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# A NOTE ON ITERATIVE COMMON SOLUTION TO MONOTONE INCLUSION AND FIXED POINT PROBLEMS

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*Honoring Prof. Simeon Reich on his 65th birthday*

Abstract. In this note we consider a problem that consists in finding a zero of the sum of two monotone mappings such that it is also a fixed point of some nonlinear mappings, simultaneously. We study an iterative method for solving such a problem recently proposed by Takahashi *et al.* and show that one condition ensuring the convergence can be completely removed.

## 1. Introduction

It is well known that the monotone inclusion problem plays an essential role in the theory of nonlinear analysis and optimization. This problem consists of finding a zero element of a monotone mapping. However, in some concrete cases including variational inequalities, the problem requires to find a zero of the sum of two monotone mappings, namely, find $\hat{x}$ in a Hilbert space $H$ so that

$$0 \in (A+B)\hat{x}, \tag{1.1}$$

where $A : H \to H$ and $B : H \rightrightarrows H$ are two monotone mappings. The zero-point set of the sum $A+B$ is denoted by $(A+B)^{-1}(0)$. A fixed point problem is to find a point $\hat{x}$ with the property:

$$\hat{x} \in C, \quad S\hat{x} = \hat{x}, \tag{1.2}$$

where $C$ is a nonempty closed convex subset of $H$ and $S : C \to C$ is a nonlinear mapping. The fixed-point set of $S$ is denoted by $\mathrm{Fix}(S)$.

In their recent paper [4], Takahashi, Takahashi and Toyoda considered a problem for finding a common solution of problem (1.1) and of problem (1.2), namely, they seek to find a point $\hat{x}$ such that

$$\hat{x} \in \mathrm{Fix}(S) \cap (A+B)^{-1}(0), \tag{1.3}$$

where $A : C \to H$ is $\kappa$-inverse strongly monotone, $B : H \rightrightarrows H$ is maximal monotone so that $\mathcal{D}(B) \subseteq C$, and $S$ is nonexpansive. Under this hypothesis, they proposed the following iteration:

$$x_{n+1} = \beta_n x_n + (1-\beta_n) S[\alpha_n u + (1-\alpha_n) J_{r_n}(x_n - r_n A x_n)], \tag{1.4}$$

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where $u \in H$ is fixed, $J_{r_n}$ denotes the resolvent of $B$ with $(r_n)$ a positive sequence, and $(\alpha_n), (\beta_n)$ are real sequences both choosing in $[0,1]$. Then the sequence $(x_n)$ generated by (1.4) can be strongly convergent to a solution of (1.3) provided that

(i) $\lim_n |r_n - r_{n+1}| = 0$;
(ii) $\lim_n \alpha_n = 0, \sum_n \alpha_n = \infty$;
(iii) $0 < a \le r_n \le b < 2\kappa, 0 < c \le \beta_n \le d < 1$.

The aim of this note is to continue the study of the above algorithm. By using the techniques developed in [3, 5], we shall show the strong convergence of algorithm (1.4) without condition (i) above, that is, conditions (ii)-(iii) are sufficient to ensure the convergence of algorithm (1.4).

## 2. Preliminary and notation

Throughout, $I$ denotes the identity mapping, and $\mathcal{D}(T)$ the domain of a mapping $T$. The notation "$\to$" stands for strong convergence, "$\rightharpoonup$" weak convergence, and $\omega_w(x_n)$ the set of the weak cluster points of $(x_n)$.

We use $P_C$ to denote the projection from $H$ onto $C$; namely, for $x \in H$, $P_C x$ is the unique point in $C$ with the property: $\|x - P_C x\| = \min_{y \in C} \|x - y\|$. It is well known that $P_C x$ is characterized by:

$$\langle x - P_C x, z - P_C x \rangle \le 0 \ \forall z \in C. \tag{2.1}$$

A mapping $T : C \to H$ is called *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\| \ \forall x, y \in C;$$

*$\alpha$-averaged* if there exist a constant $\alpha \in (0,1)$ and a nonexpansive mapping $S$ such that $T = (1-\alpha)I + \alpha S$; *firmly nonexpansive*, if

$$\langle Tx - Ty, x - y \rangle \ge \|Tx - Ty\|^2 \ \forall x, y \in C;$$

*$\kappa$-inverse strongly monotone* ($\kappa$-ism), if there exists $\kappa > 0$ so that

$$\langle Tx - Ty, x - y \rangle \ge \kappa \|Tx - Ty\|^2 \ \forall x, y \in C.$$

Nonexpansive mappings have the following essential property (see [2]).

**Lemma 2.1** (Demiclosedness principle)**.** *Let $T : C \to H$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *If $(x_n)$ is a sequence in $C$ such that $x_n \rightharpoonup x$ and $(I - T)x_n \to y$, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in$* $\mathrm{Fix}(T)$.

Firmly nonexpansive mappings are known to be both $\frac{1}{2}$-averaged and 1-ism. It is worth noting that averaged mappings have several remarkable properties that are not shared by nonexpansive mappings.

**Lemma 2.2** ([1, 8])**.** *Let $\kappa, \kappa_1$ and $\kappa_2$ be constants in* $(0,1)$.

(i) *If $T : C \to H$ is $\kappa$-averaged, then for $x \in C$ and $y \in$* $\mathrm{Fix}(T)$,
$$\|Tx - y\|^2 \le \|x - y\|^2 - \frac{1-\kappa}{\kappa} \|(I - T)x\|^2.$$
(ii) *Let $T_1 : H \to H$ and $T_2 : C \to H$ be $\kappa_1$ and $\kappa_2$-averaged, respectively. Then $T_1 T_2$ is $(\kappa_1 + \kappa_2 - \kappa_1 \kappa_2)$-averaged.*

A mapping $B : H \rightrightarrows H$ is called *monotone*, if $\langle u - v, x - y\rangle \geq 0$ for all $x, y \in \mathcal{D}(B), u \in Bx, v \in By$; *maximal monotone* if it is monotone and its graph is not properly contained in the graph of any other monotone mapping. Hereafter if no confusion occurs, denote by $J_r := (I + rB)^{-1}(r > 0)$ the resolvent of $B$. If $B$ is monotone, then $J_r$ is single-valued and firmly nonexpansive; If further $B$ is maximal monotone, then $\mathcal{D}(J_r) = H$.

**Lemma 2.3.** *Let $A : C \to H$ be a $\kappa$-ism mapping and $B : H \rightrightarrows H$ a maximal monotone mapping so that $\mathcal{D}(B) \subseteq C$. For $r \in (0, 2\kappa)$, set $T_r := J_r(I - rA)$. Then* $\mathrm{Fix}(T_r) = (A + B)^{-1}(0)$; *and for $z \in (A + B)^{-1}(0)$, it follows*

$$\|T_r x - z\|^2 \leq \|x - z\|^2 - \frac{2\kappa - r}{2\kappa + r}\|T_r x - x\|^2.$$

*Proof.* The first assertion is easy to check. To see the second, we note that $I - 2\kappa A$ is nonexpansive since $A$ is $\kappa$-ism. It then follows from

$$I - rA = \left(1 - \frac{r}{2\kappa}\right) I + \frac{r}{2\kappa}(I - 2\kappa A)$$

that $I - rA$ is $r/2\kappa$-averaged. Since $J_r$ is 1/2-averaged, using Lemma 2.2 yields the desired result. □

**Lemma 2.4.** *Let $A : C \to H$ be a $\kappa$-ism mapping and $B : H \rightrightarrows H$ a maximal monotone mapping with $\mathcal{D}(B) \subseteq C$. If $0 < r \leq s$, then for every $x \in C$,*

$$\|x - T_r x\| \leq 2\|x - T_s x\|, \tag{2.2}$$

*where $T_s := J_s(I - sA)$ and $T_r := J_r(I - rA)$.*

*Proof.* Let $z_1 = T_r x$ and $z_2 = T_s x$. By definition of $T_r$,

$$\frac{x - z_1}{r} - Ax \in Bz_1, \ \frac{x - z_2}{s} - Ax \in Bz_2.$$

The monotonicity of $B$ then implies

$$\langle z_1 - z_2, \frac{x - z_1}{r} - \frac{x - z_2}{s}\rangle \geq 0,$$

or equivalently

$$\|z_2 - z_1\|^2 \leq \left(1 - \frac{r}{s}\right) \langle z_2 - z_1, z_2 - x\rangle.$$

If $r \leq s$, then $\|z_2 - z_1\| \leq \|z_2 - x\|$. By the triangle inequality, $\|z_1 - x\| \leq \|z_1 - z_2\| + \|z_2 - x\|$, which at once yields (2.2). □

We end this section by two useful lemmas.

**Lemma 2.5** (Xu [7]). *Let $(a_n)$ be a nonnegative real sequence satisfying*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n,$$

*where $(\alpha_n) \subset (0, 1)$ and $(b_n)$ are real sequences. Then $a_n \to 0$ provided that*

(i) $\sum \alpha_n = \infty, \lim_n \alpha_n = 0$;
(ii) $\overline{\lim}_n b_n \leq 0$ *or* $\sum \alpha_n |b_n| < \infty$.

**Lemma 2.6** (Maingé [3])**.** *Let $(s_n)$ be a real sequence that does not decrease at infinity, in the sense that there exists a subsequence $(s_{n_k})$ so that*

$$s_{n_k} \leq s_{n_k+1} \text{ for all } k \geq 0.$$

*For every $n > n_0$ define an integer sequence $(\tau(n))$ as*

$$\tau(n) = \max\{n_0 \leq k \leq n : s_k < s_{k+1}\}.$$

*Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n > n_0$*

$$\max(s_{\tau(n)}, s_n) \leq s_{\tau(n)+1}. \tag{2.3}$$

## 3. Strong convergence

In this section, we consider problem (1.3) under the assumption that $A : C \to H$ is $\kappa$-ism, $B : H \rightrightarrows H$ is maximal monotone so that $\mathcal{D}(B) \subseteq C$, and $S$ is nonexpansive.

We now consider algorithm (1.4) introduced in [4]. For the convenience, we define $T_n = J_{r_n}(I - r_n A)$, and thus the algorithm has the form:

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)S[\alpha_n u + (1 - \alpha_n)T_n x_n]. \tag{3.1}$$

**Theorem 3.1.** *Let the following conditions hold:*

(i) $\lim_n \alpha_n = 0, \sum_n \alpha_n = \infty$;

(ii) $0 < a \leq r_n \leq b < 2\kappa, 0 < c \leq \beta_n \leq d < 1$.

*If the solution set $\Omega$ of problem* (1.3) *is nonempty, then the sequence $(x_n)$ generated by* (3.1) *converges strongly to $\hat{x} = P_\Omega u$.*

*Proof.* Let $y_n = \alpha_n u + (1 - \alpha_n)T_n x_n$. Hence we have

$$\begin{aligned} \|x_{n+1} - \hat{x}\| &= \|(1 - \beta_n)(Sy_n - \hat{x}) + \beta_n(x_n - \hat{x})\| \\ &\leq (1 - \beta_n)\|y_n - \hat{x}\| + \beta_n\|x_n - \hat{x}\|, \end{aligned}$$

and

$$\begin{aligned} \|y_n - \hat{x}\| &= \|\alpha_n(u - \hat{x}) + (1 - \alpha_n)(T_n x_n - \hat{x})\| \\ &\leq (1 - \alpha_n)\|x_n - \hat{x}\| + \alpha_n\|u - \hat{x}\|. \end{aligned}$$

Combining these two inequalities yields

$$\|x_{n+1} - \hat{x}\| \leq [1 - \alpha_n(1 - \beta_n)]\|x_n - \hat{x}\| + \alpha_n(1 - \beta_n)\|u - \hat{x}\|.$$

By induction, we can deduce that $(x_n)$ is bounded and so is $(y_n)$.

We next show the following key estimation:

$$\begin{aligned} s_{n+1} &\leq (1 - \sigma\alpha_n)s_n - \sigma(\|T_n x_n - x_n\|^2 + \|Sy_n - x_n\|^2) \\ &\quad + 2\alpha_n(1 - \beta_n)\langle u - \hat{x}, y_n - \hat{x}\rangle, \end{aligned} \tag{3.2}$$

where $s_n = \|x_n - \hat{x}\|^2$, and $\sigma > 0$ is chosen so that

$$\frac{(1 - \alpha_n)(1 - \beta_n)(2\kappa - r_n)}{2\kappa + r_n} \geq \sigma,$$

and $\beta_n(1 - \beta_n) \geq \sigma$ for all $n \geq 0$. Indeed, it follows from Lemma 2.3 that

$$\|T_n x_n - \hat{x}\|^2 \leq \|x_n - \hat{x}\|^2 - \frac{2\kappa - r_n}{2\kappa + r_n}\|T_n x_n - x_n\|^2.$$

By the subdifferential inequality,

$$\begin{aligned}\|y_n - \hat{x}\|^2 &= \|\alpha_n(u - \hat{x}) + (1 - \alpha_n)(T_n x_n - \hat{x})\|^2 \\ &\le (1 - \alpha_n)\|T_n x_n - \hat{x}\|^2 + 2\alpha_n\langle u - \hat{x}, y_n - \hat{x}\rangle \\ &\le (1 - \alpha_n)\|x_n - \hat{x}\|^2 + 2\alpha_n\langle u - \hat{x}, y_n - \hat{x}\rangle \\ &\quad - \frac{(1 - \alpha_n)(2\kappa - r_n)}{2\kappa + r_n}\|T_n x_n - x_n\|^2.\end{aligned}$$

Consequently,

$$\begin{aligned}\|x_{n+1} - \hat{x}\|^2 &= \beta_n\|x_n - \hat{x}\|^2 + (1 - \beta_n)\|Sy_n - \hat{x}\|^2 \\ &\quad - \beta_n(1 - \beta_n)\|Sy_n - x_n\|^2 \\ &\le \beta_n\|x_n - \hat{x}\|^2 + (1 - \beta_n)\|y_n - \hat{x}\|^2 \\ &\quad - \beta_n(1 - \beta_n)\|Sy_n - x_n\|^2 \\ &\le \beta_n\|x_n - \hat{x}\|^2 + (1 - \beta_n)(1 - \alpha_n)\|x_n - \hat{x}\|^2 \\ &\quad - \frac{(1 - \alpha_n)(1 - \beta_n)(2\kappa - r_n)}{2\kappa + r_n}\|T_n x_n - x_n\|^2 \\ &\quad + 2\alpha_n(1 - \beta_n)\langle u - \hat{x}, y_n - \hat{x}\rangle \\ &\quad - \beta_n(1 - \beta_n)\|Sy_n - x_n\|^2,\end{aligned}$$

and the desired inequality (3.2) follows.

Finally, we show $s_n \to 0$ by considering two possible cases.

CASE 1. $(s_n)$ is eventually decreasing (i.e., there exists $N \ge 0$ such that $(s_n)$ is decreasing for $n \ge N$). In this case, $(s_n)$ must be convergent, and from (3.2) it follows

$$\sigma(\|T_n x_n - x_n\|^2 + \|Sy_n - x_n\|^2) \le M\alpha_n + (s_n - s_{n+1}),$$

where $M > 0$ is a sufficient large real number. Consequently, both $\|T_n x_n - x_n\|$ and $\|Sy_n - x_n\|$ converge to zero. Let $T_a = J_a(I - aA)$. In view of Lemma 2.4, $\|x_n - T_a x_n\| \le 2\|x_n - T_n x_n\| \to 0$. Since $T_a$ is nonexpansive,

$$\omega_w(x_n) \subseteq \mathrm{Fix}(T_a) = (A + B)^{-1}(0),$$

where we use the demiclosedness principle. On the other hand, we see

$$\begin{aligned}\|x_n - y_n\| &= \|\alpha_n(u - x_n) + (1 - \alpha_n)(T_n x_n - x_n)\| \\ &\le \alpha_n\|u - x_n\| + \|T_n x_n - x_n\| \to 0,\end{aligned}$$

which implies

$$\begin{aligned}\|x_n - Sx_n\| &\le \|x_n - Sy_n\| + \|Sy_n - Sx_n\| \\ &\le \|x_n - Sy_n\| + \|y_n - x_n\| \to 0.\end{aligned}$$

Using again the demiclosedness principle, $\omega_w(x_n) \subseteq \Omega$; hence

$$\begin{aligned}\varlimsup_{n\to\infty}\langle u - \hat{x}, y_n - \hat{x}\rangle &= \varlimsup_{n\to\infty}\langle u - \hat{x}, x_n - \hat{x}\rangle \\ &= \max_{w \in \omega_w(x_n)}\langle u - \hat{x}, w - \hat{x}\rangle \le 0,\end{aligned}$$

where the inequality uses (2.1). It then follows from (3.2) that

$$s_{n+1} \leq (1-\sigma\alpha_n)s_n + 2\alpha_n(1-\beta_n)\langle u-\hat{x}, y_n-\hat{x}\rangle.$$

We therefore apply Lemma 2.5 to conclude $s_n \to 0$.

Case 2. $(s_n)$ is not eventually decreasing. Hence, we can find a subsequence $(s_{n_k})$ so that $s_{n_k} \leq s_{n_k+1}$ for all $k \geq 0$. In this case, we may define an integer sequence $(\tau(n))$ as in Lemma 2.6. In view of (2.3), we deduce from (3.2) that

$$\sigma(\|T_{\tau(n)}x_{\tau(n)} - x_{\tau(n)}\|^2 + \|Sy_{\tau(n)} - x_{\tau(n)}\|^2) \leq M\alpha_{\tau(n)} \to 0. \tag{3.3}$$

In a similar way to Case 1, we have

$$\overline{\lim_{n\to\infty}}\langle u-\hat{x}, y_{\tau(n)}-\hat{x}\rangle \leq 0.$$

Combining (2.3) and (3.2) yields

$$\sigma s_{\tau(n)} \leq 2(1-\beta_{\tau(n)})\langle u-\hat{x}, y_{\tau(n)}-\hat{x}\rangle$$

for all $n > n_0$. Taking $\overline{\lim}$ in this inequality, we get $s_{\tau(n)} \to 0$. Moreover, it follows from (3.1) that

$$\begin{aligned}\sqrt{s_{\tau(n)+1}} &= \|(x_{\tau(n)}-\hat{x}) - (x_{\tau(n)} - x_{\tau(n)+1})\| \\ &\leq \sqrt{s_{\tau(n)}} + \|x_{\tau(n)} - x_{\tau(n)+1}\| \\ &\leq \sqrt{s_{\tau(n)}} + \|x_{\tau(n)} - Sy_{\tau(n)}\|,\end{aligned}$$

which together with (3.3) implies $s_{\tau(n)+1} \to 0$. Consequently, from (2.3) the desired result $s_n \to 0$ immediately follows. □

**Remark 3.2.** *In Theorem 3.1, we remove one sufficient condition used by Takahashi, Takahashi and Toyoda* [4], *namely,* $|r_n - r_{n+1}| \to 0$.

**Remark 3.3.** *In a similar way to* [6], *we can apply our results to the variational inequalities, the split feasibility problem, and the convexly constrained linear inverse problem.*

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