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REVISITING A THEOREM ON MULTIFUNCTIONS OF ONE REAL VARIABLE

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Dedicated to Professor Simeon Reich, with esteem, on his 65th birthday

Abstract. In this paper, we intend to revisit Theorem 2 of [3] formulating it in a way that, weakening the hypotheses and, at the same time, highlighting the richer conclusion allowed by the proof, it can potentially be applicable to a broader range of different situations. Samples of such applications are also given.

Some years ago, we established a certain theorem ([3], Theorem 2) on a class of multifunctions depending on a real variable whose formulation was heavily conditioned by the application of it to minimax theory which just was the core of [3].

In the present paper, we intend to revisit that result formulating it in a way that, weakening the hypotheses and, at the same time, highlighting the richer conclusion allowed by the proof, it can potentially be applicable to a broader range of different situations.

So, after establishing the main result (Theorem 2), we give a sample of application of it (Theorem 6) that cannot be deduced by Theorem 2 of [3]. In turn, we highlight a series of consequences of Theorem 6 essentially dealing with the existence of some kind of "singular" points for functions of the type $f + \lambda g$, with $\lambda \in \mathbf{R}$.

In the sequel, the term "interval" means a non-empty connected subset of $\mathbf{R}$ with more than one point.

For a multifunction $F : I \to 2^X$, as usual, for $A \subseteq I$ and $B \subseteq X$, we set

$$F(A) = \cup_{x\in A} F(x)$$

and

$$F^-(B) = \{\lambda \in I : F(\lambda) \cap B \neq \emptyset\} \ .$$

When I is an interval, F is said to be non-decreasing (resp. non-increasing) with respect to the inclusion if $F(\lambda) \subseteq F(\mu)$ (resp. $F(\mu) \subseteq F(\lambda)$) for all $\lambda, \mu \in I$, with $\lambda < \mu$.

We start by proving the following

Proposition 1. *Let X, Y be two non-empty sets, $D \subseteq Y$, $F : X \to 2^Y$ a multifunction such that $F(x) \cap D \neq \emptyset$ for all $x \in X$. Assume also that there exist $y_0 \in Y$ and a topology on $X \setminus F^-(y_0)$ such that $X \setminus F^-(y_0)$ is sequentially compact (resp. compact) and $X \setminus F^-(\{y, y_0\})$ is sequentially closed (resp. closed) in $X \setminus F^-(y_0)$ for all $y \in D$.*

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Then, for every non-decreasing sequence $\{Y_n\}$ *of subsets of* Y*, with* $\cup_{n\in\mathbf{N}}Y_n = Y$*, there exists* $\tilde{n} \in \mathbf{N}$ *such that* $F(x) \cap Y_{\tilde{n}} \neq \emptyset$ *for all* $x \in X$*.*

Proof. Let $\{Y_n\}$ be a non-decreasing sequence of subsets of Y, with $\cup_{n\in\mathbf{N}}Y_n = Y$. Fix $\nu \in \mathbf{N}$ so that $y_0 \in Y_\nu$. Arguing by contradiction, assume that, for each $n \in \mathbf{N}$, there exists $x_n \in X$ such that

$$F(x_n) \cap Y_n = \emptyset \ . \tag{1}$$

First, consider the "sequentially compact, sequentially closed" case. Hence, for each $n \geq \nu$, one has $y_0 \notin F(x_n)$, that is $x_n \in X \setminus F^-(y_0)$. So, there exists a subsequence $\{x_{n_k}\}$ converging to a point $x^* \in X \setminus F^-(y_0)$. Now, fix $y^* \in F(x^*) \cap D$ and $h \geq \nu$ such that $y^* \in Y_h$. By assumption, $F^-(y^*) \cap (X \setminus F^-(y_0))$ is sequentially open in $X \setminus F^-(y_0)$, and hence $x_{n_k} \in F^-(y^*)$ for all k large enough. Then, if we choose k so that $n_k \geq h$, we have $y^* \in F(x_{n_k}) \cap Y_{n_k}$, against (1). Now, consider the "compact, closed" case. Let $A \subseteq D$ be a finite set. Fix $p \geq \nu$ so that $A \subseteq Y_p$. Hence, in view of (1), we have

$$X \setminus F^-(A \cup \{y_0\}) \neq 0 \ .$$

In other words, the family $\{(X \setminus F^-(y)) \cap (X \setminus F^-(y_0))\}_{y\in D}$ has the finite intersection property. But then, since each member of this family is closed in $X \setminus F^-(y_0)$ which is compact, we have

$$X \setminus F^-(D \cup \{y_0\}) \neq 0 \ .$$

This is against the assumption that $F^-(D) = X$, and the proof is complete. □

Our main result is as follows.

Theorem 2. *Let* X *be a non-empty set,* $I \subseteq \mathbf{R}$ *an interval and* $F : I \to 2^X$ *a multifunction satisfying the following conditions:*

- (i) *there exist* $\lambda_0 \in I$*, with* $F(\lambda_0) \neq \emptyset$*, and a topology on* $F(\lambda_0)$ *such that* $F(\lambda_0)$ *is sequentially compact (resp. compact);*
- (ii) *the set*

$$D =: \{\lambda \in I : F(\lambda) \cap F(\lambda_0) \text{ is sequentially closed (resp. closed) in } F(\lambda_0)\}$$

  is dense in I *;*
- (iii) *for each* $x \in X$*, the set* $I \setminus F^-(x)$ *is an interval open in* I *.*

Under such hypotheses, there exists a compact interval $[a^*, b^*] \subseteq I$ *such that either* $(F(a^*) \cap F(\lambda_0)) \setminus F(]a^*, b^*[) \neq \emptyset$ *and* $F_{|]a^*,b^*[}$ *is non-decreasing with respect to the inclusion, or* $(F(b^*) \cap F(\lambda_0)) \setminus F(]a^*, b^*[) \neq \emptyset$ *and* $F_{|]a^*,b^*[}$ *is non-increasing with respect to the inclusion. In particular, the first (resp. second) occurrence is true when* $\lambda_0 = \inf I$ *(resp.* $\lambda_0 = \sup I$*) .*

Proof. For each $x \in X$, put

$$\Phi(x) = I \setminus F^-(x) \ .$$

Clearly

$$\Phi^-(\lambda) = X \setminus F(\lambda)$$

for all $\lambda \in I$. In view of Proposition 1, there exists a compact interval $[a, b] \subseteq I$, with $\lambda_0 \in [a, b]$, such that

$$\Phi(x) \cap [a, b] \neq \emptyset$$

for all $x \in X$. Therefore, each set $\Phi(x) \cap [a, b]$ is an interval open in $[a, b]$. For each $x \in X$, put

$$\alpha(x) = \inf(\Phi(x) \cap [a, b])$$

and

$$\beta(x) = \sup(\Phi(x) \cap [a, b]) \ .$$

Clearly, for each $x_0 \in F(\lambda_0)$ and each $r \in]\alpha(x_0), \beta(x_0)[\cap D$ one has

$$x_0 \in \Phi^-(r)$$

and

$$\alpha(x) < r < \beta(x)$$

for all $x \in \Phi^-(r)$. Since, by assumption, $\Phi^-(r) \cap F(\lambda_0)$ is sequentially open (resp. open) in $F(\lambda_0)$ and D is dense in I, we then infer that $\alpha_{|F(\lambda_0)}$ is sequentially upper semicontinuous (resp. upper semicontinuous) at x_0, while $\beta_{|F(\lambda_0)}$ is sequentially lower semicontinuous (resp. lower semicontinuous) at x_0. Now, suppose that $\lambda_0 \in]a, b[$. Observe that

$$F(\lambda_0) = \alpha^{-1}([\lambda_0, +\infty[) \cup \beta^{-1}(]-\infty, \lambda_0]) \ . \tag{2}$$

Since $F(\lambda_0) \neq \emptyset$, we have either $\alpha^{-1}([\lambda_0, +\infty[) \neq \emptyset$ or $\beta^{-1}(]-\infty, \lambda_0]) \neq \emptyset$. First, assume that $\alpha^{-1}([\lambda_0, +\infty[) \neq \emptyset$. Then, since $F(\lambda_0)$ is sequentially compact (resp. compact) and $\alpha_{|F(\lambda_0)}$ is sequentially upper semicontinuous (resp. upper semicontinuous), in view of (2), there is $x^* \in F(\lambda_0)$ such that $\alpha(x^*) = \sup_X \alpha$. Since $\alpha(x^*) \geq \lambda_0$, we have $\alpha(x^*) \in]a, b[$. This implies, in particular, that $\alpha(x^*)$ does not belong to $\Phi(x^*) \cap [a, b]$, since this set is open in $[a, b]$. As a consequence, we have $x^* \in F(\alpha(x^*))$. Now, fix $\lambda, \mu \in]\alpha(x^*), \beta(x^*)[$, with $\lambda < \mu$. Clearly, $\mu \notin F^-(x^*)$ and hence $x^* \notin F(\mu)$. Next, for each $x \in \Phi^-(\mu)$, we have

$$\alpha(x) \leq \alpha(x^*) < \lambda < \mu \leq \beta(x) \ .$$

Hence, $\lambda \notin F^-(x)$ that is $x \in \Phi^-(\lambda)$. Therefore, we have

$$x^* \in F(\alpha(x^*)) \setminus F(]\alpha(x^*), \beta(x^*)[)$$

as well as

$$\Phi^-(\mu) \subseteq \Phi^-(\lambda)$$

that is

$$F(\lambda) \subseteq F(\mu) \ .$$

So, in the current case, the conclusion is satisfied taking $a^* = \alpha(x^*)$ and $b^* = \beta(x^*)$. Now, assume that $\beta^{-1}(]-\infty, \lambda_0])$ is non-empty. This time, due to the sequential lower semicontinuity (resp. lower semicontinuity) of $\beta_{|F(\lambda_0)}$, there exists $\hat{x} \in X$ such that $\beta(\hat{x}) = \inf_X \beta$. As before, one realizes that $\hat{x} \in F(\beta(\hat{x}))$. Fix $\lambda, \mu \in]\alpha(\hat{x}), \beta(\hat{x})[$ with $\lambda < \mu$. Clearly, $\hat{x} \notin F(\lambda)$. For each $x \in \Phi^-(\lambda)$, we have

$$\alpha(x) \leq \lambda < \mu < \beta(\hat{x}) \leq \beta(x)$$

and so $x \in \Phi^-(\mu)$. Therefore, we have

$$\hat{x} \in F(\alpha(\hat{x})) \setminus F(]\alpha(\hat{x}), \beta(\hat{x})[)$$

as well as

$$\Phi^-(\lambda) \subseteq \Phi^-(\mu)$$

that is

$$F(\mu) \subseteq F(\lambda) \ .$$

So, in this case, the conclusion is satisfied taking $a^* = \alpha(\hat{x})$ and $b^* = \beta(\hat{x})$. Now, assume $\lambda_0 = a$ (in particular, this occurs when $\lambda_0 = \inf I$). If $\sup_X \alpha > a$, then since

$$\alpha^{-1}([a, +\infty[) \subseteq F(a) \ ,$$

and $F(a)$ is sequentially compact (resp. compact), α attains its supremum (larger than a), and so we are exactly in the first sub-case considered when $\lambda_0 \in]a, b[$. If $\sup_X \alpha = a$, we still reach the conclusion, as in the first sub-case considered when $\lambda_0 \in]a, b[$, taking $a^* = a$ and $b^* = \beta(x^*)$, where x^* is any point in $F(a)$. In doing so, notice simply that $x^* \in F(\alpha(x^*))$ as $\alpha(x^*) = a$. Finally, let $\lambda_0 = b$ (in particular, this occurs when $\sup I = b$). If $\inf_X \beta < b$, then since

$$\beta^{-1}(] - \infty, b]) \subseteq F(b) \ ,$$

and $F(b)$ is sequentially compact (resp. compact), β attains its supremum (smaller than b), and so we are exactly in the second sub-case considered when $\lambda_0 \in]a, b[$. If $\inf_X \beta = b$, we still reach the conclusion, as in the second sub-case considered when $\lambda_0 \in]a, b[$, taking $a^* = \alpha(\hat{x})$ and $b^* = b$, where $\hat{x}$ is any point in $F(b)$. The proof is complete. $\square$

We now give a purely set-theoretical reformulation of Theorem 2 in the "compact, closed" case. We first need the following definition.

Definition 3. Let Y be a non-empty set and $\mathcal{F}$ a family of subsets of Y. We say that $\mathcal{F}$ has the compactness-like property if every subfamily of $\mathcal{F}$ satisfying the finite intersection property has a non-empty intersection.

We have the following characterization which is due to C. Costantini ([1]):

Proposition 4. *Let Y be a non-empty set, let $\mathcal{F}$ be a family of subsets of Y and let τ be the topology on Y generated by the family $\{Y \setminus C\}_{C \in \mathcal{F}}$. Then, the following assertions are equivalent:*

(a) *Each member of $\mathcal{F}$ is τ-compact.*
(b) *The family $\mathcal{F}$ has the compactness-like property.*
(c) *The space Y is τ-compact.*

Here is the reformulation of Theorem 2.

Theorem 5. *Let X be a non-empty set, $I \subseteq \mathbf{R}$ a interval and $F : I \to 2^X$ a multifunction such that, for each $x \in X$, the set $X \setminus F^-(x)$ is an interval open in I. Moreover, assume that, for some $\lambda_0 \in I$, with $F(\lambda_0) \neq \emptyset$, and some set $D \subseteq I$ dense in I, the family $\{F(\lambda) \cap F(\lambda_0)\}_{\lambda \in D}$ has the compactness-like property.*

Then, the same conclusion as that of Theorem 2 holds.

Proof. In view of Proposition 4, if we consider the topology on $F(\lambda_0)$ generated by the family $\{F(\lambda_0) \setminus F(\lambda)\}_{y \in D}$, all the assumptions of Theorem 2 (for the "compact, closed" case) are satisfied, and the conclusion follows. $\square$

Now, we are going to present an application of Theorem 2.

In the sequel, X is a non-empty set, Y is a real Hausdorff locally convex topological vector space, C is a closed subset of Y such that $Y \setminus C$ is convex, $I \subseteq \mathbf{R}$ is an interval containing 0 and f, g are two functions from X into Y. The symbol ∂ stands for boundary.

Here is the above mentioned application.

Theorem 6. *Assume that the following conditions are satisfied:*

(a_1) *the set $f^{-1}(C)$ is non-empty and the set $\{(f(x), g(x)) : x \in f^{-1}(C)\}$ is compact in $Y \times Y$;*

(a_2) *for each $x \in X$, there exists $\lambda \in I$ such that*

$$f(x) + \lambda g(x) \in Y \setminus C \ .$$

Then, there exist a compact interval $[a^*, b^*] \subseteq I$ and a point $x^* \in f^{-1}(C)$ satisfying

$$f(x^*) + \lambda g(x^*) \in Y \setminus C$$

for all $\lambda \in]a^*, b^*[$, such that, if we put

$$V = \bigcup_{\lambda \in]a^*, b^*[} \{x \in X : f(x) + \lambda g(x) \in Y \setminus C\} \ ,$$

at least one of the following assertions holds:

(p_1) *$f(x^*) + a^* g(x^*) \in \partial C$ and*

$$(f + a^* g)(V) \cap C \subseteq \partial(f + a^* g)(V) \cap \partial C \ ;$$

(p_2) *$f(x^*) + b^* g(x^*) \in \partial C$ and*

$$(f + b^* g)(V) \cap C \subseteq \partial(f + b^* g)(V) \cap \partial C \ .$$

In particular, (p_1) (resp. (p_2)) holds when $0 = \inf I$ (resp. $0 = \sup I$) .

Proof. Consider the multifunction $F : I \to 2^X$ defined by

$$F(\lambda) = \{x \in X : f(x) + \lambda g(x) \in C\}$$

for all $\lambda \in I$. Observe that, taking $\lambda_0 = 0$, F satisfies the assumptions of Theorem 2. Indeed, if we consider on $f^{-1}(C)$ the weakest topology for which both f and g are continuous in $f^{-1}(C)$, then, in view of (a_1), $f^{-1}(C)$ turns out to be compact in that topology. So, (i) is satisfied. Since Y carries a vector topology, for each $\lambda \in \mathbf{R}$, the function $f + \lambda g$ is continuous in $f^{-1}(C)$, and so also (ii) is satisfied since C is closed. Finally, for each $x \in X$, we have

$$I \setminus F^-(x) = \{\lambda \in I : f(x) + \lambda g(x) \in Y \setminus C\}$$

which is an interval open I, in view of (a_2) and of the fact that $Y \setminus C$ is open and convex. Therefore, Theorem 2 ensures the existence of a compact interval $[a^*, b^*] \subseteq I$ such that either $(F(a^*) \cap F(0)) \setminus F(]a^*, b^*[) \neq \emptyset$ and $F_{|]a^*,b^*[}$ is non-decreasing with respect to the inclusion, or $(F(b^*) \cap F(0)) \setminus F(]a^*, b^*[) \neq \emptyset$ and $F_{|]a^*,b^*[}$ is non-increasing with respect to the inclusion. Assume, for instance, that $(F(a^*) \cap F(0)) \setminus F(]a^*, b^*[) \neq \emptyset$ and $F_{|]a^*,b^*[}$ is non-decreasing with respect to the inclusion. Pick $x^* \in (F(a^*) \cap F(0)) \setminus F(]a^*, b^*[)$. So, $x^* \in f^{-1}(C)$, $f(x^*) + a^* g(x^*) \in C$ and $f(x^*) + \lambda g(x^*) \in Y \setminus C$ for all $\lambda \in]a^*, b^*[$. Now, let us show that $(f + a^* g)(V) \cap$

$C \subseteq \partial C$. So, let $x \in V$ be such that $f(x) + a^*g(x) \in C$. Fix $\lambda \in]a^*, b^*[$ such that $f(x) + \lambda g(x) \in Y \setminus C$. Arguing by contradiction, suppose that $f(x) + a^*g(x) \in \text{int}(C)$. Then, we could find $\delta \in]a^*, \lambda[$ so that $f(x) + \delta g(x) \in \text{int}(C)$. But this contradicts the fact that $f(x) + \delta g(x) \subseteq Y \setminus C$ as $F(\delta) \subseteq F(\lambda)$. Now, let us show that $(f + a^*g)(V) \cap C \subseteq \partial(f + a^*g)(V)$. So, let $z \in (f + a^*g)(V) \cap C$. Arguing by contradiction again, assume that $z \in \text{int}((f + a^*g)(V))$. Since $Y \setminus C$ is open and convex and $z \in \partial(Y \setminus C)$, there exists $\varphi \in Y^* \setminus \{0\}$ such that $\varphi(z) \leq \varphi(u)$ for all $u \in Y \setminus C$. Hence, the set $\varphi^{-1}(]-\infty, \varphi(z)[)$ is an open set contained in C which meets $\text{int}((f + a^*g)(V))$ since φ has no local minima being linear, and this is impossible for what seen above. The proof is complete. □

Remark 7. Let us recall that a function $h : X \to Y$ between topological spaces is said to be open at $x_0 \in X$ if there exists a fundamental system $\mathcal{V}$ of neighbourhoods of x_0 such that, for each $V \in \mathcal{V}$, the set $h(V)$ is a neighbourhood of $h(x_0)$. Now, in connection with Theorem 6, if τ is any topology on X containing the family

$$\{\{x \in X : f(x) + \lambda g(x) \in Y \setminus C\}\}_{\lambda \in I} \ ,$$

it follows that at least one of the functions $f + a^*g$, $f + b^*g$ is not τ-open at the point x^*.

Among the corollaries of Theorem 6, it is worth noticing the following

Theorem 8. *Let $\varphi \in Y^* \setminus \{0\}$ and $r \in \mathbf{R}$ be such that the set*

$$K := \{x \in X : \varphi(f(x)) \leq r\}$$

is non-empty. Assume also that the set $\{(f(x), g(x)) : x \in K\}$ is compact in $Y \times Y$ and that $g(K) \cap \varphi^{-1}(0) = \emptyset$.

Then, there exist a compact interval $[a^*, b^*] \subseteq \mathbf{R}$ and a point $x^* \in K$ satisfying

$$\varphi(f(x^*) + \lambda g(x^*)) > r$$

for all $\lambda \in]a^*, b^*[$, such that, if we put

$$V = \bigcup_{\lambda \in]a^*, b^*[} \{x \in X : \varphi(f(x) + \lambda g(x)) > r\} \ ,$$

at least one of the following assertions holds:

(q_1) *$\varphi(f(x^*) + a^*g(x^*)) = r$ and*

$$(f + a^*g)(V) \cap \varphi^{-1}(]-\infty, r]) \subseteq \partial(f + a^*g)(V) \cap \varphi^{-1}(r) \ ;$$

(q_2) *$\varphi(f(x^*) + b^*g(x^*)) = r$ and*

$$(f + b^*g)(V) \cap \varphi^{-1}(]-\infty, r]) \subseteq \partial(f + b^*g)(V) \cap \varphi^{-1}(r) \ .$$

Proof. It is enough to apply Theorem 6 taking $I = \mathbf{R}$ and $C = \varphi^{-1}(]-\infty, r])$, so that $\partial C = \varphi^{-1}(r)$. □

If we apply Theorem 6 jointly with [2], we obtain:

Theorem 9. *Let Y be a finite-dimensional Banach space and let $\psi : Y \to Y$ be a continuous function such that $\psi^{-1}(C)$ is non-empty and compact. Assume also that, for each $x \in Y$, there exists $\lambda \in \mathbf{R}$ such that*

$$\psi(x) + \lambda x \in Y \setminus C \ .$$

Then, there exist $x^* \in \psi^{-1}(C)$ *and* $\mu^* \in \mathbf{R}$ *such that*

$$\psi(x^*) + (1 + \mu^*)x^* \in \partial C$$

and

$$\sup_{\|x - x^*\| \leq r} \|\psi(x) - \psi(x^*) + \mu^*(x - x^*)\| \geq r$$

for each $r > 0$ *small enough.*

Proof. Apply Theorem 6 taking $X = Y$, $I = \mathbf{R}$, $f = \psi$ and $g = \mathrm{id}$. Then, there exist $x^* \in \psi^{-1}(C)$, $\lambda^* \in \mathbf{R}$ and a neighbourhood V of x^* such that

$$\psi(x^*) + \lambda^* x^* \in \partial(\psi + \lambda^* \mathrm{id})(V) \cap \partial C \ .$$

Now, from the proof of Theorem 1 of [2], we know that if $r > 0$ is such that

$$\sup_{\|x - x^*\| \leq r} \|\psi(x) - \psi(x^*) + (\lambda^* - 1)(x - x^*)\| < r \ ,$$

then, for some $r_0 > 0$ and for every $y \in X$ satisfying $\|\psi(x^*) + \lambda^* x^* - y\| < r_0$, there exists $x \in X$, with $\|x - x^*\| \leq r$, such that

$$\psi(x) + \lambda^* x = y \ .$$

As a consequence, for every $r > 0$ for which the closed ball centered at x^*, of radius r, is contained in V, we have

$$\sup_{\|x - x^*\| \leq r} \|\psi(x) - \psi(x^*) + (\lambda^* - 1)(x - x^*)\| \geq r \ .$$

Now, the conclusion follows taking $\mu^* = \lambda^* - 1$. □

The last consequence of Theorem 6 that we point out is as follows:

Theorem 10. *Let* X *be an open set in a real Banach space, let* Y *be a Banach space and let* f, g *be continuously Fréchet differentiable. Assume that* (a_1), (a_2) *hold.*

Then, there exist $x^* \in X$ *and* $\lambda^* \in I$ *such that*

$$f(x^*) + \lambda^* g(x^*) \in \partial C$$

and the continuous linear operator $f'(x^*) + \lambda^* g'(x^*)$ *is not invertible.*

Proof. By Theorem 6, there exists $x^* \in X$ and $\lambda^* \in I$ such that

$$f(x^*) + \lambda^* g(x^*) \in \partial C$$

and $f + \lambda^* g$ is not open at x^* (Remark 7). This just implies that $f'(x^*) + \lambda^* g'(x^*)$ is not invertible, since, otherwise, $f + \lambda^* g$ would be a local homeomorphism at x^* by the inverse function theorem. □

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