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ON THE CONVERGENCE OF APPROXIMANTS OF PSEUDO-CONTRACTIVE SEMIGROUPS IN BANACH SPACES

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Abstract. The purpose of this paper is to estabilish some results on the convergence of approximated fixed point sequences for uniformly lipschitzian semigroups of pseudo-contractive mappings.

1. Introduction

Let X be a Banach space and let C be a nonempty, closed and convex subset of X. Let $T : C \to C$ be a mapping, we denote by the symbol $F(T) := \{x \in C : Tx = x\}$ the set of fixed points for T and by $k(T)$ we denote, whenever it exists, the Lipschitz constant defined by

$$k(T) := \inf\{k \in [0, \infty) : \|Tx - Ty\| \leq k\|x - y\| \text{for all } x, y \in C\}.$$

We recall that T is called

(1) $L-$lipschitzian if $k(T) = L < \infty$,
(2) nonexpansive if $k(T) = 1$ and
(3) contractive if $k(T) < 1$.

One classical method to approximate fixed points for a nonexpansive mapping T is by passing through fixed points of particular contractive mappings.
More precisely, for a fixed element $u \in C$, define for each $t \in (0, 1)$, a contraction G_t by $G_t x = tu + (1 - t)Tx$ for all $x \in C$. Let x_t be the fixed point of G_t, *i.e.*,

$$x_t = tu + (1 - t)Tx_t. \tag{1.1}$$

Browder [2] proved that x_t strongly converges, as $t \to 0$, to a fixed point of the mapping T, in the setting of Hilbert spaces. Later, Reich [12] extended the result to uniformly smooth Banach spaces.
Similarly, many authors have studied the behaviour of the approximants $\{x_\varepsilon\}$ defined by

$$0 = \varepsilon R x_\varepsilon + (1 - \varepsilon)(I - T)x_\varepsilon.$$

for nonexpansive self-mappings T in Banach spaces, where $R = I - A$ and $A : C \to C$ is a contraction mapping. In [6], Gwinner proved strong convergence of inexact approximants $\{\widetilde{y}_n\}$ in a uniformly convex Banach space as follows:

Theorem 1.1. *Let X be a uniformly convex Banach space with a weakly sequentially continuous duality mapping $J : X \to X^*$. Let C be a nonempty closed convex subset and $T : C \to C$ a nonexpansive with $F(T) \neq \emptyset$. Let $R : C \to X$ be a continuous, bounded operator. Suppose R is strongly ϕ-accretive. Let $\{b_n\}$ be a sequence in*

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$(0,1)$ *and let* $\{\delta_n\}$ *be a sequence in* $(0,\infty)$ *such that* $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\delta_n}{b_n} = 0$. *If the approximate solutions* $\widetilde{y}_n \in C$ *satisfy*

$$\|b_n R\widetilde{y}_n + (1-b_n)(I-T)\widetilde{y}_n\| \leq \delta_n \text{ for all } n \in \mathbb{N}, \tag{1.2}$$

then $\{\widetilde{y}_n\}$ *converges strongly to an element* $y^* \in F(T)$ *which uniquely solves the variational inequality:*

$$\langle Ry^*, J(y^*-v)\rangle \leq 0 \text{ for all } v \in F(T). \tag{1.3}$$

Let $\mathbb{R}^+$ be the set of nonnegative real numbers and let $\mathscr{F} := \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter family of mappings from C to itself. $\mathscr{F}$ is said to be a *strongly continuous semigroup of mappings* if

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s+t) = T(s)T(t)$ for all $s,t \in \mathbb{R}^+$;
(iii) for each $x \in C$, the mapping $T(\cdot)x$ from $\mathbb{R}^+$ into C is continuous.

Moreover, $\mathscr{F}$ is said to be an *uniformly continuous semigroup of mappings*, if condition (iii) holds uniformly over any bounded subset of C.

We denote by $F(\mathscr{F})$ the set of all common fixed points of $\mathscr{F}$, i.e., $F(\mathscr{F}) := \bigcap_{t\in\mathbb{R}^+} F(T(t))$.

An interesting problem is to modify Browder's result (1.1) to approximate a common fixed point for a semigroup of nonexpansive mappings. Suzuki [15] proved the following implicit iteration process in a Hilbert space.

Theorem 1.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\mathscr{F} = \{T(t) : t \in \mathbb{R}^+\}$ be a strongly continuous semigroup of nonexpansive mappings from C into itself with $F(\mathscr{F}) \neq \emptyset$. Let $\{b_n\}$ be a sequence in $(0,1)$ and $\{t_n\}$ a sequence in $(0,\infty)$ satisfying $\lim_{n\to\infty} t_n = \lim_{n\to\infty} b_n/t_n = 0$. Fix $u \in C$ and define a sequence $\{y_n\}$ by*

$$y_n = b_n u + (1-b_n)T(t_n)y_n \text{ for all } n \in \mathbb{N}. \tag{1.4}$$

Then $\{y_n\}$ converges strongly to the element of $F(\mathscr{F})$ nearest to u.

Xu [18] extended Suzuki's result to uniformly convex Banach spaces with weakly sequentially continuous duality mappings and he posed the following question. Can the iteration sequence (1.4) provide the same result in Banach spaces that include the L_p spaces, $1 < p < \infty$?
To give a partial answer to the question, we deal with an important and widely studied generalization of nonexpansive mappings, that is the class of pseudo-contractions. We say that a mapping $T : C \to C$ is said to be

(1) *pseudo-contractive* if for all x,y in C, there exists $j(x-y)$ in $J(x-y)$ satisfying $\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2$;
(2) *ϕ-strongly pseudo-contractive* if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for all x,y in C, there exists $j(x-y)$ in $J(x-y)$ satisfying $\langle Tx-Ty, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|$;
(3) *generalized Φ-pseudo-contractive* (cf.[17]) if there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that for all x,y in C, there exists $j(x-y)$ in $J(x-y)$ satisfying $\langle Tx-Ty, j(x-y)\rangle \leq \|x-y\|^2 - \Phi(\|x-y\|)$.

We remark that $R = I - T$ is accretive (resp. ϕ-strongly accretive, uniformly accretive) if T is pseudo-contractive (resp. ϕ-strongly pseudo-contractive, generalized Φ-pseudo-contractive), where I is the identity operator.

Recently, applications of semigroups on the existence of solutions to certain partial differential equations had been explored by Hester and Morales in [7]. They proved that the semigroup result directly implies the existence of a unique global solution to a time evolution equation of the form $u' = Au$, where A is a combination of derivatives.
Our concern now is the following:

Problem 1.3. Does iteration process (1.4) provide the same result for Lipschitz pseudo-contractive semigroups $\mathscr{F}$ even in uniformly convex spaces?

In this paper, we prove a version of Theorem 1.1 for a uniformly continuous semigroup of pseudocontractive mappings in a Banach space much more general than uniformly convex spaces. This partially settles the open problem posed by Xu [18] and Problem 1.3.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers, X is a real Banach space, C is a nonempty, closed and convex subset of X, X^* is the dual space of X and J is the normalized duality mapping from X to 2^{X^*} defined by

$$J(x) := \{j \in X^* : \langle x, j \rangle = ||x||^2 = ||j||^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if X^* is strictly convex, then J is single-valued.

Recall that X is said to be *smooth* provided the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in $S_X = \{x \in X : \|x\| = 1\}$. In this case, the norm of X is said to be *Gâteaux differentiable* and it is said to be *uniformly Gâteaux differentiable* if for each $y \in$ S, this limit is attained uniformly for $x \in S$. X is said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in X$. Classical examples of uniformly smooth Banach spaces are the L_p spaces, for $1 < p < \infty$ (see e.g., [1, 4]).
Let $\{x_n\}$ be a bounded sequence in X. Consider the functional $r_a(\cdot, \{x_n\}) : X \to \mathbb{R}^+$ defined by

$$r_a(x, \{x_n\}) = \limsup_{n \to \infty} \|x_n - x\|, \ x \in X.$$

The infimum of $r_a(\cdot, \{x_n\})$ over C is said to be the *asymptotic radius* of $\{x_n\}$ with respect to C and is denoted by $r_a(C, \{x_n\})$. A point $z \in C$ is said to be an *asymptotic center* of the sequence $\{x_n\}$ with respect to C if

$$r_a(z, \{x_n\}) = \inf\{r_a(x, \{x_n\}) : x \in C\}.$$

The set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $\mathcal{Z}_a(C, \{x_n\})$.

X is said to satisfy property (I) (cf. [9]) if asymptotic center of every bounded sequence in X with respect to closed convex subsets of X consists of exactly one point.

Uniformly convex spaces are examples of this type Banach spaces (cf. [1, 5]). It is known (cf. Lim [8]) that $\mathcal{Z}_a(C, \{x_n\})$ consists of a single point if X is reflexive and uniformly convex in every direction.

We need the following known fact (cf. Morales [11, Proposition 11]).

Lemma 2.1. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm and let C be a closed and convex subset of X. Suppose $\{x_n\}$ is a bounded sequence in C and $v \in \mathcal{Z}_a(C, \{x_n\})$. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that*

$$\limsup_{k\to\infty}\langle u - v, J(x_{n_k} - v)\rangle \le 0 \text{ for all } u \in C.$$

A semigroup $\mathscr{F} := \{T(t) : t \in \mathbb{R}^+\}$ of Lipschitzian mappings from C into itself, is said *uniformly Lipschitzian* if there exists a constant $L > 0$ such that $\|T(t)x - T(t)y\| \le L\|x - y\|$ holds for any $t \in \mathbb{R}^+$ and for any $x, y \in C$.
Let C be a nonempty, closed and convex subset of a smooth Banach space X and D a nonempty subset of C. Given an accretive operator $R : C \to X$, we consider the following variational inequality $VI_D(C, R)$:

$$\text{find } z \in D \text{ such that } \langle Rz, J(z - v)\rangle \le 0 \text{ for all } v \in D.$$

We denote by $\Omega_D(C, R)$ the set of solutions of variational inequality $VI_D(C, R)$.

Remark 2.2. If R is uniformly accretive and if $\Omega_D(C, R)$ is nonempty, then this last consists of a unique element.

Proof. Let $z_1, z_2 \in \Omega_D(C, R)$. Then

$$\langle Rz_1, J(z_1 - z_2)\rangle \le 0$$

and

$$\langle Rz_2, J(z_2 - z_1)\rangle \le 0.$$

Summing the two inequalities and by the uniform accretivity of R, we get

$$\Phi(\|z_1 - z_2\|) \le \langle Rz_1 - Rz_2, J(z_1 - z_2)\rangle \le 0$$

for some strictly increasing function Φ, with $\Phi(0) = 0$. From this last, it is easily derived that $z_1 = z_2$. □

3. Main results

Firstly, we prove a result on the existence of common fixed points for a semigroup of pseudo-contractions.We assume the existence of an approximated fixed point sequence only for countable many elements of the semigroup.

Lemma 3.1. *Let X be a reflexive Banach space satisfying property (I) and let C be a nonempty closed convex subset of X. Let $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ be a strongly continuous semigroup of continuous pseudo-contractive mappings from C into itself and let $\{t_n\}$ be a sequence in $(0, \infty)$ converging to* 0.
Let $\{y_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty}\|y_n - T(t_m)y_n\| = 0$ *for all*

$m \in \mathbb{N}$ and let y^* be the unique element in $\mathcal{Z}_a(C, \{y_n\})$, then $F(\mathcal{T})$ is nonempty and $y^* \in F(\mathcal{T})$.

Proof. Fix $m \in \mathbb{N}$. Since $T(t_m)$ is continuous and pseudo-contractive, we derive from [10, Theorem 6] that $g_m := (2I - T(t_m))^{-1}$ is a nonexpansive mapping from C into itself.
Since

$$\begin{aligned}
&\limsup_{n\to\infty} \|y_n - g_m(y^*)\| \le \limsup_{n\to\infty} \|g_m(y_n) - g_m(y^*)\| + \limsup_{n\to\infty} \|y_n - g_m(y_n)\| \\
&\le \limsup_{n\to\infty} \|y_n - y^*\| + \limsup_{n\to\infty} \|(2I - T(t_m))^{-1}(2y_n - T(t_m)y_n) - (2I - T(t_m))^{-1}(y_n)\| \\
&\le \limsup_{n\to\infty} \|y_n - y^*\| + \limsup_{n\to\infty} \|y_n - T(t_m)y_n\| \\
&= \limsup_{n\to\infty} \|y_n - y^*\|,
\end{aligned}$$

it follows that $g_m(y^*) \in \mathcal{Z}_a(C, \{y_n\})$ and hence $g_m(y^*) = y^*$ for any $m \in \mathbb{N}$.
As a consequence, $y^* \in \bigcap_{n\in\mathbb{N}} F(t_n)$, where $\{t_n\} \subset (0, \infty)$ converges to 0. Applying [16, Proposition 1], it is easily derived that $y^* \in F(\mathscr{F})$. □

Our second lemma proves the existence of approximating fixed point sequences for a lipschitz semigroup under mild assumptions on the Banach space.

Lemma 3.2. *Let X be a Banach space and let C be a nonempty closed convex subset of X. Let $A : C \to X$ be a bounded mapping (i.e. A maps bounded sets into bounded sets) and let $\mathscr{F} = \{T(t) : t \in \mathbb{R}^+\}$ be a uniformly continuous semigroup of uniformly Lipschitz mappings.*
Let $\{b_n\}$ be a sequence in $(0, 1)$ and let $\{t_n\}$ and $\{\delta_n\}$ be two sequences in $(0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} b_n/t_n = \lim_{n\to\infty} \delta_n/t_n = 0.$$

If $\{y_n\} \subset C$ is a bounded sequence of approximate solutions, i.e. it satisfies

$$\|b_n(I - A)y_n + (1 - b_n)(I - T(t_n))y_n\| \le \delta_n \text{ for all } n \in \mathbb{N} \tag{3.1}$$

then $\lim_{n\to\infty} \|y_n - T(t_m)y_n\| = 0$ for all $m \in \mathbb{N}$.

Proof. Let $L > 0$ be the Lipschitz constant of the semigroup $\mathscr{F}$ and assume that $\{y_n\}$ is a bounded sequence in C satisfying (3.1).
Without loss of generality, we may assume that $\{b_n\}$ is a sequence in $(0, \delta]$ for some $\delta \in (0, 1)$. Since $\{y_n\}$ and $\{Ay_n\}$ are bounded, there exists a constant $K \ge 0$ such that $\|(I - A)y_n\| \le K$ for all $n \in \mathbb{N}$. Note that

$$\begin{aligned}
\|(I - T(t_n))y_n\| &= (1 - b_n)^{-1}\|(1 - b_n)(I - T(t_n))y_n + b_n(I - A)y_n - b_n(I - A)y_n\| \\
&\le (1 - b_n)^{-1}(\delta_n + b_n\|(I - A)y_n\|) \qquad (3.2) \\
&\le (1 - \delta)^{-1}(\delta_n + Kb_n).
\end{aligned}$$

Let $\tilde{d}$ be the metric on X defined by

$$\tilde{d}(x, y) := \sup_{s\in\mathbb{R}^+} \|T(s)x - T(s)y\|.$$

By standard arguments, it is easily derived that

$$\|x - y\| \le \tilde{d}(x, y) \le L\|x - y\| \text{ for any } x, y \in C, \tag{3.3}$$

and that for any $n \in \mathbb{N}$, $T(t_n)$ is nonexpansive with respect to $\tilde{d}$.
Let $[\cdot]$ be the integer part and fix $m \in \mathbb{N}$. Then, for any $n \geq m$,

$$
\begin{aligned}
\|y_n - T(t_m)y_n\| &\leq \tilde{d}(y_n, T(t_m)y_n) \\
&\leq \sum_{i=0}^{[t_m/t_n]-1} \tilde{d}(T(it_n)y_n, T((i+1)t_n)y_n) \\
&\quad +\tilde{d}(T([t_m/t_n]t_n)y_n, T(t_m)y_n) \\
&= \sum_{i=0}^{[t_m/t_n]-1} \tilde{d}(T^i(t_n)y_n, T^i(t_n)T(t_n)y_n) \\
&\quad +\tilde{d}(T^{[t_m/t_n]}(t_n)y_n, T^{[t_m/t_n]}(t_n)T(t_m - [t_m/t_n]t_n)y_n) \\
&\leq (t_m/t_n)\tilde{d}(y_n, T(t_n)y_n) + \tilde{d}(y_n, T(s_n)y_n),
\end{aligned}
\tag{3.4}
$$

where $s_n := t_m - [t_m/t_n]t_n \geq 0$.
Note that by (3.2) and (3.3), we have

$$\tilde{d}(y_n, T(t_n)y_n) \leq L(1-\delta)^{-1}(\delta_n + Kb_n),$$

thus (3.4) becomes

$$\|y_n - T(t_m)y_n\| \leq L(t_m(1-\delta)^{-1}(\delta_n/t_n + Kb_n/t_n) + \sup_{y\in\{y_n\}} \|y - T(s_n)y\|). \tag{3.5}$$

Observe that

$$s_n = t_m - [t_m/t_n]t_n \leq t_n \to 0$$

and hence

$$\lim_{n\to\infty} \sup_{y\in\{y_n\}} \|y - T(s_n)y\| = 0, \tag{3.6}$$

by the uniform continuity of $\mathscr{F}$.
On the other hand and by hypothesis,

$$\lim_{n\to\infty} t_m(1-\delta)^{-1}(\delta_n/t_n + Kb_n/t_n) = 0,$$

which, together with (3.6) and (3.5), implies $\lim_{n\to\infty} \|y_n - T(t_m)y_n\| = 0$ for any fixed $m \in \mathbb{N}$. $\square$

We now prove our main result.

Theorem 3.3. *Let X be a uniformly smooth Banach space, which satisfies property (I). Let $C \subset X$ be nonempty, closed and convex. Let $A : C \to X$ be a bounded and continuous generalized Φ-pseudo-contractive mapping and $\mathscr{F} = \{T(t) : t \in \mathbb{R}^+\}$ a uniformly continuous semigroup of uniformly Lipschitz pseudo-contractive mappings from C into itself. Let $\{b_n\}$ be a sequence in $(0,1)$ and let $\{t_n\}$ and $\{\delta_n\}$ be two sequences in $(0,\infty)$ such that*

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} b_n/t_n = \lim_{n\to\infty} \delta_n/t_n = \lim_{n\to\infty} \delta_n/b_n = 0. \tag{3.7}$$

If the approximate solutions $y_n \in C$ satisfy (3.1) and $\{y_n\}$ is bounded, then

(a) *$F(\mathscr{F})$ is nonempty,*
(b) *$F(\mathscr{F}) \cap \Omega_{F(\mathscr{F})}(I - A, C)$, is nonempty and*

(c) $\{y_n\}$ *converges strongly to the unique element* $y^* \in F(\mathscr{F}) \cap \Omega_{F(\mathscr{F})}(I-A, C)$

Proof. (a) Assume that the approximate solutions $y_n \in C$ satisfy (3.1) and $\{y_n\}$ is bounded. By Lemma 3.2, we have $y_n - T(t_m)y_n \to 0$ as $n \to \infty$ for all $m \in \mathbb{R}$. Since X has property (I), it follows from Lemma 3.1 that $F(\mathscr{F}) \cap \mathscr{Z}_a(C, \{y_n\})$ is nonempty and singleton. In particular, $F(\mathscr{F}) \neq \emptyset$.

(b) Let $v \in F(\mathscr{F})$. Set $\beta_n = \langle b_n(I-A)y_n + (1-b_n)(I-T(t_n))y_n, J(y_n - v)\rangle$ and $c_v = \sup_{n\in\mathbb{N}} \|y_n - v\|$. Observe that $\beta_n \leq \delta_n c_v$ and

$$\begin{aligned}\langle y_n - T(t_n)y_n, J(y_n - v)\rangle &= \langle y_n - v + T(t_n)v - T(t_n)y_n, J(y_n - v)\rangle \\ &= \|y_n - v\|^2 - \langle T(t_n)y_n - T(t_n)v, J(y_n - v)\rangle \\ &\geq 0 \text{ for all } n \in \mathbb{N}.\end{aligned}$$

Thus,

$$\begin{aligned}\langle (I-A)y_n, J(y_n - v)\rangle &= b_n^{-1}\langle b_n(I-A)y_n + (1-b_n)(I-T(t_n))y_n \\ &\quad -(1-b_n)(I-T(t_n))y_n, J(y_n - v)\rangle \\ &= b_n^{-1}\beta_n - b_n^{-1}(1-b_n)\langle (I-T(t_n))y_n, J(y_n - v)\rangle \\ &\leq b_n^{-1}\beta_n \\ &\leq b_n^{-1}\delta_n c_v. \end{aligned} \tag{3.8}$$

Let y^* be the unique element of $\mathscr{Z}_a(C, \{y_n\})$, which also lies in $F(\mathscr{F})$. By Lemma 2.1, there exists a subsequence $\{y_{n_k}\}$ such that

$$\limsup_{k\to\infty} \langle Ay^* - y^*, J(y_{n_k} - y^*)\rangle \leq 0. \tag{3.9}$$

From (3.8), we have

$$\begin{aligned}\|y_{n_k} - y^*\|^2 &= \langle y_{n_k} - Ay_{n_k} + Ay_{n_k} - Ay^* + Ay^* - y^*, J(y_{n_k} - y^*)\rangle \\ &\leq b_n^{-1}\delta_{n_k} c_{y^*} + \|y_{n_k} - y^*\|^2 - \Phi(\|y_{n_k} - y^*\|) + \langle Ay^* - y^*, J(y_{n_k} - y^*)\rangle,\end{aligned}$$

which gives us that

$$\Phi(\|y_{n_k} - y^*\|) \leq b_{n_k}^{-1}\delta_{n_k} c_{y^*} + \langle Ay^* - y^*, J(y_{n_k} - y^*)\rangle. \tag{3.10}$$

Together with (3.9), this last implies that $\{y_{n_k}\}$ strongly converges to y^*.
Let $v \in F(\mathscr{F})$ and observe that, by (3.8),

$$\begin{aligned}\langle y^* - Ay^*, J(y^* - v)\rangle &= \langle (I-A)y^*, J(y^* - v)\rangle - \langle (I-A)y^*, J(y_{n_k} - v)\rangle \\ &\quad + \langle (I-A)y^*, J(y_{n_k} - v)\rangle - \langle (I-A)y_{n_k}, J(y_{n_k} - v)\rangle \\ &\quad + \langle (I-A)y_{n_k}, J(y_{n_k} - v) \\ &\leq |\langle (I-A)y^*, J(y^* - v)\rangle - \langle (I-A)y^*, J(y_{n_k} - v)\rangle| \\ &\quad + \|(I-A)y_{n_k} - (I-A)y^*\|\|J(y_{n_k} - v)\| + b_{n_k}^{-1}\delta_{n_k} c_v.\end{aligned}$$

Since the duality mapping J is single-valued and norm to weak* uniformly continuous on any bounded subset of a Banach space X with a uniformly Gâteaux differentiable norm and $\{y_{n_k}\}$ converges to y^*, we have

$$\langle y^* - Ay^*, J(y^* - v)\rangle \leq 0 \text{ for any } v \in F(\mathscr{F}),$$

i.e. $y^* \in F(\mathscr{F}) \cap \Omega_{F(\mathscr{F})}(I-A, C) \neq \emptyset$.

(c) Suppose that the sequence $\{y_n\}$ does not converge to y^*. As a consequence, there exists $\varepsilon_0 > 0$ and a subsequence $\{y_{n_m}\}$, such that for any $m \in \mathbb{N}$,

$$\|y_{n_m} - y^*\| \geq \varepsilon_0. \tag{3.11}$$

Let z^* be the unique element in $\mathcal{Z}_a(C, \{y_{n_m}\})$ and note that by Lemma 3.1, z^* also belongs to $F(\mathscr{F})$. By Lemma 2.1 and passing to a further subsequence, if necessary, we can assume that

$$\limsup_{m\to\infty} \langle Az^* - z^*, J(y_{n_m} - z^*)\rangle \leq 0.$$

Following the same arguments as in (b), we then derive that $y_{n_m} \to z^*$ and that $z^* \in F(\mathscr{F}) \cap \Omega_{F(\mathscr{F})}(I - A, C)$. Since $\Omega_{F(\mathscr{F})}(I - A, C)$ is singleton, we obtain that $z^* = y^*$, which contradicts (3.11). Hence $\lim_{n\to\infty} y_n = y^*$. □

By the next proposition, we prove the existence of a sequence satisfying (3.1). Moreover we obtain an answer to the problem posed by Xu in [18].

Proposition 3.4. Let C be a nonempty closed convex subset of a uniformly smooth Banach space X, $A : C \to C$ a continuous generalized Φ-pseudo-contractive mapping and $\mathscr{F} = \{T(t) : t \in \mathbb{R}^+\}$ a semigroup of pseudo-contractive mappings from C into itself. Let $\{b_n\}$ be a sequence in $(0, 1)$ and $\{t_n\}$ a sequence in $(0,\infty)$. For each $n \in \mathbb{N}$, define $G_n : C \to C$ by $G_n z := b_n Az + (1 - b_n)T(t_n)z$, $y \in C$. Then, there exists exactly one fixed point z_n in C of G_n defined by

$$z_n = b_n Az_n + (1 - b_n)T(t_n)z_n \text{ for all } n \in \mathbb{N}. \tag{3.12}$$

Proof. Set $\Phi_n(\cdot) := b_n\Phi(\cdot)$ for each $n \in \mathbb{N}$. Then the mapping $G_n : C \to C$ is continuous and generalized Φ_n-pseudo-contractive. Indeed, for x, y in C, we have

$$\begin{aligned}\langle G_n x - G_n y, J(x - y)\rangle &= b_n\langle Ax - Ay, J(x - y)\rangle \\ &\quad + (1 - b_n)\langle T(t_n)x - T(t_n)y, J(x - y)\rangle \\ &\leq b_n(\|x - y\|^2 - \Phi(\|x - y\|)) + (1 - b_n)\|x - y\|^2 \\ &= \|x - y\|^2 - \Phi_n(\|x - y\|).\end{aligned}$$

Note also that $\Phi_n(\cdot)$ is a strictly increasing function with $\Phi_n(0) = 0$. By Xiang [17, Theorem 2.1], G_n has a unique fixed point z_n in C. □

Corollary 3.5. Let X be a uniformly smooth Banach space, which satisfies property (I). Let $C \subset X$ be nonempty, closed and convex. Let $A : C \to X$ be a bounded and continuous generalized Φ-pseudo-contractive mapping and $\mathscr{F} = \{T(t) : t \in \mathbb{R}^+\}$ a uniformly continuous semigroup of uniformly Lipschitz pseudo-contractive mappings from C into itself. Let $\{b_n\}$ be a sequence in $(0, 1)$, let $\{t_n\}$ and $\{\delta_n\}$ be two sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} b_n/t_n = \lim_{n\to\infty} \delta_n/t_n = \lim_{n\to\infty} \delta_n/b_n = 0 \tag{3.13}$$

and let $\{z_n\}$ be defined by (3.12).
If $\{z_n\}$ is bounded then

(a) $F(\mathscr{F})$ is nonempty,
(b) $F(\mathscr{F}) \cap \Omega_{F(\mathscr{F})}(I - A, C)$, is nonempty and
(c) $\{z_n\}$ converges strongly to the unique element $y^* \in F(\mathscr{F}) \cap \Omega_{F(\mathscr{F})}(I - A, C)$

Remark 3.6. We remark that,

(a) in both Lemma 3.2 and Theorem 3.3, if the sequence $\{t_n\}$ can be chosen so that, for $n \geq m$, $t_m/t_n \in \mathbb{N}$ (e.g. $t_n = a^{-n}$ for some $a \in \mathbb{N}$), the uniform continuity condition on the semigroup can be weakened by only assuming strong continuity;

(b) in Theorem 3.3, we prove the existence of a solution of a variational inequality problem on the set $F(\mathscr{F})$, which can fail to be convex.

(c) the asymptotic center technique is used in Theorem 3.3. Therefore, our approach is different from the results recently studied in Sahu, Wong and Yao [13].

References

[1] R. P. Agarwal, Donal O'Regan and D. R. Sahu, *Fixed Point Theory for Lipschitzian-Type Mappings with Applications*, Series: Topological Fixed Point Theory and Its Applications, 6, Springer New York, 2009.

[2] F. E. Browder, *Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces*, Archive for Rational Mechanics and Analysis **24** (1967), 82–90.

[3] C. E. Chidume and H. Zegeye, *Strong convergence theorems for common fixed points of uniformly L-Lipschitzian pseudocontractive semigroups*, Appl. Anal. **86** (2007), 353–366.

[4] L. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Aacademic Publishers, Dordrecht, 1990.

[5] M. Edelstein, Fixed point theorems in uniformly convex Banach spaces, Proc. Amer. Math. Soc. **44** (1974), 369–374.

[6] J. Gwinner, *On the Convergence of Some Iteration Processes in Uniformly Convex Banach Spaces*, Proc. Amer. Math. Soc. 71 (1978), 29–35.

[7] A. Hester and C. H. Morales, *Semigroups generated by pseudo-contractive mappings under the Nagumo condition*, J. Diff. Eq. **245** (2008), 994–1013.

[8] T. C. Lim, *On asymptotic centres and fixed points for nonexpansive mappings*, Canad. J. Math. **32** (1980), 421–430.

[9] G. Marino and H. K. Xu, *Asymptotic centers, inward sets and fixed points*, Comm. Pure Appl. Anal. **10**(2003), 55–63.

[10] R. H. Martin, *Differential equations on closed subsets of a Banach space*, Trans. Amer. Math. Soc. **179** (1973), 399–414.

[11] C. H. Morales, *Variational inequalities for Φ-pseudo-contractive mappings*, Nonlinear Analysis **75** (2012), 477–484.

[12] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), 287–292.

[13] D. R. Sahu, N. C. Wong and J.C. Yao, *A unified hybrid iterative method for solving variational inequalities involving generalized pseudo-contractive mappings*, SIAM J. Control Opt., 2012, in press.

[14] D. R. Sahu, *Fixed points of demicontinuous nearly Lipschitzian mappings in Banach spaces*, Comment. Math. Univ. Carolin. **46** (2005), 653–666.

[15] T. Suzuki, *On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces*, Proc. Amer. Math. Soc., **131** (2002), 2133–2136.

[16] T. Suzuki, *The set of common fixed points of a one-parameter continuous semigroup of mappings is $F(T(1)) \cap F(T(\sqrt{2}))$* Proc. Amer. Math. Soc. **134** (2006), 673–681

[17] C. H. Xiang, *Fixed point theorem for generalized Φ-pseudo-contractive mappings*, Nonlinear Anal. **70** (2009), 277–279.

[18] H. K. Xu, *A strong convergence theorem for contraction semigroups in Banach spaces*, Bull. Austral. Math. Soc., **72** (2005), 371–379.

[19] N. C. Wong, D. R. Sahu and J.C. Yao, *Solving variational inequalities involving nonexpansive type mappings*, Nonlinear Anal. **69** (2008) 4732–4753.

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