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ON THE EXISTENCE OF A NEUTRAL REGION II: THE IMPLICIT RELATION CASE

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Dedicated to Simeon Reich for his nonlinear way of life and for his multivalued (past and future) contributions

Abstract. Interesting objects, lying on the borderline between fixed point theory and computational geometry, have appeared recently. These objects, which are variations of the widely used object called “the Voronoi diagram”, are called “zone diagrams”, “double zone diagrams”, and “(double) territory diagrams”. They induce a decomposition of a given space into regions. This decomposition is obtained from an implicit relation, a collection of subsets (sites), and a distance function. In several works it was claimed without providing a proof that the corresponding decomposition must contain a special region: the neutral region. This paper shows that this assertion is true in a wide class of cases but not in general. It also shows that this region can be used to justify the interpretation of a zone diagram as a certain equilibrium between mutually hostile opponents. The relation between this equilibrium and the Nash equilibrium from game theory and the relation between the neutral region and forbidden zones are explained.

1. Introduction

In recent years a new and interesting connection between computational geometry and fixed point theory has been established. In order to better understand this connection consider first the object called “the Voronoi diagram” (“the Voronoi tessellation”, “the Voronoi decomposition”, “the Dirichlet tessellation”). This object is a widely used decomposition of a given space X into regions, a decomposition which is induced by a distance function and a collection $(P_k)_{k\in K}$ of nonempty subsets (sites). More concretely, the Voronoi region (Voronoi cell) R_k associated with the site P_k is the set of all the points in X whose distance to P_k is not greater than their distance to the other sites P_j. For an illustration in a simple setting, see Figure 1. It turns out that these diagrams appear in a large number of fields in science and technology and have many applications. For more details about them, see, e.g., [5, 6, 10, 19].

A few years ago Asano, Matoušek, and Tokuyama [2, 3] have introduced a variation of Voronoi diagrams called zone diagrams. As in the case of Voronoi diagrams, these geometric objects induce a decomposition of the given space into regions, but in contrast to the Voronoi diagram, in which the region R_k associated with the site P_k is the set of all points in the space whose distance to P_k is not greater than

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Some of the results of this paper were announced in a preliminary form in the extended abstract [23].

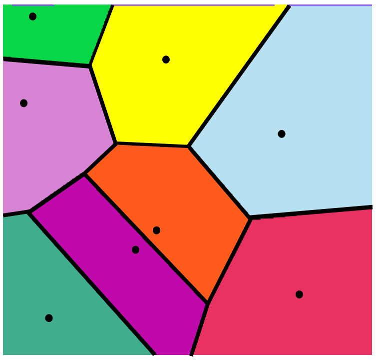

Figure 1: Voronoi diagram of 8 point sites in a square in $(\mathbb{R}^2, \ell_2)$. The neutral region does not exist.

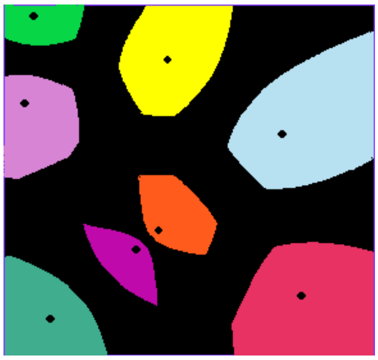

Figure 2: The zone diagram [and hence a double zone diagram and a (double) territory diagram], of the same sites given in Figure 1. The (black) neutral region is clearly seen.

their distance to the other sites $P_j, j \neq k$, in the case of a zone diagram the region R_k is the set of all the points in the space whose distance to P_k is not greater to their distance to the other regions $R_j, j \neq k$. This somewhat implicit definition implies, after some thinking, that a zone diagram is a solution to a certain fixed point equation involving sets.

The above mentioned pioneering works have opened the way to a considerable amount of related investigation. New properties have been established and new objects have been introduced. See, for instance, [1, 8, 9, 11, 13–15, 22, 25].

Returning back to zone diagrams, although their existence is not obvious in advance (even in the original setting of the Euclidean plane with point sites), it seems intuitively clear that if a zone diagram does exist, then it should induce a decomposition of the space into the regions (zones) R_k, and an additional region: the neutral one. See Figure 2. As a matter of fact, the very first works discussing the concept of a zone diagram used the terminology "a Voronoi diagram with neutral zones" [4] and "Voronoi diagram with neutral zone" (see the 2006 conference version of [2] and the footnote on [3, p. 1182]) for describing the concept of zone diagrams. However, neither there nor in other places it has been formally proved that a neutral region must exist in the case of zone diagrams.

In Section 3 we prove that the above claim about the existence of a neutral region holds in a wide class of spaces (geodesic metric spaces) but not in general. We discuss similar phenomena occurring with variations (actually generalizations) of zone diagrams called double zone diagrams [25], territory diagrams [9] (called "subzone diagrams" or "mollified zone diagrams" in the conference version of [9]), and double territory diagrams which are introduced here (we also generalize the definition of territory diagrams from the original setting of the Euclidean plane with point sites). As in the case of zone diagrams, the existence of a neutral zone in the case of territory diagrams was claimed (in [9]) without providing a proof.

An interesting interpretation of the concept of zone diagrams is a certain equilibrium between mutually hostile opponents. This interpretation was mentioned without full justification in [3, 25]. As shown in Section 4, at least in the setting of geodesic metric spaces the neutral region can be used to justify this interpretation. The relation between this kind of equilibrium and the well-known Nash equilibrium is explained. Another concept related to the neutral region is the concept of "forbidden zone" [7, 9]. The relation is briefly explained in Section 5.

2. Preliminaries

In this section we present our notation and basic definitions, as well as additional details and examples related to the relevant notions. Throughout the text we will make use of tuples, the components of which are sets (which are subsets of a given set X). Every operation or relation between such tuples, or on a single tuple, is done component-wise. Hence, for example, if $K \neq \emptyset$ is a set of indices, and if $R = (R_k)_{k\in K}$ and $S = (S_k)_{k\in K}$ are two tuples of subsets of X, then $R \subseteq S$ means $R_k \subseteq S_k$ for each $k \in K$. When R is a tuple, the notation $(R)_k$ is the k-th component of R, i.e, $(R)_k = R_k$.

Definition 2.1. *Let (X, d) be a metric space. Given two nonempty subsets $P, A \subseteq X$, the dominance region* $\operatorname{dom}(P, A)$ *of P with respect to A is the set of all $x \in X$ whose distance to P is not greater than their distance to A, i.e.,*

$$\operatorname{dom}(P, A) = \{x \in X : d(x, P) \leq d(x, A)\}. \tag{2.1}$$

Here

$$d(x, A) = \inf\{d(x, a) : a \in A\} \tag{2.2}$$

and in general, for any subsets A_1, A_2 we denote

$$d(A_1, A_2) = \inf\{d(a_1, a_2) : a_1 \in A_1,\ a_2 \in A_2\}.$$

with the agreement that $d(A_1, A_2) = \infty$ if $A_1 = \emptyset$ or $A_1 = \emptyset$.

Definition 2.2. *Let (X, d) be a metric space. Let K be a set of at least 2 elements (indices), possibly infinite. Given a tuple $(P_k)_{k\in K}$ of nonempty subsets $P_k \subseteq X$, called the generators or the sites, the Voronoi diagram induced by this tuple is the tuple $(R_k)_{k\in K}$ of nonempty subsets $R_k \subseteq X$, such that for all $k \in K$,*

$$R_k = \operatorname{dom}(P_k, \bigcup_{j\neq k} P_j) = \{x \in X : d(x, P_k) \leq d(x, P_j)\ \forall j \neq k,\ j \in K\}. \tag{2.3}$$

In other words, each R_k, called a Voronoi cell or a Voronoi region, is the set of all $x \in X$ whose distance to P_k is not greater than their distance to the other sites P_j, $j \neq k$. The set $X\backslash(\bigcup_{k\in K} R_k)$ is called the neutral region.

Definition 2.3. *Let (X, d) be a metric space. Let K be a set of at least 2 elements (indices), possibly infinite. Given a tuple $(P_k)_{k\in K}$ of nonempty subsets $P_k \subseteq X$, a* *zone diagram with respect to that tuple is a tuple $R=(R_k)_{k\in K}$ of nonempty subsets $R_k\subseteq X$ satisfying*

$$R_k=\mathrm{dom}(P_k,\bigcup_{j\neq k}R_j)\quad \forall k\in K.$$

In other words, if we define $X_k=\{C: P_k\subseteq C\subseteq X\}$, then a zone diagram is a fixed point of the mapping $\mathrm{Dom}:\prod_{k\in K}X_k\to\prod_{k\in K}X_k$, *defined by*

$$\mathrm{Dom}(R)=(\mathrm{dom}(P_k,\bigcup_{j\neq k}R_j))_{k\in K}. \tag{2.4}$$

A tuple $R=(R_k)_{k\in K}$ is called a double zone diagram if it is the fixed point of the second iteration $\mathrm{Dom}\circ\mathrm{Dom}$, *i.e., $R=\mathrm{Dom}^2(R)$. A tuple $R=(R_k)_{k\in K}$ is called a territory diagram if $R\subseteq\mathrm{Dom}(R)$ and it is called a double territory diagram if $R\subseteq\mathrm{Dom}^2(R)$.*

Remark 2.4. Some of the concepts mentioned in Definition 2.3 are related (see also Figures 3-6). Any zone diagram is obviously a territory diagram. It is also a double zone diagram as can be seen by applying Dom on $R=\mathrm{Dom}(R)$. Any double zone diagram is obviously a double territory diagram. A double territory diagram is not necessarily a territory diagram: take $X=\{-1,0,1\}\subset\mathbb{R}$, let d be the absolute value distance, and let $(P_1,P_2)=(\{-1\},\{1\})$, $R=(\{-1,0\},\{0,1\})$; then $\mathrm{Dom}(R)=(\{-1\},\{1\})$ and $\mathrm{Dom}^2(R)=R$, but $R\subsetneqq\mathrm{Dom}(R)$. A territory diagram is not necessarily a double territory diagram: take $X=[-1,1]\subset\mathbb{R}$, $(P_1,P_2)=(\{-1\},\{1\})$, $R=([-1,0],\{1\})$; then $\mathrm{Dom}(R)=([-1,0],[0.5,1])$, $\mathrm{Dom}^2(R)=([-1,-0.25],[0.5,1])$, and hence $R\subsetneqq\mathrm{Dom}^2(R)$. But $R\subseteq\mathrm{Dom}(R)$.

Remark 2.5. The components of any territory and double territory diagrams are contained in the Voronoi regions of their sites. Indeed, the Voronoi regions corresponding to the tuple $P=(P_k)_{k\in K}$ of sites are nothing but the components of $\mathrm{Dom}(P)$. By the definition Dom and the space $\prod_{k\in K}X_k$ of tuples we have $P\subseteq R$ and $P\subseteq\mathrm{Dom}(R)$ for each tuple R in this space. Thus the anti monotonicity of Dom (see Lemma 3.1(a)) implies that $\mathrm{Dom}(R)\subseteq\mathrm{Dom}(P)$ and $\mathrm{Dom}^2(R)\subseteq\mathrm{Dom}(P)$ and the assertion follows by taking R to be a territory or a double territory diagram.

Remark 2.6. Examples (illustrations) of zone diagrams in various settings can be found in [3, 11, 14, 22, 25]. Examples of double zone diagrams which are not zone diagrams can be found in [22, 25]. Examples (including illustrations) of territory diagrams which are not zone diagrams can be found in [9]. Additional illustrations can be found in Figures 2, 3-6.

Existence (and sometimes uniqueness) proofs of zone diagrams in certain settings can be found in [3, 13, 14, 25]. Double zone diagrams always exist [25] (in a setting called m-spaces which is even general than a metric space: the distance function can be negative and does not necessarily satisfy the triangle inequality). However, for our purposes we only need to know that territory diagrams and double territory diagrams always exist, and, as a matter of fact, it is quite easy to construct explicit examples of them. Indeed, we can simply start with $P=(P_k)_{k\in K}$ and iterate it using Dom. As explained in Remark 2.5, for each tuple R one has $P\subseteq\mathrm{Dom}(R)$ and $P\subseteq\mathrm{Dom}^2(R)$. Now, since Dom is antimonotone and since Dom^2 is monotone the inequality

$$P \subseteq \mathrm{Dom}^2(P) \subseteq \mathrm{Dom}^4(P) \subseteq \cdots \subseteq \cdots \subseteq \mathrm{Dom}^3(P) \subseteq \mathrm{Dom}(P)$$

follows. Hence any even power is a territory and a double territory diagram [which is usually not a (double) zone diagram].

We finish this section with the definition of geodesic metric spaces.

Definition 2.7. *Let $x, y \in S \subseteq X$. The subset S is called a geodesic segment (or a metric segment) between x and y if there exists an isometric function γ (that is, a distance preserving mapping) which maps a real line segment $[r_1, r_2]$ onto S and satisfying $\gamma(r_1) = x$ and $\gamma(r_2) = y$. We denote $S = [x, y]_\gamma$, or simply $S = [x, y]$. If between all points $x, y \in X$ there exists a geodesic segment, then (X, d) is called a geodesic metric space. The sets $[x, y) = [x, y]\backslash\{y\}$, $(x, y] = [x, y]\backslash\{x\}$, and $(x, y) = [x, y]\backslash\{x, y\}$ represent the half open segments and open segments respectively.*

Simple and familiar examples of geodesic metric spaces are: the Euclidean plane, any normed space, any convex subset of a normed space, Euclidean spheres, complete Riemannian manifolds [12, pp. 25-28], and hyperbolic spaces [26, pp. 538-9].

3. The existence of a neutral region

In this section we prove the existence of a neutral region (zone) in the context of zone diagrams, double zone diagrams, and (double) territory diagrams. The proof is based on several lemmas, the first of them is taken (each part separately) from [25, Lemma 5.4] (see also [3, p. 1184]) and [22] (Lemma 9.3(c) and Lemma 9.9(c) in the current arXiv version (v3)).

Lemma 3.1. *Let (X, d) be a metric space and let $P = (P_k)_{k\in K}$ be a tuple of nonempty subsets in X.*

(a) Dom *is antimonotone (order reversing), i.e.,* $\mathrm{Dom}(R) \subseteq \mathrm{Dom}(S)$ *whenever* $S \subseteq R$; Dom^2 *is monotone, that is,* $R \subseteq S \Rightarrow \mathrm{Dom}^2(R) \subseteq \mathrm{Dom}^2(S)$.

(b) $\mathrm{Dom}(\overline{R}) = \mathrm{Dom}(R)$.

(c) *Suppose that (X, d) is a geodesic metric space and that*

$$r_k := \inf\{d(P_k, P_j) : j \neq k\} > 0 \quad \forall k \in K. \tag{3.1}$$

Then $(r_k/8) + (r_j/8) \leq d((\mathrm{Dom}^\gamma P)_k, (\mathrm{Dom}^\gamma P)_j)$ for any $j, k \in K, j \neq k$ and any $\gamma \geq 2$.

Lemma 3.2. *Let (X, d) be a metric space, let $P = (P_k)_{k\in K}$ be a tuple of nonempty subsets of X. Suppose that $R = (R_k)_{k\in K}$ satisfies $P_k \subseteq R_k \subseteq X$ for each $k \in K$.*

(a) *Suppose that $R \subseteq \mathrm{Dom}(R)$. If $\overline{P_k} \bigcap \overline{P_j} = \emptyset$ whenever $j \neq k$, then $R_k \bigcap R_j = \emptyset$ for each $j, k \in K$, $j \neq k$.*

(b) *Suppose that* (3.1) *holds. If $R \subseteq \mathrm{Dom}(R)$, then the components of R satisfy $\max\{r_k, r_j\}/3 \leq d(R_k, R_j)$ for each $j, k \in K, k \neq j$.*

(c) *Suppose that $R \subseteq \mathrm{Dom}^2(R)$, that (X, d) is a geodesic metric space, and that* (3.1) *holds. Then the components of R satisfy $(r_k/8) + (r_j/8) \leq d(R_k, R_j)$ for each $j, k \in K, k \neq j$.*

Proof. (a) Suppose by way of contradiction that $x \in R_k \bigcap R_j$ for some $j, k \in K$, $j \neq k$. Since $x \in R_k \subseteq (\mathrm{Dom}R)_k$ we have $d(x, P_k) \leq d(x, \bigcup_{i \neq k} R_i) \leq d(x, R_j) = 0$, so $x \in \overline{P_k}$. In the same way $x \in \overline{P_j}$, a contradiction.

(b) Let $j, k \in K$, $j \neq k$ and $x \in R_k \subseteq \mathrm{dom}(P_k, \bigcup_{i \neq k} R_i)$, $y \in R_j \subseteq \mathrm{dom}(P_j, \bigcup_{i \neq j} R_i)$. This implies that $d(x, P_k) \leq d(x, R_j) \leq d(x, y)$ and $d(y, P_j) \leq d(x, y)$. Therefore

$$r_k \leq d(P_k, P_j) \leq d(P_k, x) + d(x, y) + d(y, P_j) \leq 3d(x, y).$$

Thus $r_k/3 \leq d(R_k, R_j)$. Similarly, $r_j/3 \leq d(R_k, R_j)$.

(c) From the monotonicity of Dom^2 (Lemma 3.1(a)) we have $R \subseteq \mathrm{Dom}^2(R) \subseteq \mathrm{Dom}^4(R)$. This, Lemma 3.1 parts (a)-(b), the inclusion $P \subseteq R \subseteq (X)_{k \in K}$, and $\overline{P} = \mathrm{Dom}(X)_{k \in K}$ imply that $R \subseteq \mathrm{Dom}^4(X)_{k \in K} = \mathrm{Dom}^3(\overline{P}) = \mathrm{Dom}^3(P)$. From Lemma 3.1(c) we conclude that

$$d(R_k, R_j) \geq d((\mathrm{Dom}^3 P)_k, (\mathrm{Dom}^3 P)_j) \geq (r_k/8) + (r_j/8)$$

for each $j, k \in K, k \neq j$.

□

Lemma 3.3. *Let $B = (B_k)_{k \in K}$ be a tuple of nonempty subsets in a geodesic metric space (X, d) and suppose that*

$$\rho_k := \inf\{d(B_k, B_j) : j \in K,\ j \neq k\} > 0 \quad \forall k \in K. \tag{3.2}$$

Then $N := X \backslash \bigcup_{k \in K} B_k \neq \emptyset$. Moreover, $\bigcup_{k \in K} S_k \subseteq N$ where

$$S_k = \{x \in X : d(x, B_k) < \rho_k,\ x \notin B_k\}. \tag{3.3}$$

Proof. Let $j, k \in K$, $j \neq k$ and let $x \in B_k$, $y \in B_j$. Since X is a geodesic metric space there exists an isometry $\gamma : [0, d(x, y)] \to X$ satisfying $\gamma(0) = x$ and $\gamma(d(x, y)) = y$. Let E be the inverse image of the part of the (geodesic) segment $[x, y]$ which does not meet $\overline{B_k}$ anymore, i.e.,

$$E := \{t \in [0, d(x, y)] :\ [\gamma(s), y] \cap \overline{B_k} = \emptyset \quad \forall s \in [t, d(x, y)]\}.$$

Since $y \in B_j$ and $y \notin \overline{B_k}$ (by (3.2)) it follows that $d(x, y) \in E$. Thus $E \neq \emptyset$. Let $a = \inf E$. If $a = 0$, then $\gamma(a) = x \in \overline{B_k}$. Otherwise $a > 0$. Assume by way of contradiction that $\gamma(a) \notin \overline{B_k}$. Since $\overline{B_k}$ is closed it follows that a small ball around $\gamma(a)$ does not intersect $\overline{B_k}$. Because γ is continuous, for all t in a small segment around a the point $\gamma(t)$ is inside this ball and thus does not belong to $\overline{B_k}$. This contradicts the minimality of a. Therefore $\gamma(a) \in \overline{B_k}$.

Consider the segment $(\gamma(a), y]$. Its length is at least ρ_k by (3.2) (since the distance between two sets is the distance between their closures). Since γ is an isometry the length of $[a, d(x, y)]$ is at least ρ_k. Let $s \in (0, \rho_k)$ and let $z = \gamma(a + s)$. Then $z \in (\gamma(a), y]$ and $d(z, \overline{B_k}) \leq d(z, \gamma(a)) = s < \rho_k$. From the definition of a there exists $b \in (a, a + s) \cap E$. Thus $[\gamma(b), y] \cap \overline{B_k} = \emptyset$ and in particular $z \notin B_k$. Since $d(z, \overline{B_k}) < \rho_k$ it follows from (3.2) that $z \notin \bigcup_{i \neq k} B_i$. Therefore $z \in N$ and in particular $N \neq \emptyset$.

Finally, let S_k be the shell defined in (3.3) and let $x \in S_k$. From (3.2) we see that $x \notin B_j$ for $j \neq k, j \in K$. In addition, $x \notin B_k$ by the definition of S_k. Hence $x \in N$ and $S_k \subseteq N$ for each $k \in K$. □

Theorem 3.4. *Let (X,d) be a geodesic metric space and let $(P_k)_{k\in K}$ be a tuple of nonempty subsets of X. Assume that* (3.1) *holds. Let $R=(R_k)_{k\in K}$ satisfy $P_k\subseteq R_k\subseteq X$ for each $k\in K$ and suppose that either $R\subseteq \mathrm{Dom}(R)$ or $R\subseteq \mathrm{Dom}^2(R)$. Then there exists a neutral region in X, i.e., $N:=X\backslash\bigcup_{k\in K}R_k\neq\emptyset$. In particular this is true when R is a zone or a double zone diagram. Moreover, let*

$$\beta_k=\begin{cases} r_k/3, & \text{if } R\subseteq \mathrm{Dom}(R),\\ (r_k+\inf\{r_j : j\in K,\, j\neq k\})/8, & \text{if } R\subseteq \mathrm{Dom}^2(R)\end{cases}$$

for each $k\in K$. Then $\bigcup_{k\in K}S_k\subseteq N$, where for each $k\in K$,

$$S_k=\{x\in X : d(x,R_k)<\beta_k,\ x\notin R_k\}. \tag{3.4}$$

Proof. This is a simple consequence of Lemma 3.3 with $B=R$ and $\rho_k=\beta_k$ since (3.2) is satisfied by Lemma 3.2(b)-(c). ☐

Example 3.5. An illustration of Theorem 3.4 is given in Figures 3-6 which also show some of the differences between the various notions. In all of these figures the setting is $X=[-6,6]^2$, $P_1=\{(2,1),(-2,-1)\}$, $P_2=\{(-2,1),(2,-1)\}$, and the distance is the 2-dimensional ℓ_1 distance. The (black) neutral region is clearly seen. Figures 3, 5, and 6 were produced using the method described in [22] (which is based on the algorithm for computing Voronoi diagrams in a general setting described in [21]) and Figure 4 was produced directly.

Example 3.6. From the proof of Lemma 3.3 and Theorem 3.4 one obtains points in the neutral region by looking at certain parts of geodesic segments connecting points located in different sites. The example described here shows that sometimes the neutral zone is nothing more than such a segment. In particular this example shows that the shells S_k located around the components of the (double) territory diagram (see (3.4)) can be very small.

Indeed, let $X_1=\{0\}\times(-2,3]$, $X_2=\{x\in\mathbb{R}^2 : \|x-(0,-3)\|\leq 1\}$, and $X=X_1\cup X_2$, where $\|\cdot\|$ is the Euclidean norm. Define a metric d on X by $d(x,y)=\|x-y\|$ if x and y belong to the same component X_i, $i=1,2$, and $d(x,y)=\|x-(0,-2)\|+\|y-(0,-2)\|$ otherwise. Then (X,d) is a geodesic metric space. Now let $P_1=\{(0,3)\}$, $P_2=\{(0,-3)\}$, $R_1=\{0\}\times[1,3]$, and $R_2=X_2\cup(\{0\}\times(-2,-1])$. Then $R=(R_1,R_2)$ is a zone diagram with respect to $P=(P_1,P_2)$ and the neutral region is $\{0\}\times(-1,1)$. See Figure 7.

Example 3.7. Let $X=\{-1,0,1\}$ be a subset of $\mathbb{R}$ with the standard absolute value metric. Let $P_1=\{-1\}$, $P_2=\{1\}$. Let $R_1=P_1$, $R_2=\{0,1\}$. Then $R=(R_1,R_2)$ is a zone diagram (and hence also a territory diagram) but $R_1\cup R_2=X$, violating Theorem 3.4. This is not surprising since X is not a geodesic metric space. However, $R_1\cap R_2=\emptyset$, as predicted by Lemma 3.2(a). This setting was mentioned in a different context in [25, Example 2.3].

In the same way, if $S_1=\{-1,0\}$ and $S_2=\{0,1\}$, then $S=(S_1,S_2)$ is a double zone diagram as a simple check shows (starting with observing that $\mathrm{Dom}(S)=(P_1,P_2)$). Now not only $S_1\cup S_2=X$, but also $S_1\cap S_2\neq\emptyset$.

Example 3.8. Condition (3.1) is necessary. Indeed, let $X=\mathbb{R}$ with the standard absolute value metric $d(x,y)=|x-y|$, let $K=X$, and let $P_k=k$, $k\in K$.

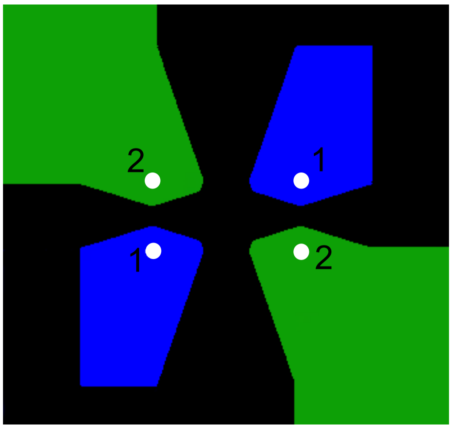

Figure 3: The neutral region induced by a zone diagram of two sites, each consists of 2 points, in a square in $(\mathbb{R}^2, \ell_1)$ (Example 3.5).

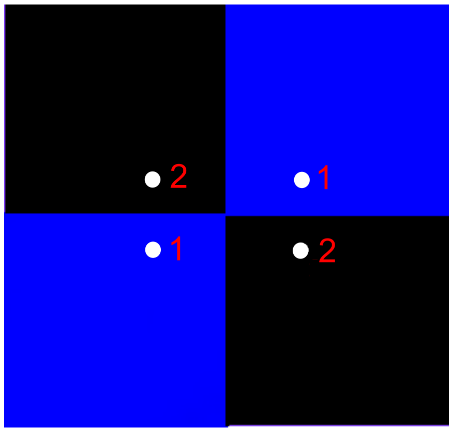

Figure 4: The neutral region induced by a territory diagram R of the setting of Example 3.5. The second component of R is P_2 and R is not a double territory diagram.

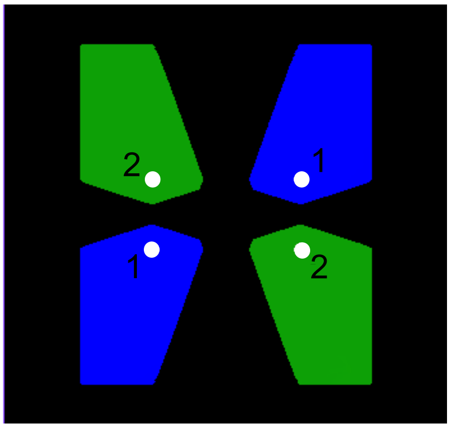

Figure 5: The neutral region induced by the least double zone diagram R of the setting of Example 3.5. R is not a zone diagram.

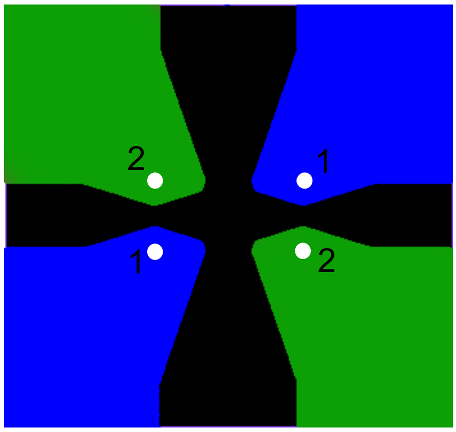

Figure 6: The neutral region induced by the greatest double zone diagram R of the setting of Example 3.5. R is a double territory diagram but not a territory diagram.

Let $R = (P_k)_{k\in K}$. Then (X, d) is a geodesic metric space, $R = \mathrm{Dom}(R)$, but $X\backslash(\bigcup_{k\in K} R_k) = \emptyset$.

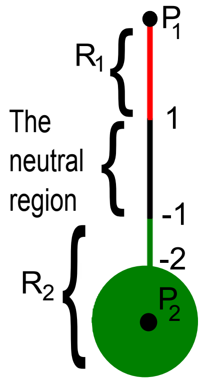

Figure 7: The neutral region described in Example 3.6.

4. Justifying the equilibrium interpretation of zone diagrams

One of the interpretations of zone diagrams, first suggested in [3] and then extended in [25], is a a certain equilibrium between mutually hostile kingdoms competing over territory. Kingdom number k has a territory R_k which has to be defended against attacks from the other kingdoms. Its site P_k is interpreted as a castle, or, more generally, as a collection of army camps, castles, cities, and so forth. The sites remain unchanged and they are assumed to be located inside the kingdom and hence separated from each other. Due to various considerations (resources, field conditions, etc.), the defending army is located only in (part of) the corresponding site (unless the kingdom moves forces to attack another kingdom).

Assuming the time to move armed forces between two points is proportional to the distance between the points (with the same proportion relation allover X), it seems intuitively clear that if $R=(R_k)_{k\in K}$ is a zone diagram, then each point in each kingdom can be defended at least as fast as it takes to attack it from any other kingdom, and no kingdom can enlarge its territory without violating this condition. More precisely, given any index $k\in K$ and any nonempty subset $A_k\subset X$ satisfying

$$A_k\bigcap R_k=\emptyset=A_k\bigcap(\bigcup_{j\neq k}P_j), \tag{4.1}$$

if we let $\widetilde{R_k}=R_k\bigcup A_k$ and $\widetilde{R_j}=R_j\backslash A_k$ for any $j\neq k$, then there exist points in $\widetilde{R_k}$ which cannot be defended fast enough by armed forces emanating from P_k: there is some kingdom $\widetilde{R_j}$, $j\neq k$ which can send secretly its forces to its borders, and, after exiting R_j, they will arrive to these points before the defending forces from P_k will arrive. In other words, it is not true that $\widetilde{R_k}\subseteq \mathrm{dom}(P_k,\bigcup_{j\neq k}\widetilde{R_j})$. (The second assumption in (4.1), namely $A_k\bigcap(\bigcup_{j\neq k}P_j)=\emptyset$, is assumed in order to make sure that the original sites do not change after the enlarging attempt. This casts a slight restriction on the full generality of the possible enlarging attempt, but at least intuitively this condition seems quite clear because the sites are located strictly inside their corresponding kingdoms and a kingdom cannot send its forces to a site by jumping on them "out of the blue".)

It also seems clear that the various territories are separated by a no-man's land: the neutral territory. This was said explicitly in [3, p. 1183] where the setting was the Euclidean plane and each site was a point. In [25] the setting was general and it was noted that counterexamples may exist in a discrete setting, but no further investigation of the whole interpretation has been carried out.

The goal of this section is to give a more rigorous justification to the above interpretation. It turns out that when the setting is similar to that of Theorem 3.4, then the interpretation holds.

Theorem 4.1. *Let (X,d) be a geodesic metric space and let $P=(P_k)_{k\in K}$ be a tuple of nonempty subsets of X. Assume that* (3.1) *holds. Suppose that $R=(R_k)_{k\in K}$ is a zone diagram corresponding to P. Then R is an equilibrium in the above mentioned sense and there exists a neutral region separating its components.*

Proof. The existence of a neutral region N was proved in Theorem 3.4. The proof actually shows that N separates the regions R_k, $k\in K$ in the sense that each region R_k is surrounded by a shell S_k (defined in (3.4)) contained in N and any path connecting two points located in different regions goes through N.

As for the equilibrium interpretation, let x be a point in some region R_k. By definition, $d(x,P_k)\leq d(x,\bigcup_{j\neq k}R_j)$. Since the time to move armed forces between any two points is proportional to the distance between them, this shows that armed forces emanating from P_k will arrive at x before any armed forces emanating from another kingdom will arrive at x. This last fact is true in general, even in m-spaces [25] (in which the distance function can be, e.g., negative) and even if the sites are not mutually disjoint, although in this general case the interpretation looses something from its intuitiveness.

It remains to prove that no kingdom can enlarge its territory without violating the fast defense condition. As already mentioned (around (4.1)), the precise meaning of this is that given any index $k\in K$ and any nonempty subset $A_k\subset X$ satisfying

$$A_k\bigcap R_k=\emptyset=A_k\bigcap(\bigcup_{j\neq k}P_j), \tag{4.2}$$

if we let $\widetilde{R_k}=R_k\bigcup A_k$ and $\widetilde{R_j}=R_j\backslash A_k$ for all $j\neq k$, then it is not true that $\widetilde{R_k}\subseteq \mathrm{dom}(P_k,\bigcup_{j\neq k}\widetilde{R_j})$.

To prove this, let $A_k\neq\emptyset$ satisfy (4.2) and suppose for a contradiction that for some $x\in A_k$ we have

$$d(x,P_k)\leq d(x,\bigcup_{j\neq k}\widetilde{R_j}). \tag{4.3}$$

First, by (4.2) it follows that $x\notin R_k$. It must be that $x\notin R_j$ for each $j\neq k$. Indeed, assume by way of negation that $x\in R_j$ for some $j\neq k$. In particular $d(x,P_j)\leq d(x,R_k)$ since R is a zone diagram. By Lemma 3.2 we also know that $x\notin R_k$. Now observe the simple fact that the neighborhood $B(P_k,r_k/4)=\{y\in X: d(y,P_k)<r_k/4\}$ is contained in R_k. A proof can be found in [22] (Lemma 9.9(a) and Lemma 9.2(b) in the current arXiv version (v3)) and a related claim also

in [13, Observation 2.2]. Let $p \in P_k$ satisfy $d(x,p) < d(x,P_k)+(r_k/16)$. The segment $[p,x]$ starts at a point in $B(P_k, r_k/4)$ and ends at a point outside this neighborhood and therefore the intermediate value theorem implies that it intersects the boundary of $B(P_k, r_k/4)$ in at least one point y. The point y is of distance at least $r_k/4$ from p, otherwise it will be strictly inside $B(P_k, r_k/4)$. The discussion above implies that

$$(r_k/16) + d(x,P_k) > d(x,p) = d(x,y) + d(y,p) \geq d(x,R_k) + (r_k/4)$$

and hence, recalling that $d(x,R_k) \geq d(x,P_j)$, we have

$$d(x,P_k) > d(x,R_k) + (3r_k/16) \geq d(x,P_j) + (3r_k/16) > d(x,P_j).$$

But this is impossible since we assumed in (4.3) that $d(x,P_k) \leq d(x, \bigcup_{i\neq k} \widetilde{R_i})$ and from (4.2) we know that $P_j \subseteq \widetilde{R_j} \subseteq \bigcup_{i\neq k} \widetilde{R_i}$. This contradiction proves that $x \notin R_j$ for each $j \neq k$ and hence $A_k \bigcap (\bigcup_{j\neq k} R_j) = \emptyset$.

Finally x cannot be in the (original) neutral region $N = X \backslash (\bigcup_{j\in K} R_j)$. Indeed, if x is there, then in particular $x \notin R_k = \mathrm{dom}(P_k, \bigcup_{j\neq k} R_j)$, i.e., $d(x,R_j) < d(x,P_k)$ for some $j \neq k$. But $R_j = \widetilde{R_j}$ since $A_k \cap R_j = \emptyset$ as proved above. Thus $d(x, \widetilde{R_j}) < d(x,P_k)$, a contradiction to (4.3). In conclusion, $x \notin R_k \bigcup (\bigcup_{j\neq k} R_j) \bigcup N = X$, an obvious contradiction. Thus (4.3) does not hold, i.e., $d(x,P_k) > d(x, \bigcup_{j\neq k} \widetilde{R_j})$ and x cannot be defended fast enough, as initially claimed. □

Remark 4.2. When the space is no geodesic anymore a kingdom can enlarge its territory without violating the fast defense condition: just consider for instance Example 3.7 where $X = \{-1, 0, 1\}$ (or, if we allow attacks on the sites, even the more simple example where $X = \{-1, 1\}$, $P_1 = R_1 = \{-1\}$, $P_2 = R_2 = \{1\}$). Here it is worthwhile to kingdom 1 to try to capture the point 0. However, one can argue against this example that the armed forces must jump out of the space in order to arrive at the other kingdoms and if they do manage to do this, then they seem to appear there "out of the blue". Hence it is implicitly assumed in the original interpretation that the space is "continuous", or, in more precise terms, that it is a geodesic metric space or even a convex subset of a normed space.

Remark 4.3. Another type of interpretation was suggested to zone diagrams: again as a certain equilibrium, but now in a discrete setting related to a certain combinatorial game. See [25, Section 4].

Remark 4.4. It is interesting to compare the equilibrium induced by a zone diagram (as described in Theorem 4.1) to the well-known Nash equilibrium [16–18, 20]. A Nash equilibrium is a certain equilibrium between $n \geq 2, n \in \mathbb{N}$ players. More precisely, each player has his (or her) own set of strategies and each player obtains a payoff for each vector of strategies (that is, the payoff of player k depends on the player's own chosen strategy and on all the other strategies). Each player knows the other players strategies and he can decide to change his strategy in order to enlarge his payoff. This may result in an instable situation where each player tries to maximize his own payoff just to realize that the payoff has changed because the other players change their strategies too and the payoff depends on all strategies. However, if, given a vector of strategies, each player cannot enlarge his payoff by changing his own strategy while the remaining strategies (the remaining components of the given vector) are fixed, then this vector of strategies is called a Nash equilibrium: instead of the previous mentioned dynamical situation the obtained situation is static. Nash equilibrium can be defined as a fixed point of a certain multivalued function and its common proofs are based on either the Kakutani [16, 17, 20] or the Brouwer [18] fixed point theorems.

Similarly, the situation leading to the interpretation of a zone diagram can be considered as a game between given (possibly infinite) players. Instead of a strategy each player has a kingdom. The payoff each player obtains is somewhat abstract. For example, if we consider player k, then he “obtains more” for a kingdom $R_{k,2}$ than for a kingdom $R_{k,1}$ whenever $R_{k,1} \subsetneq R_{k,2}$. (One can try to use also a kind of a measure function such as area together with the abstract payoff of set inclusion; a measure function alone cannot replace set inclusion, because, for instance, the area of one set can be greater than another even when neither of these sets is contained in the other one.)

A given player may want to enlarge his kingdom. However, because of the hostility between the players and the way they move their forces, the new enlarged kingdom may have parts which cannot be efficiently defended. In this case there is a certain (very large) penalty which enforces the player to abandon these parts from the kingdom. This will lead to a dynamical situation in which the players enlarge and reduce their kingdoms depending on the other players’ kingdoms. A zone diagram is a special vector (tuple) of kingdoms in which each player cannot obtain a larger kingdom than he has now while still having a kingdom whose all parts are safe.

5. A short remark about forbidden zones

The goal of this short section is to discuss very briefly the relation between the neutral region and a concept called “the forbidden zone”, a concept which was introduced in [9] and was further investigated in [7] . Given two nonempty subsets P (the “site”) and R (the “region”), the forbidden zone with region R and a site P is the set

$$F(R,P) = \{z \in X : d(z,y) < d(y,P) \text{ for some } y \in R\}. \tag{5.1}$$

Originally [9], the setting was a subset R in the Euclidean space $\mathbb{R}^m$ and P was a point in R, but the same definition holds with respect to any given subsets contained in any metric space. In fact, the distance function can be completely general (for instance, for the sake of the proof of the following proposition, it merely needs to be symmetric; positivity is not needed), but we will confine ourselves to metric spaces. The relation to the neutral region is described in the following proposition.

Proposition 5.1. *Let (X,d) be a metric space. Suppose that $(R_k)_{k\in K}$ is a territory diagram with respect to the tuple $(P_k)_{k\in K}$. Then the union $\bigcup_{k\in K} F(R_k,P_k)\backslash R_k$ is contained in the (possibly empty) neutral region.*

Proof. Suppose for a contradiction that there exist $k \in K$ and $z \in F(R_k,P_k)\backslash R_k$ such that z is not in the neutral region. Hence $z \in \bigcup_{j\neq k} R_j$, i.e., $z \in R_j$ for some $j \neq k$. Because $z \in F(P_k,R_k)$ there exists $y \in R_k$ satisfying $d(z,y) < d(y,P_k)$. Because $(R_i)_{i\in K}$ is a territory diagram we have $y \in R_k \subseteq \text{dom}(P_k, \bigcup_{i\neq k} R_i)$. This and $z \in R_j$ imply that $d(y, P_k) \leq d(y, R_j) \leq d(y, z)$. Hence $d(z, y) < d(y, z)$, a contradiction which proves the assertion. $\square$

6. Concluding remarks

We finish this note with some remarks about possible future lines of investigation. First, we believe that it is possible to prove the existence of a neutral region in a context more general than Theorem 3.4, with some caution (because of the counterexamples), e.g., in the case where several sites intersect, but at the moment we do not have any result in this direction. It may be of interest to establish additional properties of the neutral region and to find applications to this concept. Investigating more the interpretation of a zone diagram as a certain equilibrium, including making some computer based simulations related to it and to the dynamical system it induces, may be of interest too. Finding other relations between the neutral region and other concepts and also suggesting interesting interpretations to the other concepts (double zone diagrams, territory and double territory diagrams, forbidden zones, and also new interpretations to zone diagrams) and investigating them can be interesting too. Introducing the neutral region in other settings and investigating this concept there might be interesting too, and in fact we do have something in this direction (concerning a neutral Voronoi region) which will be considered in a companion paper [24].

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