set of a monotone inclusion problem in a Hilbert space. Using this result, we also prove a strong convergence theorem for finding an element of the solutions set of generalized split feasibility problem ($\mathbf{GSFP_{FF}}$). As applications, we study the solutions and algorithms for the convex feasibility problems, split feasibility problems. To be the best of our knowledge, there are no researchers consider generalized split feasibility problem ($\mathbf{GSFP_{FF}}$) by using these methods in the infinite dimensional real Hilbert spaces and finite dimensional Euclidean spaces.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to H$ be mapping, and let $Fix(T) := \{x \in C : Tx = x\}$ denote the set of fixed points of T. A mapping $T : C \to H$ is said to be nonexpansive if $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in C$; T is said to be quasi-nonexpansive mapping if $Fix(T) \neq \emptyset$ and $||Tx - y|| \leq ||x - y||$ for all $x \in C$ and $y \in Fix(T)$. For $\alpha > 0$, a mapping $A : H \to H$ is called $\alpha-$inverse-strongly monotone (α-ism) if

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \forall x, y \in H.$$

If $0 < \lambda \leq 2\alpha$, $A : H \to H$ is a $\alpha-$inverse-strongly monotone mapping, then $I - \lambda A : H \to H$ is nonexpansive. A mapping $T : C \to H$ is said to be a firmly nonexpansive mapping if

$$||Tx - Ty||^2 \leq ||x - y||^2 - ||(I - T)x - (I - T)y||^2$$

for every $x, y \in C$.

A mapping $g : H \to H$ is a contraction if there exists $k \in (0, 1)$ such that $\|g(x)-g(y)\| \leq k\|x-y\|$, for all $x, y \in H$. We call such a mapping g a $k-$contraction. A nonlinear operator $V : H \to H$ is called strongly monotone if there exists $\bar{\gamma} > 0$ such that $\langle x - y, Vx - Vy\rangle \geq \bar{\gamma}\|x - y\|^2$ for all $x, y \in H$. Such V is also called $\bar{\gamma}-$strongly monotone. A nonlinear operator $V : H \to H$ is called Lipschitzian

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continuous if there exists $L > 0$ such that $\|Vx - Vy\| \leq L\|x - y\|$ for all $x, y \in H$. Such V is also called L-Lipschitzian continuous.

Let $B : H \multimap H$ be a multivalued mapping. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. A multivalued mapping B is said to be a monotone operator on H if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in D(B), u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1} : H \to D(B)$, which is called the resolvent of B for r, and define the set $B^{-1}0$ as $B^{-1}0 = \{x \in H : 0 \in Bx\}$.

In 2011, Lin and Takahashi [15] proved the following strong convergence theorem.

Theorem 1.1. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\alpha > 0$ and let F be a $\alpha-$inverse-strongly monotone mapping of C into H. Let B be a maximal monotone mapping on H and let G be a maximal monotone mapping on H such that the domain of G is included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ and $T_r = (I + rG)^{-1}$ for each $\lambda > 0$ and $r > 0$. Let $0 < k < 1$ and let g be a $k-$contraction of H into itself. Let V be a $\bar{\gamma}-$ strongly monotone and $L-$Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$. Soppose that $(A + B)^{-1}0 \bigcap G^{-1}0 \neq \emptyset$. Take $\mu, \gamma \in \mathbb{R}$ as follows:*

$$0 < \mu < \frac{2\bar{\gamma}}{L^2},\ 0 < \gamma < \frac{\bar{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Let $x_1 = x \in H$ and let $\{x_n\} \subset H$ be defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n \gamma f(x_n) + (1 - \beta_n V)J_{\lambda_n}(I - \lambda_n F)T_{r_n} x_n)$$

for each $n \in \mathbb{N}$, $\lambda_n \subset (0, \infty)$, $\alpha_n \subset (0, 1)$, $\beta_n \subset (0, 1)$, and $r_n \subset (0, \infty)$. Assume that:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$*;*
(ii) $\lim\limits_{n\to\infty} \beta_n = 0$*, and* $\sum_{n=1}^{\infty} \beta_n = \infty$*;*
(iii) $0 < a \leq \lambda_n \leq b < 2\alpha$*, and* $\liminf_{n\to\infty} r_n > 0$.

Then $\lim\limits_{n\to\infty} x_n = \bar{x}$*, where* $\bar{x} = P_{(F+B)^{-1}0 \bigcap G^{-1}0}(I - V + \gamma g)\bar{x}$.

On the other hand, the split feasibility problem can be formulated as the following problem:

(**SFP**) Find $\bar{x} \in H_1$ such that $\bar{x} \in C$ and $A\bar{x} \in Q$,

where C and Q are nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is an operator.

The split feasibility problem (**SFP**) in finite dimensional Hilbert spaces was first introduced by Censor and Elfving [7] for modeling inverse problems which arise from medical image reconstruction. Since then, the split feasibility problem (**SFP**) has received much attention due to its applications in signal processing, image reconstruction, with particular progress in intensity-modulated radiation therapy, approximation theory, control theory, biomedical engineering, communications, and geophysics. For examples, one can refer to [5, 7, 8, 16, 20] and related literatures. Since then, many researchers study (**SFP**) in finite dimensional or infinite dimensional Hilbert spaces. For examples, one can see [5, 6, 11, 18, 19, 25, 27, 26, 28, 29, 30].

A special case of problem (**SFP**) is the convexly constrained linear inverse problem in the finite dimensional Hilbert space [12]:

$$(\textbf{CLIP}) \text{ Find } \bar{x} \in C \text{ such that } A\bar{x} = b, \quad \text{where } b \in H_2,$$

which has extensively been investigated by using the Landweber iterative method [17]:

$$x_{n+1} := x_n + \gamma A^T(b - Ax_n), n \in \mathbb{N}.$$

In 2002, Byrne [5] first introduced the so-called CQ algorithm which generates a sequence $\{x_n\}$ by the following recursive procedure:

$$x_{n+1} = P_C(x_n - \rho_n A^*(I - P_Q)Ax_n), \tag{1.1}$$

where the stepsize ρ_n is chosen in the interval $(0, 2/||A||^2)$, and P_C and P_Q are the metric projections onto $C \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$, respectively. Compared with Censor and Elfving's algorithm [7] where the matrix inverse A is involved, the CQ algorithm (1.1) seems more easily executed since it only deals with metric projections with no need to compute matrix inverses.

In 2010, Xu [26] modified Byrne's CQ algorithm and proved the following weak convergence theorem in infinite Hilbert spaces for their modified algorithm.

Theorem 1.2 ([26]). *Suppose that the solution set of (*__SFP__*) is nonempty. Let $\{x_n\} \subset H$ be defined by*

$$x_{n+1} = P_C((1 - \rho\epsilon_n)x_n - \rho A^*(I - P_Q)Ax_n), \tag{1.2}$$

*for each $n \in \mathbb{N}$ and $\varepsilon_n \subset (0, 1)$. Assume that $0 < \rho < \frac{2}{||A||^2}$ and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Then $\{x_n\}$ converges weakly to a solution of (*__SFP__*).*

Besides, we know that the equilibrium problem is to find $z \in C$ such that

$$(\textbf{EP}) \; g(z, y) \geq 0 \text{ for each } y \in C,$$

where $g : C \times C \to \mathbb{R}$ is a bifunction.

This problem includes fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, minimax inequalities, and saddle point problems as special cases. (For examples, one can see [3] and related literatures.)

The solution set of equilibrium problem (**EP**) is denoted by $EP(g)$. For solving the equilibrium problem, let us assume that the bifunction $g : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $g(x, x) = 0$ for each $x \in C$;
(A2) g is monotone, i.e., $g(x, y) + g(y, x) \leq 0$ for any $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} g(tz + (1 - t)x, y) \leq g(x, y)$;
(A4) for each $x \in C$, the scalar function $y \to g(x, y)$ is convex and lower semicontinuous.

We have the following result from Blum and Oettli [3].

Theorem 1.3 ([3])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $g : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4). Then for each $r > 0$ and each $x \in H$, there exists $z \in C$ such that*

$$g(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$$

for all $y \in C$.

In 2005, Combettes and Hirstoaga [9] estabilshed the following important properties of resolvent operator.

Theorem 1.4 ([9])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and let $g : C \times C \to \mathbb{R}$ be a function satisfying conditions (A1)–(A4). For $r > 0$, define $T_r^g : H \to C$ by*

$$T_r^g x = \left\{ z \in C : g(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall\ y \in C \right\}.$$

for all $x \in H$. Then the following hold:

(i) *T_r^g is single-valued;*

(ii) *T_r^g is firmly nonexpansive, that is, $||T_r^g x - T_r^g y||^2 \leq \langle x - y, T_r^g x - T_r^g y\rangle$ for all $x, y \in H$;*

(iii) *$\{x \in H :\ T_r^g x = x\} = \{x \in C :\ g(x, y) \geq 0,\ \forall y \in C\}$;*

(iv) *$\{x \in C :\ g(x, y) \geq 0,\ \forall y \in C\}$ is a closed and convex subset of C.*

We call such T_r^g the resolvent of g for $r > 0$.

Motivated by Theorem 1.1 and Theorem 1.2, we first consider the following algorithm for finding a point $\bar{x} = P_{(F+B)^{-1}0 \bigcap G^{-1}0}(0)$:

Let J_ρ, T_r and F be defined as Theorem 1.1. Suppose that $(F + B)^{-1}0 \bigcap G^{-1}0 \neq \emptyset$. Let $\{x_n\} \subset H$ be defined by

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ x_{n+1} = J_\rho(I - \rho(F + \beta_n I))T_r x_n \end{cases}$$

for each $n \in \mathbb{N}$, $\rho \subset (0, \infty)$, $\beta_n \subset (0, 1)$, and $r \subset (0, \infty)$. Assume that:

(i) $0 < a \leq \rho < \frac{2}{\alpha^2+2}$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ and $\Sigma_{n=1}^{\infty} \beta_n = \infty$.

Then $\lim\limits_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{(F+B)^{-1}0 \bigcap G^{-1}0}(0)$.

Let C_1, C_2 and Q be nonempty closed convex subsets of Hilbert spaces H_1, H_1 and H_2, respectively. Let $g_1 : C_1 \times C_1 \to \mathbb{R}$, $g_2 : C_2 \times C_2 \to \mathbb{R}$ and $g_3 : Q \times Q \to \mathbb{R}$ be three bifunctions which satisfies conditions (A1)-(A4). Let F_1 be a firmly nonexpansive mapping of H_2 into H_2. Let $A : H_1 \to H_2$ be a bounded linear operator. Then we apply a strong convergence theorem for finding a element of the solutions set of a monotone inclusion problem in a Hilbert space to prove a strongly convergence theorem for the following generalized feasibility problem :

($\mathbf{GSFP_{FF}}$) Find $\bar{x} \in H_1$ such that $\bar{x} \in Fix(J_\lambda) \bigcap Fix(T_r)$ and $A\bar{x} \in Fix(F_1)$.

The generalized split feasibility problem ($\mathbf{GSFP_{FF}}$) contains many important problems as special cases.

(i) If $J_{\rho_1} = T_{\rho_1}^{g_1}$, $T_{\rho_2} = T_{\rho_2}^{g_2}$ and $F_1 = T_{\rho_3}^{g_3}$, then ($\mathbf{GSFP_{FF}}$) is reduced to generalized split feasibility equilibrium problem: ($\mathbf{GSFP_{EE}}$).

($\mathbf{GSFP_{EE}}$) Find $\bar{x} \in H_1$ such that $\bar{x} \in EP(g_1) \bigcap EP(g_2)$ and $A\bar{x} \in EP(g_3)$.

(ii) If $C_1 = C_2$, $g_2(x,y) = 0$ for each $(x,y) \in C_1 \times C_1$, then ($\mathbf{GSFP_{EE}}$) is reduced to the split equilibrium problem ($\mathbf{SFP_{EE}}$):

($\mathbf{SFP_{EE}}$) Find $\bar{x} \in H_1$ such that $\bar{x} \in EP(g_1)$ and $A\bar{x} \in EP(g_3)$.

(iii) If $g_1(x,y) = 0$, and $g_2(u,v) = 0$ for each $(x,y) \in C_1 \times C_1$ and each $(u,v) \in C_2 \times C_2$, then ($\mathbf{GSFP_{EE}}$) is reduced to ($\mathbf{GSFP_{CE}}$):

($\mathbf{GSFP_{CE}}$) Find $\bar{x} \in H_1$ such that $\bar{x} \in C_1 \bigcap C_2$ and $A\bar{x} \in EP(g_3)$.

(iv) If $g_3(x,y) = 0$ for each $(x,y) \in Q \times Q$, then ($\mathbf{GSFP_{CE}}$) is reduced to ($\mathbf{GSFP_{CQ}}$):

($\mathbf{GSFP_{CQ}}$) Find $\bar{x} \in H_1$ such that $\bar{x} \in C_1 \bigcap C_2$ and $A\bar{x} \in Q$.

(v) If $C_1 = C_2$, then ($\mathbf{GSFP_{CQ}}$) is reduced to split feasibility problem($\mathbf{SFP_{CQ}}$):

($\mathbf{SFP_{CQ}}$) Find $\bar{x} \in H_1$ such that $\bar{x} \in C_1$ and $A\bar{x} \in Q$.

In this paper, we first establish a strong convergence theorem with a Regularized-like method to find a element of the solutions set of a monotone inclusion problem in a Hilbert space. Using this result, we also prove a strong convergence theorem for finding a element of the solutions set of generalized split feasibility problem ($\mathbf{GSFP_{FF}}$). As applications, we study the solutions and algorithms for the convex feasibility problems, split feasibility problems. To be the best of our knowledge, there are no researchers consider generalized split feasibility problem ($\mathbf{GSFP_{FF}}$) by using these methods in the infinite dimensional real Hilbert spaces and finite dimensional Euclidean spaces.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers and let $\mathbb{R}$ be the set of real numbers. Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $||\cdot||$, respectively. We denote the strongly convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [24], for each $x, y \in H$ and $\lambda \in [0,1]$, we have

$$||\lambda x + (1-\lambda)y||^2 = \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)||x-y||^2.$$

Hence, we also have

$$2\langle x-y, u-v\rangle = ||x-v||^2 + ||y-u||^2 - ||x-u||^2 - ||y-v||^2 \tag{2.1}$$

for all $x, y, u, v \in H$.

Let C be a nonempty subset of a real Hilbert space H, and let $T : C \to H$ is said to be a firmly nonexpansive mapping if

$$||Tx - Ty||^2 \leq ||x-y||^2 - ||(I-T)x - (I-T)y||^2$$

for every $x, y \in C$, that is,

$$||Tx - Ty||^2 \leq \langle x-y, Tx-Ty\rangle$$

for every $x, y \in C$. The following results are needed in this paper.

Lemma 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$, F is a $\frac{1}{\alpha^2}-$inverse-strong-monotone mapping of C into H, and $\gamma \in \mathbb{R}$, then $F + \gamma I$ is a $\frac{1}{\gamma+\alpha^2}-$inverse-strong-monotone mapping.*

Proof. Since F is a $\frac{1}{\alpha^2}-$inverse-strong-monotone mapping, we have

$$\langle Fx - Fy, x - y\rangle \geq \frac{1}{\alpha^2}\|Fx - Fy\|^2.$$

for all $x, y \in C$. This implies that

$$\begin{aligned}
&(\gamma + \alpha^2)\langle (F + \gamma I)x - (F + \gamma I)y, x - y\rangle \\
&= (\gamma + \alpha^2)[\gamma\|x - y\|^2 + \langle Fx - Fy, x - y\rangle] \\
&= \gamma^2\|x - y\|^2 + \gamma\langle Fx - Fy, x - y\rangle + \gamma\alpha^2\|x - y\|^2 + \alpha^2\langle Fx - Fy, x - y\rangle \\
&\geq \gamma^2\|x - y\|^2 + 2\gamma\langle Fx - Fy, x - y\rangle + \|Fx - Fy\|^2 \\
&= \|\gamma(x - y) + Fx - Fy\|^2 = \|(F + \gamma I)x - (F + \gamma I)y\|^2.
\end{aligned} \tag{2.2}$$

Thus, we obtain that $F + \gamma I$ be a $\frac{1}{\gamma+\alpha^2}-$inverse-strong-monotone mapping. □

Lemma 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$, F is a $\frac{1}{\alpha^2}-$inverse-strong-monotone mapping of C into H, $\gamma \in \mathbb{R}, \lambda \in (0, 1)$ and $0 < \lambda \leq \frac{2}{\alpha^2+2\gamma}$, then $I-\lambda(F+\gamma I)$ is a contractive mapping with coefficient $(1 - \lambda\gamma)$.*

Proof. Since F is a $\frac{1}{\alpha^2}-$inverse-strong-monotone mapping, we have

$$\langle Fx - Fy, x - y\rangle \geq \frac{1}{\alpha^2}\|Fx - Fy\|^2.$$

This implies that

$$\begin{aligned}
&||(I - \lambda(F + \gamma I))x - (I - \lambda(F + \gamma I))y||^2 \\
&\leq ||(1 - \lambda\gamma)(x - y) - \lambda(Fx - Fy)||^2 \\
&\leq (1 - \lambda\gamma)^2||x - y\|^2 - 2(1 - \lambda\gamma)\lambda\langle x - y, Fx - Fy\rangle + \lambda^2\|Fx - Fy||^2 \\
&\leq (1 - \lambda\gamma)^2||x - y\|^2 - \frac{2}{\alpha^2}(1 - \lambda\gamma)\lambda\|Fx - Fy\|^2 + \lambda^2\|Fx - Fy||^2 \\
&\leq (1 - \lambda\gamma)^2||x - y\|^2 - \lambda(\frac{2(1 - \lambda\gamma)}{\alpha^2} - \lambda)\|Fx - Fy\|^2 \\
&\leq (1 - \lambda\gamma)^2||x - y\|^2. \\
&\leq (1 - \lambda\gamma)^2||x - y\|^2.
\end{aligned} \tag{2.3}$$

So, $I - \lambda(F + \gamma I)$ is a contractive mapping with coefficient $(I - \lambda\gamma)$. □

Lemma 2.3. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator, and A^* be the adjoint of A. Let C be a nonempty closed convex subset of H_2, and let $G : H_2 \to H_2$ be a firmly nonexpansive mapping. Then $A^*(I - G)A$ is a $\frac{1}{\|A\|^2}-$ism, that is,*

$$\frac{1}{\|A\|^2}||A^*(I - G)Ax - A^*(I - G)Ay||^2 \leq \langle x - y, A^*(I - G)Ax - A^*(I - G)Ay\rangle$$

for all $x, y \in H_1$.

Proof. Since G is a firmly nonexpansive mapping. Hence,

$$\begin{aligned}
&||A^*(I-G)Ax - A^*(I-G)Ay||^2\\
&\leq \|A\|^2||(I-G)Ax-(I-G)Ay||^2\\
&= \|A\|^2(||Ax-Ay||^2 + ||GAx-GAy||^2 - 2\langle Ax-Ay, GAx-GAy\rangle)\\
&\leq \|A\|^2(||Ax-Ay||^2 - \langle Ax-Ay, GAx-GAy\rangle)\\
&= \|A\|^2(\langle Ax-Ay, (I-G)Ax-(I-G)Ay\rangle)\\
&= \|A\|^2(\langle x-y, A^*(I-G)Ax - A^*(I-G)Ay\rangle)
\end{aligned}$$

for all $x, y \in H$. Therefore, $A^*(I-G)A$ is $\frac{1}{\|A\|^2}-$ ism. □

Lemma 2.4 ([2])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G : H \to H$ be a firmly nonexpansive mapping. Suppose that $F(G) \neq \emptyset$. Then $\langle x - Gx, Gx - w\rangle \geq 0$ for each $x \in H$ and each $w \in Fix(G)$.*

Lemma 2.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G : H \to H$ be a firmly nonexpansive mapping. Suppose that $Fix(G) \neq \emptyset$. Then $\|x - Gx\|^2 + \|Gx - w\|^2 \leq \|x - w\|^2$ for each $x \in H$ and each $w \in Fix(G)$.*

Proof. By Lemma 2.4, we have

$$\langle x - Gx, Gx - w\rangle \geq 0$$

for each $x \in H$ and each $w \in Fix(G)$. Using (2.1), we have that

$$2\langle x - Gx, Gx - w\rangle = -\|x - Gx\|^2 + \|x - w\|^2 - \|Gx - w\|^2 \geq 0,$$

for each $x \in H$ and each $w \in Fix(G)$. Hence, we have that

$$\|x - Gx\|^2 + \|Gx - w\|^2 \leq \|x - w\|^2$$

for each $x \in H$ and each $w \in Fix(G)$. □

We also know that the metric projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x$ satisfying the property $||x - P_C x|| = \inf_{y\in C} ||x - y||$. The following Lemma is a special case of Lemma 2.4.

Lemma 2.6 ([23])**.** *Let C be a nonempty closed convex subset of a Hilbert space H. Let P_C be the metric projection from H onto C. Then for each $x \in H$, $\langle x - P_C x, P_C x - y\rangle \geq 0$ for all $y \in C$.*

Proof. Since P_C is a firmly nonexpansive mapping. It is easy to see that $Fix(P_C) = C$. Put $Gx = P_C x$ in Lemma 2.4, for all $x \in H$. Then Lemma 2.6 follows from Lemma 2.4. □

In 2013, He and Du [14] gave the following result which is an special case of Lemma 2.5.

Lemma 2.7 ([14])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4). Take any $\alpha > 0$ and let α be fixed. Suppose that $EP(G) \neq \emptyset$. Then $||x - T^G_\alpha x||^2 + ||T^G_\alpha x - \bar{x}||^2 \leq ||x - \bar{x}||^2$ for each $x \in H$ and each $\bar{x} \in EP(G)$.*

Proof. Lemma 2.7 follows immediately from Lemma 2.5 and Theorem 1.3. □

Lemma 2.8 ([4]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive mapping of C into itself, and let $\{x_n\}$ be a sequence in C. If $x_n \rightharpoonup w$ and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, then $Tw = w$.*

Lemma 2.9 ([21]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X, and let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Suppose that $x_{n+1} = \alpha_n y_n + (1-\alpha_n)x_n$ for each $n \in \mathbb{N}$, and $\limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0$. Then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.*

We also know the following lemma from [23].

Lemma 2.10 ([23]). *Let H be a Hilbert space and B be a maximal monotone mapping on H. Let J_r is the resolvent of B defined by $J_r = (I + rB)^{-1}$ for each $r > 0$.*

(i) *For each $r > 0$, J_β is single-valued and firmly nonexpansive;*
(ii) *$\mathcal{D}(J_\beta) = H$ and $Fix(J_\beta) = \{x \in \mathcal{D}(A) : 0 \in Ax\}$.*

Lemma 2.11 ([10, 26]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a mapping. Then the following satisfied:*

(i) *T is nonexpansive if and only if the complement $(I - T)$ is $1/2$-ism.*
(ii) *If S is υ-ism, then for $\gamma > 0$, γS is υ/γ-ism.*
(iii) *S is averaged if and only if the complement $I - S$ is υ-ism for some $\upsilon > 1/2$.*
(iv) *If S and T are both averaged, then the product (composite) ST is averaged.*
(v) *If the mappings $\{T_i\}_{i=1}^n$ are averaged and have a common fixed point, then $\bigcap_{i=1}^n Fix(T_i) = Fix(T_1 \cdots T_n)$. The notation $Fix(T)$ denotes the set of all fixed points of the mapping T, that is, $Fix(T) = \{x \in H : Tx = x\}$.*

Lemma 2.12 ([1]). *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0,1]$ with $\sum_{n=1}^\infty \alpha_n = \infty$, $\{u_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^\infty u_n < \infty$, $\{t_n\}$ a sequence of real numbers with* $\limsup t_n \leq 0$. *Suppose that $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n t_n + u_n$ for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n = 0$.*

3. Main results

In this section, we first establish a strong convergence theorem with a Regularized-like method to find an element of the set of solutions for a monotone inclusion problem in a Hilbert space.

Theorem 3.1. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\alpha > 0$, F is a $\frac{1}{\alpha^2}$-inverse-strongly monotone mapping of C into H. Let B be a maximal monotone mapping on H and let G be a maximal monotone mapping on H such that the domains of B and G are included in C. Let $J_\rho = (I + \rho B)^{-1}$ and $T_r = (I + rG)^{-1}$ for each $\rho > 0$ and $r > 0$. Suppose that $(F + B)^{-1}0 \bigcap G^{-1}0 \neq \emptyset$. Let $\{x_n\} \subset H$ be defined by*

$$(3.1)\quad \begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ x_{n+1} = J_\rho(I - \rho(F + \beta_n I))T_r x_n \end{cases}$$

for each $n \in \mathbb{N}$, $\rho \in (0,\infty)$, $\beta_n \in (0,1)$, *and* $r \in (0,\infty)$. *Assume that:*

(i) $0 < a \leq \rho < \frac{2}{\alpha^2+2}$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\Sigma_{n=1}^{\infty}\beta_n = \infty$.

Then $\lim\limits_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{(F+B)^{-1}0\bigcap G^{-1}0}(0)$.

Proof. By Lemma 2.10, we know that $J_\rho = (I+\rho B)^{-1}$ and $T_r = (I+rG)^{-1}$ are firmly nonexpansive mappings, for each $\rho > 0$ and $r > 0$. It follows from $0 < a \leq \rho < \frac{2}{\alpha^2+2}$ and F is a $\frac{1}{\alpha^2}-$ism that we have

(3.1)

$$\begin{aligned}
&||J_\rho(I-\rho(F+\beta_n I))T_r x - J_\rho(I-\rho(F+\beta_n I))T_r y||^2\\
\leq\ & ||(I-\rho(F+\beta_n I))T_r x - (I-\rho(F+\beta_n I))T_r y||^2\\
\leq\ & ||(1-\rho\beta_n)(T_r x - T_r y) - \rho(FT_r x - FT_r y)||^2\\
\leq\ & (1-\rho\beta_n)^2||T_r x - T_r y\|^2 - 2(1-\rho\beta_n)\rho\langle T_r x - T_r y, FT_r x - FT_r y\rangle\\
&+\ \rho^2\|FT_r x - FT_r y||^2\\
\leq\ & (1-\rho\beta_n)^2||T_{r_n} x - T_{r_n} y\|^2 - \frac{2}{\alpha^2}(1-\rho\beta_n)\rho\|FT_r x - FT_r y\|^2 + \rho^2\|FT_r x - FT_r y||^2\\
\leq\ & (1-\rho\beta_n)^2||T_r x - T_r y\|^2 - \rho\Big(\frac{2}{\alpha^2}(1-\rho) - \rho\Big)\|FT_r x - FT_r y\|^2\\
\leq\ & (1-\rho\beta_n)^2||T_r x - T_r y\|^2\\
\leq\ & (1-\rho\beta_n)^2||x - y\|^2.
\end{aligned}$$

Let $\bar{x} \in (F+B)^{-1}0\bigcap G^{-1}0$. It follows from Lemma 2.10(ii), we have

$$\bar{x} = J_\rho(I-\rho F)\bar{x} \text{ and } \bar{x} = T_r\bar{x}. \tag{3.2}$$

Let $u_n = T_r x_n$. For each $n \in \mathbb{N}$, we have from (3.1), and (3.1) that

$$\begin{aligned}
||x_{n+1} - \bar{x}|| &= ||J_\rho(I-\rho(F+\beta_n I))T_r x_n - J_\rho(I-\rho F)T_r\bar{x}||\\
&\leq ||J_\rho(I-\rho(F+\beta_n I))T_r x_n - J_\rho(I-\rho(F+\beta_n I))T_r\bar{x}||\\
&\quad + ||J_\rho(I-\rho(F+\beta_n I))T_r\bar{x} - J_\rho(I-\rho F)T_r\bar{x}||\\
&\leq (1-\rho\beta_n)||x_n - \bar{x}\| + \rho\beta_n\|\bar{x}||\\
&\leq \max\{\|x_n - \bar{x}\|, \|\bar{x}||\}.
\end{aligned} \tag{3.3}$$

By induction, we deduce

$$||x_n - \bar{x}|| \leq \max\{\|x_1 - \bar{x}\|, \|\bar{x}||\}. \tag{3.4}$$

This indicates that the sequence $\{x_n\}$ is bounded. Furthermore, $\{u_n\}$ is bounded.

Since F be a $\frac{1}{\alpha^2}-$ism, and $\beta_n \in \mathbb{R}$, it follows from Lemma 2.11 that ρF is $\frac{1}{\rho\alpha^2}-$ism and $I-\rho F$ is $\frac{\rho\alpha^2}{2}-$averaged. That is,

$$I - \rho F = \Big(1 - \frac{\rho\alpha^2}{2}\Big)I + \frac{\rho\alpha^2}{2}T$$

for some nonexpansive mapping T. Since J_ρ is $1/2$ averaged, $J_\rho = (I+S)/2$ for some nonexpansive mapping S, T_r is also $1/2$ averaged, $T_r = (I+A)/2$ for some nonexpansive mapping A. Then, we can rewrite x_{n+1} as

$$x_{n+1} = \frac{2-\rho\alpha^2}{8}x_n + \frac{6+\rho\alpha^2}{8}y_n,$$

where

$$y_n = \frac{8}{6+\rho\alpha^2}\Big(\frac{2-\rho\alpha^2}{8}Ax_n + \frac{\rho\alpha^2}{4}TT_rx_n - \frac{1}{2}\rho\beta_nT_rx_n + \frac{1}{2}S(I-\rho(F+\beta_nI))T_rx_n\Big).$$

Hence, we have that

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \Big\|\frac{8}{6+\rho\alpha^2}\Big(\frac{2-\rho\alpha^2}{8}Ax_{n+1} + \frac{\rho\alpha^2}{4}TT_rx_{n+1} \\
&\quad -\frac{1}{2}\rho\beta_{n+1}T_rx_{n+1} + \frac{1}{2}S(I-\rho(F+\beta_{n+1}I))T_rx_{n+1}\Big) \\
&\quad -\frac{8}{6+\rho\alpha^2}\Big(\frac{2-\rho\alpha^2}{8}Ax_n + \frac{\rho\alpha^2}{4}TT_rx_n - \frac{1}{2}\rho\beta_nT_rx_n \\
&\quad +\frac{1}{2}S(I-\rho(F+\beta_nI))T_rx_n\Big)\Big\| \\
&\le \frac{8}{6+\rho\alpha^2}\Big(\frac{2-\rho\alpha^2}{8}\|Ax_{n+1}-Ax_n\| + \Big\|\frac{\rho\alpha^2}{4}TT_rx_{n+1} - \frac{\rho\alpha^2}{4}TT_rx_n\Big\| \\
&\quad +\frac{1}{2}\rho\beta_{n+1}\|T_rx_{n+1}\| + \frac{1}{2}\rho\beta_n\|T_rx_n\|\Big) \\
&\quad +\frac{4}{6+\rho\alpha^2}\|(I-\rho(F+\beta_{n+1}I))T_rx_{n+1} - (I-\rho(F+\beta_nI))T_rx_n\|.
\end{aligned}$$

Now, we choose a constant M such that

$$\sup_n\Big\{\|x_n\|, \|T_rx_n\|\Big\}\le M.$$

We have the following estimates:

$$\frac{2-\rho\alpha^2}{8}\|Ax_{n+1}-Ax_n\| \le \frac{2-\rho\alpha^2}{8}\|x_{n+1}-x_n\|,$$

$$\Big\|\frac{\rho\alpha^2}{4}TT_rx_{n+1} - \frac{\rho\alpha^2}{4}TT_rx_n\Big\|\le \frac{\rho\alpha^2}{4}\|TT_rx_{n+1}-TT_rx_n\| \le \frac{\rho\alpha^2}{4}\|x_{n+1}-x_n\|,$$

and

$$\begin{aligned}
&\|(I-\rho(F+\beta_{n+1}I))T_rx_{n+1} - (I-\rho(F+\beta_nI))T_rx_n\| \\
\le\ & \|(I-\rho F)T_rx_{n+1}) - (I-\rho F)T_rx_n)\| + \rho\beta_{n+1}\|T_rx_{n+1}\| + \rho\beta_n\|T_rx_n\| \\
\le\ & \|x_{n+1}-x_n\| + (\rho\beta_{n+1}+\rho\beta_n)M.
\end{aligned}$$

Thus, we deduce that

$$
\begin{aligned}
\|y_{n+1}-y_n\| \le{}& \frac{2-\rho\alpha^2}{6+\rho\alpha^2}\|x_{n+1}-x_n\| + \frac{2\rho\alpha^2}{6+\rho\alpha^2}\|x_{n+1}-x_n\| \\
&+ \frac{4}{6+\rho\alpha^2}(\|x_{n+1}-x_n\| + 2(\rho\beta_{n+1}+\rho\beta_n)M) \\
={}& \|x_{n+1}-x_n\| + \frac{8}{6+\rho\alpha^2}(\rho\beta_{n+1}+\rho\beta_n)M.
\end{aligned} \tag{3.5}
$$

By (3.5) and assumption,

$$
\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \le 0. \tag{3.6}
$$

By (3.6) and Lemma 2.9,

$$
\lim_{n\to\infty}\|y_n - x_n\| = 0.
$$

Consequently,

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}\frac{6+\rho\alpha^2}{8}||x_n - y_n|| = 0. \tag{3.7}
$$

Since $\{x_n\}$ is bounded, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \hat{w}$. Next, we have

$$
\begin{aligned}
\big\|x_{n_j} - J_\rho(I-\rho F)T_r x_{n_j}\big\| \le{}& \big\|x_{n_j} - x_{n_j+1}\big\| + \big\|x_{n_j+1} - J_\rho(I-\rho F)T_r x_{n_j}\big\| \\
={}& \big\|J_\rho(I-\rho(F+\beta_{n_j}I))T_r x_{n_j} - J_\rho(I-\rho F)T_r x_{n_j}\big\| \\
&+ \|x_{n_j} - x_{n_j+1}\big\| \\
\le{}& \beta_{n_j}\rho\big\|x_{n_j}\big\| + \big\|x_{n_j} - x_{n_j+1}\big\|.
\end{aligned} \tag{3.8}
$$

By (3.7), (3.8), and assumptions,

$$
\lim_{n_j\to\infty}\|x_{n_j} - J_\rho(I-\rho F)T_r x_{n_j}\big\| = 0. \tag{3.9}
$$

Since $J_\rho(I-\rho F)T_r$ is nonexpansive, it follows from Lemma 2.8 that $\hat{w} \in \mathrm{Fix}(J_\rho(I-\rho F)T_r)$. From (3.2), we have

$$
Fix(J_\rho(I-\rho F))\bigcap Fix(T_r) \neq \emptyset. \tag{3.10}
$$

Since $J_\rho(I-\rho F)$ and T_r are averaged, it follows from (3.10) and Lemma 2.11 that $\hat{w} \in \mathrm{Fix}(J_\rho(I-\rho F)T_r) = Fix(J_\rho(I-\rho F))\bigcap Fix(T_r)$. Hence, it follows from Lemma 2.10(ii), we have $\bar{w} \in (F+B)^{-1}0\bigcap G^{-1}0$.

Let $\hat{x}$ be the minimum norm solution of Ω. That is, $\hat{x} = P_\Omega 0$, where $\Omega = (F+B)^{-1}0\bigcap G^{-1}0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z$ and

$$
\limsup_{n\to\infty}\langle -\hat{x}, x_n - \hat{x}\rangle = \lim_{j\to\infty}\langle -\hat{x}, x_{n_j} - \hat{x}\rangle.
$$

As the above proof, we know that $z \in \Omega$. Hence,

$$
\limsup_{n\to\infty}\langle -\hat{x}, x_n - \hat{x}\rangle = \lim_{j\to\infty}\langle -\hat{x}, x_{n_j} - \hat{x}\rangle = \langle -\hat{x}, z - \hat{x}\rangle \le 0. \tag{3.11}
$$

Since J_ρ be a firmly nonexpansive, and by Lemma 2.2, we have the following:

(3.12)

$$\begin{aligned}
||x_{n+1}-\hat{x}||^2 &= ||J_\rho(I-\rho(F+\beta_n I))T_r x_n - J_\rho(I-\rho F)T_r\hat{x}||^2 \\
&\le \langle (I-\rho(F+\beta_n I))T_r x_n - (I-\rho F)T_r\hat{x}, x_{n+1}-\hat{x}\rangle \\
&\le \langle (I-\rho(F+\beta_n I))T_r x_n - (I-\rho(F+\beta_n I))T_r\hat{x}, x_{n+1}-\hat{x}\rangle \\
&\quad + \rho\beta_n\langle -T_r\hat{x}, x_{n+1}-\hat{x}\rangle \\
&\le \|(I-\rho(F+\beta_n I))T_r x_n - (I-\rho(F+\beta_n I))T_r\hat{x}\| \cdot \|x_{n+1}-\hat{x}\| \\
&\quad + \beta_n\rho\langle -T_r\hat{x}, x_{n+1}-\hat{x}\rangle \\
&\le (1-\rho\beta_n)||x_n-\hat{x}\| \cdot \|x_{n+1}-\hat{x}\| + \beta_n\rho\langle -\hat{x}, x_{n+1}-\hat{x}\rangle \\
&\le \frac{(1-\rho\beta_n)}{2}||x_n-\hat{x}\|^2 + \frac{1}{2}\|x_{n+1}-\hat{x}\|^2 + \beta_n\rho\langle -\hat{x}, x_{n+1}-\hat{x}\rangle.
\end{aligned}$$

It follows that

$$\|x_{n+1}-\hat{x}\|^2 \le (1-\beta_n\rho)\|x_n-\hat{x}\|^2 + 2\beta_n\rho\langle -\hat{x}, x_{n+1}-\hat{x}\rangle. \tag{3.13}$$

By assumptions, (3.11), (3.13), and Lemma 2.12, we know that $x_n \to \hat{x}$. Therefore, the proof is completed. □

Let C_1 and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let F_1 be a firmly nonexpansive mapping of H_2 into H_2. Let B be a maximal monotone mapping on H_1 and let G be a maximal monotone mapping on H_1 such that the domains of B and G are included in C_1. Let $J_\lambda = (I+\lambda B)^{-1}$ and $T_r = (I+rG)^{-1}$ for each $\lambda > 0$ and $r > 0$. Let $A : H_1 \to H_2$ be a bounded linear operator, and let A^* be the adjoint of A. Now, we recall the following problem:

(**GSFP$_{\mathbf{FF}}$**) Find $\bar{x} \in H_1$ such that $\bar{x} \in Fix(J_\lambda) \bigcap Fix(T_r)$ and $A\bar{x} \in Fix(F_1)$.

In order to to study the convergence theorems for the solution set of generalized split feasibility problem (**GSFP$_{\mathbf{FF}}$**), we must give an essential result in this paper.

Theorem 3.2. *Given any $\bar{x} \in H_1$.*

(i) *If $\bar{x}$ is a solution of* (**GSFP$_{\mathbf{FF}}$**), *then $J_\lambda(I-\rho A^*(I-F_1)A)T_r\bar{x}) = \bar{x}$.*

(ii) *Suppose that $J_\lambda(I-\rho A^*(I-F_1)A)T_r\bar{x}) = \bar{x}$ with $0 < \rho < \frac{2}{\|A\|^2+2}$ and the solution set of* (**GSFP$_{\mathbf{FF}}$**) *is nonempty. Then $\bar{x}$ is a solution of* (**GSFP$_{\mathbf{FF}}$**).

Proof. (i) Suppose that $\bar{x} \in H_1$ is a solution of (**GSFP$_{\mathbf{FF}}$**). Then $\bar{x} \in Fix(J_\lambda) \bigcap Fix(T_r)$ and $A\bar{x} \in Fix(F_1)$. It is easy to see that

$$J_\lambda(I-\rho A^*(I-F_1)A)T_r\bar{x} = J_\lambda(\bar{x}-\rho A^*(I-F_1)A\bar{x}) = J_\lambda\bar{x} = \bar{x}.$$

(ii)Suppose that $J_\lambda(I-\rho A^*(I-F_1)A)T_r\bar{x}) = \bar{x}$ with $0 < \rho < \frac{2}{\|A\|^2+2}$ and the solution set of (**GSFP$_{\mathbf{FF}}$**) is nonempty.

Since the solution set of (**GSFP$_{\mathbf{FF}}$**) is nonempty, there exists $\bar{w} \in H_1$ such that $\bar{w} \in Fix(J_\lambda) \bigcap Fix(T_r)$ and $A\bar{w} \in Fix(F_1)$. So,

$$\bar{w} \in Fix(J_\lambda) \bigcap Fix(I-\rho A^*(I-F_1)A)) \bigcap Fix(T_r) \ne \emptyset. \tag{3.14}$$

By Lemma 2.3, we have that

$$A^*(I-F_1)A \text{ is} \frac{1}{\|A\|^2} - ism. \tag{3.15}$$

By (3.15), $0 < \rho < \frac{2}{\|A\|^2+2}$, and lemma 2.11(ii),(iii), we know that

$$I - \rho A^*(I - F_1)A \text{ is averaged.} \tag{3.16}$$

On the other hand, since J_λ, and T_r are firmly nonexpansive mappings, it is easy to see that

$$J_\lambda \text{ and } T_r \text{ are } \frac{1}{2} \text{ avereged.} \tag{3.17}$$

Hence, by (3.14), (3.16), (3.17) and Lemma 2.11(v), we have that

$$\bar{x} \in J_\lambda(I - \rho A^*(I - F_1)A)T_r\bar{x}) = Fix(J_\lambda) \bigcap Fix(I - \rho A^*(I - F_1)A)) \bigcap Fix(T_r).$$

By Lemma 2.4,

$$\langle(\bar{x} - \rho A^*(I - F_1)A\bar{x}) - \bar{x}, \bar{x} - w\rangle \geq 0 \text{ for each}$$
$$w \in Fix(J_\lambda) \bigcap Fix(T_r) \text{ and } Aw \in Fix(F_1).$$

That is,

$$\langle A^*(I - F_1)A\bar{x}, \bar{x} - w\rangle \leq 0 \text{ for each } w \in Fix(J_\lambda) \bigcap Fix(T_r) \text{ and } Aw \in Fix(F_1). \tag{3.18}$$

By (3.18) and A^* is the adjoint of A,

$$\langle A\bar{x} - F_1A\bar{x}, A\bar{x} - Aw\rangle \leq 0 \text{ for each } w \in Fix(J_\lambda) \bigcap Fix(T_r) \text{ and } Aw \in Fix(F_1). \tag{3.19}$$

On the other hand, by Lemma 2.4 again,

$$\langle A\bar{x} - F_1A\bar{x}, v - F_1A\bar{x}\rangle \leq 0 \text{ for each } v \in Fix(F_1). \tag{3.20}$$

By (3.19) and (3.20),

$$\langle A\bar{x} - F_1A\bar{x}, v - F_1A\bar{x} + A\bar{x} - Aw\rangle \leq 0 \tag{3.21}$$

for each $w \in Fix(J_\lambda) \bigcap Fix(T_r)$ and $Aw \in Fix(F_1)$ and each $v \in Fix(F_1)$. That is,

$$||A\bar{x} - F_1A\bar{x}||^2 \leq \langle A\bar{x} - F_1A\bar{x}, Aw - v\rangle \tag{3.22}$$

for each $w \in Fix(J_\lambda) \bigcap Fix(T_r)$ and $Aw \in Fix(F_1)$ and each $v \in Fix(F_1)$. Since $\bar{w}$ is a solution of generalized split feasibility problem ($\mathbf{GSFP_{FF}}$), we know that $\bar{w} \in Fix(J_\lambda) \bigcap Fix(T_r)$ and $A\bar{w} \in Fix(F_1)$. So, it follows from (3.22) that $A\bar{x} = Fix(F_1)$. Further, $\bar{x} \in Fix(J_\lambda)$ and $\bar{x} \in Fix(T_r)$. Therefore, $\bar{x}$ is a solution of ($\mathbf{GSFP_{FF}}$). □

Apply Theorem 3.1, and Theorem 3.2, we can find the solution of ($\mathbf{GSFP_{FF}}$).

Theorem 3.3. *Let C_1 and C_2 be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let F_1 be a firmly nonexpansive mapping of H_2 into H_2. Let B be a maximal monotone mapping on H_1 and let G be a maximal monotone mapping on H_1 such that the domains of B and G are included in C_1. Let $J_\lambda = (I+\lambda B)^{-1}$ and $T_r = (I+rG)^{-1}$ for each $\lambda > 0$ and $r > 0$. Let $A : C_1 \to H_2$ be a bounded linear operator, and let A^* be the adjoint of A. Suppose that the solution set of* ($\mathbf{GSFP_{FF}}$) *is nonempty. Let $\{x_n\} \subset H$ be defined by*

$$\begin{cases} x_1 \in H \text{ chosen arbitrarily,} \\ x_{n+1} := J_\rho((1 - \beta_n\rho)I - \rho A^*(I - F_1)A)T_rx_n \end{cases} \tag{3.2}$$

for each $n \in \mathbb{N}$, $\rho \in (0,\infty)$, $\beta_n \in (0,1)$, *and* $r \in (0,\infty)$. *Assume that:*

(i) $0 < a \leq \rho < \frac{2}{\|A\|^2+2}$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\Sigma_{n=1}^{\infty}\beta_n = \infty$.

Then $\lim\limits_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x}$ *is a solution of* $(\mathbf{GSFP_{FF}})$.

Proof. Since F_1 is a firmly nonexpansive, it follow from Lemma 2.3 that we have that $A^*(I-F_1)A : C_1 \to H_1$ is $\frac{1}{\|A\|^2}$−ism. Put $F = A^*(I-F_1)A$ in Theorem 3.1. Then algorithm (3.1) in Theorem 3.1 follows immediately from algorithm (3.2) in Theorem 3.3. Since the solution set of $(\mathbf{GSFP_{FF}})$ is nonempty, there exist $w \in C_1$, such that $w \in Fix(J_\lambda)\bigcap Fix(T_r)$ and $Aw \in Fix(F_1)$. Hence, we have that $w \in Fix(J_\rho(I-\rho A^*(I-F_1)A)T_r) = Fix(J_\rho(I-\rho F)T_r)$. Therefore, we have that $w \in (F+B)^{-1}0\bigcap G^{-1}0 \neq \emptyset$. It follow from Theorem 3.1 that $\lim\limits_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{(F+B)^{-1}0\bigcap G^{-1}0}(0)$. that is,

$$\bar{x} = J_\rho(I-\rho F)T_r\bar{x} = J_\rho(I-\rho A^*(I-F_1)A)T_r\bar{x}. \tag{3.23}$$

By assumptions, (3.23), and Theorem 3.2(ii), we know that $\bar{x}$ is a solution of $(\mathbf{GSFP_{FF}})$. Therefore, the proof is completed. □

Remark 3.4. In Theorem 3.3, we establish a strongly convergence theorem of generalized split feasibility problem $(\mathbf{GSFP_{FF}})$ without calculating the inverse of the operator we consider.

4. Application

Takahashi, Takahashi and Toyoda [22] showed the following lemma.

Lemma 4.1 ([22])**.** *Let* C *be a nonempty closed convex subset of a Hilbert space* H *and let* $g : C \times C \to \mathbb{R}$ *be a bifunction satisfying the conditions (A1)-(A4). Define* A_g *as follows:*

$$A_g x = \begin{cases} \{z \in H : g(x,y) \geq \langle y-x, z\rangle, \forall y \in C\}, \forall x \in C \\ \emptyset, \forall x \notin C \end{cases} \tag{4.1}$$

Then, $EP(g) = A_g^{-1}0$ *and* A_g *is a maximal monotone operator with the domain of* $A_g \subset C$. *Furthermore, for any* $x \in H$ *and* $r > 0$, *the resolvent* T_r^g *of* g *coincides with the resolvent of* A_g, *i.e.,* $T_r^g x = (I + rA_g)^{-1}x$.

Now, we recall the following problem:

$(\mathbf{GSFP_{EE}})$ Find $\bar{x} \in H_1$ such that $\bar{x} \in EP(g_1)\bigcap EP(g_2)$ and $A\bar{x} \in EP(g_3)$.

Apply Theorems 1.4, and 3.3, Lemma 4.1, we get the following result.

Theorem 4.2. *Let* C_1, C_2 *and* Q *be three nonempty closed convex subsets of three Hilbert spaces* H_1, H_1 *and* H_2, *respectively. Let* $g_1 : C_1 \times C_1 \to \mathbb{R}$, $g_2 : C_2 \times C_2 \to \mathbb{R}$ *and* $g_3 : Q \times Q \to \mathbb{R}$ *with conditions (A1)-(A4), and let* $T_{\rho_i}^{g_i}$ *the resolvent of* g_i *for* $\rho_i > 0, i = 1,2,3$. *Let* $A : H_1 \to H_2$ *be a bounded linear operator, and let* A^* *denote the adjoint of* A. *Suppose that the solution set of* $(\mathbf{GSFP_{EE}})$ *is nonempty. Let* $\{x_n\}$ *be defined by*

$$\begin{cases} x_1 \in H_1 \text{ chosen arbitrarily,} \\ x_{n+1} := T_{\rho_1}^{g_1}((1-\beta_n\rho_1)I - \rho_1 A^*(I - T_{\rho_3}^{g_3})A)T_{\rho_2}^{g_2}x_n \end{cases} \tag{4.2}$$

for each $n \in \mathbb{N}$, $\rho_i \subset (0,\infty), i = 1,2,3,$ *and* $\beta_n \subset (0,1)$. *Assume that:*

(i) $0 < a \le \rho_1 < \frac{2}{\|A\|^2+2}$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\Sigma_{n=1}^{\infty}\beta_n = \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $\hat{x}$, *where* $\hat{x}$ *is an element of the solution set of* $(\mathbf{GSFP_{EE}})$.

Proof. Define A_g as (4.1). By Lemma 4.1, we know that $EP(g) = A_g^{-1}0$ and A_g is a maximal monotone operator with the domain of $A_g \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r^g of g coincides with the resolvent of A_g, i.e.,

$$T_r^g x = (I + rA_g)^{-1}x.$$

By Theorem 1.4, $T_{\rho_3}^{g_3}$ is a firmly nonexpansive mapping.

Put $B = A_{g_1}$, $G = A_{g_2}$ and $F_1 = T_{\rho_3}^{g_3}$ in Theorem 3.3. Then $J_{\rho_1}x = (I + \rho_1 A_{g_1})^{-1}x = T_{\rho_1}^{g_1}x$, $T_{\rho_2}x = (I + \rho_2 A_{g_2})^{-1}x = T_{\rho_2}^{g_2}x$. By Theorem 1.4, we have that $Fix(J_{\rho_1}) = Fix(T_{\rho_1}^{g_1}) = EP(g_1)$, $Fix(T_{\rho_2}) = Fix(T_{\rho_2}^{g_2}) = EP(g_2)$ and $Fix(F_1) = Fix(T_{\lambda}^{g_3}) = EP(g_3)$. So, we have that the solution set of $(\mathbf{GSFP_{EE}})$ coincides with the solution set of $(\mathbf{GSFP_{FF}})$, we get the result. □

Now, we recall the following problem:

$(\mathbf{SFP_{EE}})$ Find $\bar{x} \in C_1$ such that $\bar{x} \in EP(g_1)$ and $A\bar{x} \in EP(g_3)$.

Apply Theorem 4.2, we can finding the solution of $(\mathbf{SFP_{EE}})$.

Theorem 4.3. Let C_1 and Q be three nonempty closed convex subsets of H_1, H_1 and H_2, respectively. Let $g_1 : C_1 \times C_1 \to \mathbb{R}$ and $g_3 : Q \times Q \to \mathbb{R}$ with conditions (A1)-(A4), and let $T_{\rho_i}^{g_i}$ the resolvent of g_i for $\rho_i > 0, i = 1,3$. Let $A : H_1 \to H_2$ be a bounded linear operator, and let A^* denote the adjoint of A. Suppose that the solution set of $(\mathbf{SFP_{EE}})$ is nonempty. Let $\{x_n\}$ be defined by by

$$(4.3)\quad \begin{cases} x_1 \in C_1 \text{ chosen arbitrarily,} \\ x_{n+1} := T_{\rho_1}^{g_1}((1-\beta_n\rho_1)I - \rho_1 A^*(I - T_{\rho_3}^{g_3})A)x_n. \end{cases}$$

for each $n \in \mathbb{N}$, $\rho_i \in (0,\infty), i = 1,2,3,$ and $\beta_n \in (0,1)$. Assume that:

(i) $0 < a \le \rho_1 < \frac{2}{\|A\|^2+2}$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ and $\Sigma_{n=1}^{\infty}\beta_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to $\hat{x}$, where $\hat{x}$ is an element of the solution set of $(\mathbf{SFP_{EE}})$.

Proof. Put $g_2(x,y) = 0, \forall x,y \in C_1$ and $C_1 = C_2$ in Theorem 4.2, Then $T_{\rho_2}^{g_2}x = P_{C_1}x$. By Theorem 4.2, we get the result. □

Now, we recall the following problem:

$(\mathbf{GSFP_{CE}})$ Find $\bar{x} \in H_1$ such that $\bar{x} \in C_1 \bigcap C_3$ and $A\bar{x} \in EP(g_3)$.

Apply Theorem 4.2, we can find the solution of $(\mathbf{GSFP_{CE}})$.

Theorem 4.4. *Let* C_1, C_2 *and* Q *be three nonempty closed convex subsets of* H_1, H_1 *and* H_2, *respectively. Let* $g_3 : Q \times Q \to \mathbb{R}$ *with conditions (A1)-(A4), and let* $T_{\rho_3}^{g_3}$ *the resolvent of* g_3 *for* $\rho_3 > 0$. *Let* $A : H_1 \to H_2$ *be a bounded linear operator, and let* A^* *denote the adjoint of* A. *Suppose that the solution set of* $(\mathbf{GSFP_{CE}})$ *is nonempty. Let* $\{x_n\}$ *be defined by by*

$$(4.4) \begin{cases} x_1 \in H_1 \text{ chosen arbitrarily,} \\ x_{n+1} := P_{C_1}((1-\beta_n\rho_1)I - \rho_1 A^*(I - T^{g_3}_{\rho_3})A)P_{C_2}x_n \end{cases}$$

for each $n \in \mathbb{N}$, $\rho_i \in (0,\infty), i = 1,2,3,$ *and* $\beta_n \in (0,1)$. *Assume that:*

(i) $0 < a \le \rho_1 < \frac{2}{\|A\|^2+2}$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\Sigma_{n=1}^{\infty}\beta_n = \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $\hat{x}$, *where* $\hat{x}$ *is an element of the solution set of* $(\mathbf{GSFP_{CE}})$.

Proof. Put $g_1(x,y) = 0, \forall x,y \in C_1$ and $g_2(x,y) = 0, \forall x,y \in C_2$ in Theorem 4.2, Then $T^{g_1}_{\rho_1}x = P_{C_1}x$. and $T^{g_2}_{\rho_2}x = P_{C_2}x$. By Theorem 4.2, we get the result. □

Now, we recall the following problem:

$(\mathbf{GSFP_{CQ}})$ Find $\bar{x} \in H_1$ such that $\bar{x} \in C_1 \bigcap C_2$ and $A\bar{x} \in Q$.

Apply Theorem 4.2, we can find the solution of $(\mathbf{GSFP_{CQ}})$.

Theorem 4.5. *Let* C_1, C_2 *and* Q *be three nonempty closed convex subsets of* H_1, H_1 *and* H_2, *respectively. Let* $A : H_1 \to H_2$ *be a bounded linear operator, and let* A^* *denote the adjoint of* A. *Suppose that the solution set of* $(\mathbf{GSFP_{CQ}})$ *is nonempty. Let* $\{x_n\}$ *be defined by by*

$$(4.4) \begin{cases} x_1 \in H_1 \text{ chosen arbitrarily,} \\ x_{n+1} := P_{C_1}((1-\beta_n\rho_1)I - \rho_1 A^*(I - P_Q)A)P_{C_2}x_n \end{cases}$$

for each $n \in \mathbb{N}$, $\rho_1 \in (0,\infty)$, *and* $\beta_n \in (0,1)$. *Assume that:*

(i) $0 < a \le \rho_1 < \frac{2}{\|A\|^2+2}$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\Sigma_{n=1}^{\infty}\beta_n = \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $\hat{x}$, *where* $\hat{x}$ *is an element of the solution set of* $(\mathbf{GSFP_{CQ}})$.

Proof. Put $g_1(x,y) = 0, \forall x,y \in C_1$, $g_2(x,y) = 0, \forall x,y \in C_2$ and $g_3(x,y) = 0, \forall x,y \in Q$ in Theorem 4.2, Then $T^{g_1}_{\rho_1}x = P_{C_1}x$, $T^{g_2}_{\rho_2}x = P_{C_2}x$ and $T^{g_3}_{\rho_3}x = P_Q x$. By Theorem 4.2, we get the result. □

Now, we recall the following problem:

$(\mathbf{SFP_{CQ}})$ Find $\bar{x} \in H_1$ such that $\bar{x} \in C_1$ and $A\bar{x} \in Q$.

Apply Theorem 4.2, we can find the solution of $(\mathbf{SFP_{CQ}})$.

Theorem 4.6. *Let* C_1 *and* Q *be three nonempty closed convex subsets of* H_1 *and* H_2, *respectively. Let* $A : H_1 \to H_2$ *be a linear and bounded operator, and let* A^* *denote the adjoint of* A. *Suppose that the solution set of* $(\mathbf{GSFP_{CQ}})$ *is nonempty. Let* $\{x_n\}$ *be defined by by*

$$(4.5) \begin{cases} x_1 \in C_1 \text{ chosen arbitrarily,} \\ x_{n+1} := P_{C_1}((1-\beta_n\rho_1)I - \rho_1 A^*(I - P_Q)A)x_n \end{cases}$$

for each $n \in \mathbb{N}$, $\rho_1 \in (0,\infty)$, *and* $\beta_n \in (0,1)$. *Assume that:*

(i) $0 < a \le \rho_1 < \frac{2}{\|A\|^2+2}$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\Sigma_{n=1}^{\infty}\beta_n = \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $\hat{x}$, *where* $\hat{x}$ *is an element of the solution set of* $(\mathbf{GSFP_{CQ}})$.

Proof. Put $g_1(x,y) = 0, \forall x, y \in C_1$, $g_2(x,y) = 0, \forall x, y \in C_2$, $g_3(x,y) = 0, \forall x, y \in Q$ and $C_1 = C_2$ in Theorem 4.2, Then $T^{g_1}_{\rho_1} = P_{C_1}$, $T^{g_2}_{\rho_2} = P_{C_2}$ and $T^{g_3}_{\rho_3} = P_Q$. By Theorem 4.2, we get the result. □

Remark 4.7. (i) Theorem 4.6 is different CQ method; (ii) Theorem 4.6 give a strongly convergent theorem, but Theorem 3.7 in [26] only study weak convergence theorem of the split feasibility problem. (iii) Theorem 4.6 also different from Theorem 3.7 in [26].

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