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WEAK CONTINUITY OF THE NORMALIZED DUALITY MAP

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This paper is dedicated to Professor Simeon Reich on the occasion of his 65th Birthday

Abstract. The duality maps play a crucial role in solving nonlinear problems in the setting of Banach spaces, due to the lack of an inner product on Banach spaces. Weak continuity of duality maps is required occasionally, and even weak continuity of the normalized duality is assumed by some authors. We will show that weak continuity of the normalized duality in infinite-dimensional spaces is an inappropriate assumption as the normalized duality map of the sequential space ℓ_p for $1 < p < \infty$ fails to be weakly continuous unless $p = 2$. Consequently, assuming weak continuity of a generalized duality map is more appropriate in dealing with nonlinear problems in Banach spaces.

1. Introduction

The use of duality maps as a tool to solve nonlinear problems in Banach spaces is initiated by Browder in a series of papers [4]-[10] in the 1960's. It is now realized that duality maps play a key role in solving both linear and nonlinear problems in Banach spaces such as nonlinear differential equations and semigroups [1, 2], nonlinear monotone mappings [25], optimization and control in Banach spaces [3], Since Banach spaces lack the inner product structure as opposed to Hilbert spaces, they only possess the so-called semi-inner product structure which is determined by duality maps. On the other hand, many geometrical properties of Banach spaces such as convexity and smoothness are characterized by duality maps.

Recall that a gauge is a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with the properties: (i) $\varphi(0) = 0$, (ii) φ is continuous and strictly increasing, and (iii) $\lim_{t\to\infty} \varphi(t) = \infty$. Associated with a gauge φ is the duality map $J_\varphi : X \to X^*$ ([5]) defined by

$$J_\varphi(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\| \cdot \|x^*\|,\ \varphi(\|x\|) = \|x^*\|\}. \tag{1.1}$$

If $\varphi(t) = t^{p-1}$ for all $t \geq 0$ and some $1 < p < \infty$, then the associated duality map is referred to as the generalized duality map of order p and is denoted by J_p. In particular, J_2 is known as the normalized duality map which is written as J. Thus it turns out that

$$J_p(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^p,\ \|x^*\| = \|x\|^{p-1}\} \tag{1.2}$$

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and

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}. \tag{1.3}$$

The different sorts of continuity of duality maps are crucial. The normalized duality of a Hilbert space H is the identity operator (here the dual space H^* is identified with H through the Riesz canonical embedding) and is therefore always continuous in either weak or strong topology. However, a general duality map J_φ is not so lucky; it is not always continuous in either weak or strong topology unless certain smoothness condition is satisfied. For instance, J_φ is norm-to-norm uniformly continuous over bounded sets if the space X is uniformly smooth. However, (sequential) weak continuity of J_φ is much more subtle. We say that a duality map J_φ of a Banach space X is sequentially weakly continuous if a sequence (x_n) in X is weakly convergent to x, then the sequence $(J_\varphi(x_n))$ in X^* is weak*ly convergent to $J_\varphi(x)$. Though the sequence space ℓ_p with $1 < p < \infty$ has a sequentially weakly continuous duality map J_φ with $\varphi(t) = t^{p-1}$, most of the useful Banach spaces such as $L^p[0,1]$ fail to have a sequentially weakly continuous duality map. [For the sake of simplicity, by 'weakly continuous' we always mean 'sequentially weak continuous' hereafter; also we will always assume, unless otherwise specified, that Banach spaces are infinite-dimensional throughout the rest of this paper.] However a weakly continuous duality map J_φ does provide a convenient tool for argument of weakly convergent sequences. As a matter of fact, we have the following result [19].

Proposition 1.1. *If a Banach space X has a weakly continuous duality map J_φ, then for any sequence (x_n) in X weakly convergent to x, there holds the identity*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|x - y\|), \quad y \in X, \tag{1.4}$$

where Φ is defined by

$$\Phi(t) = \int_0^t \varphi(s)ds. \tag{1.5}$$

It should be pointed out that even if a Banach space X has a weakly continuous duality map J_φ with some gauge φ, it does not mean that for another gauge ψ, the duality map J_ψ remains weakly continuous. It is easily understood that the weak continuity of the duality map J_φ depends on the selection of the gauge φ. Several authors (see e.g. [11, 15, 29, 18, 33]) use the assumption in their papers that the normalized duality map J of a Banach space is weakly continuous in order to extend some results from the setting of Hilbert spaces to that of Banach spaces. This assumption is however quite sensitive as we shall see that even the sequential space ℓ_p fails to have a weakly continuous normalized duality map (unless $p = 2$) although the pth-order generalized duality map J_p of ℓ_p is indeed weakly continuous. This indicates that the assumption of the weak continuity of the normalized duality map J seems an unappropriate condition imposed on Banach spaces; instead the weak continuity of a general duality map J_φ should be employed and assumed to be weakly continuous.

2. Basic properties of duality maps

Duality maps have extensively been studied and employed to solve problems in Banach spaces such as nonlinear equations, fixed point problems, and optimization; see [13, 14, 16, 17, 19, 21, 22, 23, 27, 18, 26, 40, 37, 36, 34, 35, 41, 42]. In this section we collect some basic properties of duality maps.

Proposition 2.1. ([10]) *Let X be a Banach space and φ a gauge function.*

(a) *For each $x \in X$, $J_\varphi(x)$ is a non-empty weak* closed convex subset of X^*.*
(b) *J_φ is an upper semi-continuous mapping of X into 2^{X^*} with X^* equipped with its weak* topology.*
(c) *J_φ is the subdifferential of the convex functional $\Phi(\|\cdot\|)$ (see (1.5)), that is,*

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad x \in X.$$

(d) *If ψ is another gauge, then $\varphi(\|x\|)J_\psi(x) = \psi(\|x\|)J_\varphi(x)$ for $x \in X$.*
(e) *Let $J_p : \ell_p \to \ell_q$, with $q = p/(p-1)$, be the pth-order generalized duality map of ℓ_p with $1 < p < \infty$, then*

$$J_p(x) = \{|x_n|^{p-1}\mathrm{sgn}(x_n)\} \tag{2.1}$$

for all $x = (x_n) \in \ell_p$; hence it is weakly continuous [4].
(f) *The pth-order generalized duality map of $L^p[0,1]$ with $1 < p < \infty$ is given by*

$$J_p^{L^p[0,1]}(f)(t) = \left(\frac{|f(t)|}{\|f\|_p}\right)^{p-1} \mathrm{sgn}(f(t)), \quad \text{a.e. } t \in [0,1] \tag{2.2}$$

for all $0 \neq f \in L^p[0,1]$. Hence, it fails to be weakly continuous [4, 24].

Recently there appeared a few paper assuming the weak continuity of J. This seems not quite reasonable since even for l^p this is false if $p \neq 2$. Note that l^p for $1 < p < \infty$ is the only known space which is reflexive and has a weak continuous generalized duality map [24].

Let X be a Banach space and let $S_X = \{x \in X : \|x\| = 1\}$ be its unit sphere. Consider the limit:

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}. \tag{2.3}$$

Recall we say that X is

- smooth (or Gateaux differentiable) if the limit (2.3) exists for each $x, y \in S_X$;
- uniformly Gateaux differentiable if it is smooth and the limit (2.3) is attained uniformly in $x \in S_X$ for each fixed $y \in S_X$;
- Frechet differentiable if it is smooth and the limit (2.3) is attained uniformly in $y \in S_X$ for each fixed $x \in S_X$;
- uniformly smooth (or uniformly Frechet differentiable) if it is smooth and the limit (2.3) is attained uniformly in $x, y \in S_X$.

Proposition 2.2. ([12], [31]) *Let X be a real Banach space. Then its normalized duality map J satisfies the following properties:*

(i) *J is homogeneous, i.e., $J(\lambda x) = \lambda J(x)$ for $\lambda \in \mathbb{R}$ and $x \in X$.*

(ii) *J is additive if and only if X is a Hilbert space.*
(iii) *J is single-valued if and only if X is smooth*
(iv) *J is surjective if and only if X is reflexive*
(v) *J is injective or strictly monotone if and only if X is strictly convex.*
(vi) *J is single-valued and norm-to-norm continuous if and only if X is Frechet differentiable.*
(vii) *if X is smooth (i.e., Gateaux differentiable), then J is single-valued and norm-to-weak* continuous.*
(viii) *if X is uniformly Gateaux differentiable, then J is single-valued and norm-to-weak* uniformly continuous on bounded sets of X.*
(ix) *J is single-valued and norm-to-norm uniformly continuous on bounded sets of X if and only if X is uniformly smooth.*

3. Weak continuity of duality maps

Let X be a (infinite-dimensional) Banach space and consider a duality map J_φ induced by gauge φ. In the sequel, we will denote $\to$ for strong convergence, $\rightharpoonup$ for weak convergence, and $\overset{*}{\rightharpoonup}$ for weak* convergence.

Definition 3.1. The duality map J_φ is said to be (sequentially) weak continuous [4] if J_φ maps weakly convergent sequences in X to weak*ly convergent sequences in X^*, that is, if $x_n \rightharpoonup x$ in X, then $J_\varphi(x_n) \overset{*}{\rightharpoonup} J_\varphi(x)$ in X^*.

Weak continuity of a duality map J_φ plays an important role in the fixed point theory for nonexpansive mappings. For instance, it is proved [23] that if X is a Banach space having a weakly continuous duality map J_φ, then it possesses Reich's property. This means that for each closed convex subset C of X and each nonexpansive mapping $T : C \to C$ with fixed points (recall that T is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$), if $x_\tau \in C$ is the unique fixed point of the contraction T_τ defined by

$$T_\tau x := \tau u + (1-\tau)Tx, \quad x \in C,$$

where $u \in C$ and $\tau \in (0,1)$, then (x_τ) converges in norm, as $\tau \to 0$, to a fixed point of T. Reich [30] proved that a uniformly smooth Banach space enjoys Reich's property. It is an interesting problem to find more Banach spaces which have Reich's property.

It is interesting to know what kind of (infinite-dimensional) Banach spaces can have a weakly continuous duality map. Browder [4] shows that for each $1 < p < \infty$, the pth generalized duality map of ℓ_p is weakly continuous, and that of L^p fails to be weakly continuous. Next we show that even for ℓ_p, its normalized duality map fails to be weakly continuous (except for $p = 2$).

Proposition 3.2. *For $1 < p < \infty$ and $p \neq 2$, the normalized duality map J of l^p is not weakly continuous.*

Proof. Recall, for any Banach space X, the relationship between its normalized duality map J and pth generalized duality map J_p is given by

$$J(x) = \frac{1}{\|x\|^{p-2}} J_p(x), \quad 0 \neq x \in X. \tag{3.1}$$

Now return to the space $X = \ell_p$. Define a sequence (x_n) as follows

$$x_n = \begin{cases} e_1, & \text{if } n \text{ is odd,} \\ e_n + e_1, & \text{if } n \text{ is even.} \end{cases} \tag{3.2}$$

Here (e_n) is the standard basis of ℓ_p. Then $x_n \rightharpoonup e_1$ and $J_p(x_n) \rightharpoonup J_p(e_1)$ by the weak continuity of J_p for ℓ_p. However, since

$$\|x_n\|_p = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 2^{1/p}, & \text{if } n \text{ is even.} \end{cases}$$

we get

$$J(x_n) = \frac{1}{\|x_n\|^{p-2}} J_p(x_n) \begin{cases} \rightharpoonup J_p(e_1), & \text{if } n \text{ (odd)} \to \infty, \\ \rightharpoonup 2^{(2-p)/p} J_p(e_1), & \text{if } n \text{ (even)} \to \infty. \end{cases}$$

As $p \neq 2$, $2^{(2-p)/p} > 1$. We conclude that $J(x_n)$ does not converge weakly. □

Proposition 3.3. *Let H be a real (infinite-dimensional) Hilbert space and let J_p be the generalized duality map of H for $1 < p < \infty$. Then J_p is weakly continuous if and only $p = 2$.*

Proof. It suffices to prove that J_p is not weakly continuous if $p \neq 2$. By (3.1) and the fact the normalized duality map of a Hilbert space is the identity, we get

$$J_p(x) = \|x\|^{p-2} x, \quad x \in H. \tag{3.3}$$

Let (e_n) be an orthonormal sequence in H converging weakly to 0. Define a sequence (x_n) by (3.2). It follows that

$$J_p(x_n) = \|x_n\|^{p-2} x_n \begin{cases} \rightharpoonup e_1, & \text{if } n \text{ (odd)} \to \infty, \\ \rightharpoonup 2^{(p-2)/2} e_1, & \text{if } n \text{ (even)} \to \infty. \end{cases}$$

This clearly shows that $J_p(x_n) \not\rightharpoonup J_p(e_1)$ as $2^{(p-2)/2} \neq 1$ for $p \neq 2$. □

Basing upon the result of Proposition 3.3, one may conjecture that if the normalized duality J of a Banach space X is weakly continuous, then the space X is a Hilbert space. The answer to this conjecture is nevertheless negative, as shown by the following simple example.

Example 3.4. *Consider the space*

$$X = H + V$$

endowed with the norm

$$\|x\| = \sqrt{\|h\|^2 + \|v\|^2}, \quad x = (h, v),\ h \in H,\ v \in V,$$

where H is a Hilbert space and V is a finite-dimensional smooth normed space with a norm not induced by an inner product. Then X is non-Hilbert; however its normalized duality map is weakly continuous.

4. Duality maps of product spaces

In this section we consider the permanence of weak continuity of duality maps. Let X and Y be two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. If no confusions would occur, we use $\|\cdot\|$ to denote norm of any Banach spaces. Let $1 < p < \infty$. We shall use

$$Z := X \oplus_p Y$$

to denote the product space $X \times Y$ equipped with the norm

$$\|z\| \equiv \|z\|_Z := (\|x\|_X^p + \|y\|_Y^p)^{\frac{1}{p}}, \quad z = (x, y) \in Z. \tag{4.1}$$

We write J_p^X, J_p^Y and J_p^Z for the pth generalized duality maps of X, Y and Z, respectively.

Proposition 4.1. *Let φ be a gauge and $1 < p < \infty$. We have*

$$J_p^Z(z) = (J_p^X(x), J_p^Y(y)), \quad z = (x, y) \in Z \tag{4.2}$$

and

$$J_\varphi^Z(z) = \left(\left(\frac{\|x\|_X}{\|z\|_Z} \right)^{p-1} \frac{\|z\|_Z}{\|x\|_X} J_\varphi^X(x), \left(\frac{\|y\|_Y}{\|z\|_Z} \right)^{p-1} \frac{\|z\|_Z}{\|y\|_Y} J_\varphi^Y(y) \right) \tag{4.3}$$

for $z = (z, y) \in Z,\ x \neq 0,\ y \neq 0$.

Proof. It is known that the dual space $Z^* = X^* \oplus_q Y^*$, $q = p/(p-1)$, with norm

$$\|z^*\|_{Z^*} = (\|x^*\|_{X^*}^q + \|y^*\|_{Y^*}^q)^{\frac{1}{q}}, \quad z^* = (x^*, y^*) \in Z^*.$$

Now put $z^* = (x^*, y^*)$ with $x^* \in J_p^X(x)$ and $y^* \in J_p^Y(y)$. Then it is easy to see that

$$\begin{aligned} \langle z, z^* \rangle &= \langle x, x^* \rangle + \langle y, y^* \rangle \\ &= \|x\|_X^p + \|y\|_Y^p = \|z\|_Z^p. \end{aligned} \tag{4.4}$$

It is also not hard to find that

$$\begin{aligned} \|z^*\|_{Z^*} &= (\|x^*\|_{X^*}^q + \|y^*\|_{Y^*}^q)^{\frac{1}{q}} \\ &= (\|x\|_X^{(p-1)q} + \|y\|_Y^{(p-1)q})^{\frac{1}{q}} \\ &= (\|x\|_X^p + \|y\|_Y^p)^{\frac{1}{q}} \\ &= \|z\|_Z^{p-1}. \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5) proves (4.2).

By Proposition 2.1(d), we have the relation, for any Banach space $(V, \|\cdot\|)$,

$$J_\varphi^V(v) = \frac{\varphi(\|v\|)}{\|v\|} J_p^V(v), \quad v \in V,\ v \neq 0.$$

This together with (4.2) immediately implies (4.3). □

Corollary 4.2. *Keep the notion in Proposition 4.1. Suppose both J_p^X and J_p^Y are weakly continuous. Then J_p^Z is also weakly continuous, where Z is equipped with the ℓ_p product norm (4.1). However, even if the normalized duality maps J^X and J^Y are both weakly continuous, the normalized duality map J^Z of Z endowed with the ℓ_p product norm (4.1) is not necessarily weakly continuous unless $p = 2$.*

Proof. The weak continuity of J_p^Z follows immediately from that of J_p^X and J_p^Y via (4.2).

To see the second part, we first observe from (3.1) that, for $z = (x, y) \neq (0, 0)$,

$$J^Z(z) = \left(\left(\frac{\|x\|_X}{\|z\|_Z} \right)^{p-2} J^X(x), \left(\frac{\|y\|_Y}{\|z\|_Z} \right)^{p-2} J^Y(y) \right). \tag{4.6}$$

This indicates that, due to the weak lower semicontinuity of norms, J^Z may fail to be weakly continuous even though J^X and J^Y are weakly continuous. A counterexample is constructed as follows. Take X and Y to be separable Hilbert spaces and let $Z = X \oplus_p Y$. Notice that $J^X = I$ and $J^Y = I$ are the identity operators of X and Y, respectively. Let (e_n^X) and (e_n^Y) be two orthonormal bases of X and Y, respectively. For the sake of convenience, we shall use $\|\cdot\|$ to denote all norms on the spaces X, Y and Z. Define (for $n \geq 1$)

$$x_n = e_{n+1}^X + e_1^X \rightharpoonup x := e_1^X, \quad \|x_n\| = 2^{\frac{1}{2}}, \ \|x\| = 1,$$
$$y_n = e_{n+1}^Y + \frac{1}{2} e_1^Y \rightharpoonup y := \frac{1}{2} e_1^Y, \quad \|y_n\| = \frac{\sqrt{5}}{2}, \ \|y\| = \frac{1}{2},$$
$$z_n = (x_n, y_n) \rightharpoonup z := (x, y), \quad \|z_n\| = \left(2^{\frac{p}{2}} + \left(\frac{\sqrt{5}}{2} \right)^p \right)^{1/p}, \ \|z\| = \left(1 + \frac{1}{2^p} \right)^{1/p}.$$

We now show that $J^Z(z_n) \not\rightharpoonup J^Z(z)$ for $p \neq 2$. As a matter of fact, since

$$\frac{\|x_n\|}{\|z_n\|} = \frac{\sqrt{2}}{\left(2^{\frac{p}{2}} + \left(\frac{\sqrt{5}}{2} \right)^p \right)^{1/p}} =: c_p, \quad \frac{\|y_n\|}{\|z_n\|} = \frac{\sqrt{5}}{2 \left(2^{\frac{p}{2}} + \left(\frac{\sqrt{5}}{2} \right)^p \right)^{1/p}} =: d_p,$$

it turns out from (4.6) that

$$J^Z(z_n) = \left(c_p^{p-2} x_n, d_p^{p-2} y_n \right) \rightharpoonup \left(c_p^{p-2} x, d_p^{p-2} y \right).$$

However we have (again by (4.6)),

$$J^Z(z) = \left(\left(\frac{\|x\|}{\|z\|} \right)^{p-2} x, \left(\frac{\|y\|}{\|z\|} \right)^{p-2} y \right) = (\hat{c}_p^{p-2} x, \hat{d}_p^{p-2} y),$$

where

$$\hat{c}_p = \left(1 + \frac{1}{2^p} \right)^{-\frac{1}{p}}, \quad \hat{d}_p = \frac{1}{2} \hat{c}_p = \frac{1}{2} \left(1 + \frac{1}{2^p} \right)^{-\frac{1}{p}}.$$

As it is evident that $\hat{c}_p^{p-2} \neq c_p^{p-2}$ and $\hat{d}_p^{p-2} \neq d_p^{p-2}$ since $p \neq 2$, we conclude that $J^Z(z_n) \not\rightharpoonup J^Z(z)$ if $p \neq 2$. □

Remark 4.3. Proposition 4.14 of [12] claimed that in ℓ_p spaces, $1 < p < \infty$, every duality map is sequentially weak-to-weak continuous. This is a misstatement. The normalized duality map J^{ℓ_p} is not sequentially weak-to-weak continuous unless $p = 2$ though the generalized duality map $J_p^{\ell_p}$ is indeed sequentially weak-to-weak continuous.

5. Concluding remarks and open questions

Recall that a Banach space X is said to satisfy Opial's property [24] if for any sequence $x_n \rightharpoonup x$, it follows that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - z\|, \quad z \in X, \; z \neq x. \tag{5.1}$$

It is known that Opial's property plays an important role in the fixed point theory of nonexpansive and asymptotically mappings [28, 20, 39, 38]. It is easy to find that a Banach X possesses Opial's property if it has a weakly continuous duality map J_φ, but not vice versa.

In [32], van Dulst profoundly proves that any separable Banach space can equivalently be renormed to satisfy Opial's property.

Question One. Whether a separable and smooth Banach space can equivalently be renormed to have a weakly continuous duality map J_φ with some gauge φ?

Question Two. If X is a Banach space such that its normalized duality map J is weakly continuous, can X be decomposed as $X = H \oplus_2 Y$? Here H is a Hilbert space and Y is finite-dimensional Banach space.

Question Three. If X is a Banach space such that there exists a closed subspace M of X with the properties: (i) M has a weakly continuous duality map J_φ^M with a gauge φ and (ii) the quotient space X/M is finite-dimensional, does X have a weakly duality map J_ψ^X for some gauge ψ?

References

[1] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International Publishing, Leyden, The Netherlands, 1976.

[2] V. Barbu, *Nonlinear differential equations of monotone types in Banach spaces*, Springer Science+Business Media, LLC 2010.

[3] V. Barbu and T. Precupanu, *Convexity and Optimization in Banach Spaces (4th ed.)*, Springer Monographs in Mathematics, Springer Science+Business Media B.V. 2012.

[4] F. E. Browder, *Fixed point theorems for nonlinear semicontractive mappings in Banach spaces*, Arch. Rational Mech. Anal. **21** (1966), 259–269.

[5] F. E. Browder, *Nonlinear equations of evolution and nonlinear accretive operators in Banach spaces*, Bull. Amer. Math. Soc. **73** (1967), 867–874.

[6] F. E. Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc. **73** (1967), 875–882.

[7] F. E. Browder, *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Z. **100** (1967), 201–225.

[8] F. E. Browder, *Convergence of approximants to fixed points of nonexpansive non-linear mappings in Banach spaces*, Arch. Rational Mech. Anal. **24** (1967), 82–90.

[9] F. E. Browder, *Nonlinear maximal monotone operators in Banach space*, Math. Ann. **175** (1968), 213–231.

[10] F. E. Browder, *Nonlinear variational inequalities and maximal monotone mappings in Banach spaces*, Math. Ann. **183** (1969), 213–231.

[11] R. E. Bruck, *Approximating Fixed Points and Fixed Point Sets of Nonexpansive Mappings in Banach Spaces*, Ph.D. Thesis, The University of Chicago, 1969.

[12] I. Cioranescu, *Geometry of Banach Spaces*, Duality Mappings and Nonlinear Problems, Kluwer Academic Publishers, Dordrecht, 1990.

[13] J.-P. Chancelier, *Iterative schemes for computing fixed points of nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **353** (2009), 141–153.

[14] F. Kohsakai and W. Takahashi, *Strong convergence of an iterative sequence for maximal monotone operators in a Banach space*, Abstract Applied Anal. **3** (2004), 239–249.

[15] J. S. Jung, *Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **302** (2005), 509–520.

[16] S. Kamimura and W. Takahashi, *Strong convergence of a proximinal-type algorithm in a Banach space*, SIAM J. Optim. **13** (2002), 938–945.

[17] T. H. Kim and H. K. Xu, *Strong convergence of modified Mann iterations*, Nonlinear Anal. **61** (2005), 51–60.

[18] S. Lahrech, A. Mbarki, A. Ouahab, and S. Rais, *On strong convergence to common fixed points in Banach spaces*, Rend. Sem. Mat. Univ. Pol. Torino **66** (2008), 23–28.

[19] T. C. Lim and H. K. Xu, *Fixed point theorems for asymptotically nonexpansive mappings*, Nonlinear Anal. **22** (1994), 1345–1355.

[20] P. K. Lin, K. K. Tan and H. K. Xu, *Demiclosedness principle and asympttic behavior for asymptotically nonexpansive mappings*, Nonlinear Anal. **24** (1995), 929–946.

[21] S. Matsushita and W. Takahashi, *A strong convergence theorem for rel- atively nonexpansive mappings in a Banach space*, J. Approx. Theory **134** (2005), 257–266.

[22] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), 372–379.

[23] J. G. O'Hara, P. Pillay and H. K. Xu, *Iterative approaches to convex feasibility problems in Banach spaces*, Nonlinear Anal. **64** (2006), 2022–2042.

[24] Z. Opial, *Weak convergence of the sequence of successive approximations of nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 595–597.

[25] D. Pascali and S. Sburlan, *Nonlinear Mappings of Monotone Type*, Editura Academiei, Bucuresti, Romania, 1978. Sijthoff & Noordhoff International Publishers, Alphen aan den Rijn, the Netherlands.

[26] J. J. Peiris, *On the duality mapping of L^∞ spaces*, Hiroshima Math. J. **29** (1999), 89–115.

[27] J. Pruss, *A characterization of uniform convexity and applications to accretive operators*, Hiroshima Math. J. **11** (1981), 229–234.

[28] S. Prus, *Banach spaces with the uniform Opial property*, Nonlinear Anal. *18* (1992), 697–704.

[29] Y. S. Song and R. D. Chen, *Viscosity approximation methods for nonexpansive nonself-mappings*, J. Math. Anal. Appl. **321** (2006), 316–326.

[30] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), 287–292.

[31] W. Takahashi, *Nonlinear Functional Analysis: Fixed Point Theory and Its Applications*, Yokohama Publishers, Yokohama, 2000.

[32] D. van Dulst, *Equivalent norms and the fixed point property for nonexpansive mappings*, J. London Math. Soc. **25** (1982), 139–144.

[33] Y. Yao and M. A. Noor, *On viscosity iterative methods for variational inequalities*, J. Math. Anal. Appl. **325** (2007), 776–787.

[34] H. K. Xu, *Strong convergence of approximating fixed point sequences for nonexpansive mappings*, Bull. Austral. Math. Soc. **74** (2006), 143–151.

[35] H. K. Xu, *Strong convergence of an iterative method for nonexpansive and accretive operators*, J. Math. Anal. Appl. 314 (2006) 631-643.

[36] H. K. Xu, *A strong convergence theorem for contraction semigroups in Banach spaces*, Bull. Austral. Math. Soc. **72** (2005), 371–379.

[37] H. K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl. **298** (2004), 279–291.

[38] H. K. Xu, *Banach space properties of Opial's type and fixed point theorems of nonlinear mappings*, Ann. Univ. Mariae Curie-Sklodowska Sect. A **51** (1997), 293–303.

[39] H. K. Xu, *Geometrical coefficients of Banach spaces and nonlinear mappings*, in Recent Advances on Metric Fixed Point Theory (Seville, 1995), Ciencias, 48, Univ. Sevilla, Seville, 1996, pp. 161–178.

[40] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1991), 1127–1138.

[41] Z. B. Xu and G. F. Roach, *Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces*, J. Math. Anal. Appl. **157** (1991), 189–210.
[42] V. Zizler, *A note on duality mapping in Banach spaces*, Archiv der Math. **26** (1975), 94–97.

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