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NON-CONVEX PROXIMAL PAIRS ON HILBERT SPACES AND BEST PROXIMITY POINTS

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Abstract. A sufficient condition is given for a non-convex proximal pair to be a proximal parallel pair on Hilbert spaces. Let (A, B) be a nonempty weakly compact non-convex proximal parallel pair in a Hilbert space X over the real field and $T : A \cup B \to X$ be a relatively nonexpansive map. We prove that there exists $x \in A \cup B$ such that $\|x - Tx\| = dist(A, B)$ whenever $A \cup B$ is a cyclic $T-$regular set. We also establish that there exists $(x, y) \in A \times B$ such that $Tx = x$, $Ty = y$ and $\|x - y\| = dist(A, B)$, if $A \cup B$ is a $T-$regular set, $T(A) \subseteq A$ and $T(B) \subseteq B$. In the above cases, we prove that the Kransnoel'skiĭ's iteration process yields a convergence result under suitable assumption.

1. Introduction

Let A and B be nonempty weakly compact convex subsets of a Banach space X such that (A, B) is a proximal pair having proximal normal structure. Let $T : A \cup B \to A \cup B$ be a map satisfying:

$$\|Tx - Ty\| \leq \|x - y\|, \; x \in A \text{ and } y \in B \tag{1.1}$$

$$T(A) \subseteq B \text{ and } T(B) \subseteq A. \tag{1.2}$$

In [2] it is shown that there exists a point $x \in A \cup B$ such that $\|x - Tx\| = dist(A, B)$. Also, it is proved that [2, Theorem 2.2] if $T : A \cup B \to A \cup B$ satisfies the conditions:

$$\|Tx - Ty\| \leq \|x - y\|, \; x \in A \text{ and } y \in B$$

$$T(A) \subseteq A \text{ and } T(B) \subseteq B. \tag{1.3}$$

Then there exists $(x, y) \in A \times B$ such that $Tx = x$, $Ty = y$ and $\|x - y\| = dist(A, B)$.

If $A = B$, then the above problems boil down to the well known Browder-Göhde-Kirk fixed point theorem. Also, it is easy to see that the pair (A, A) has proximal normal structure if and only if A has normal structure. Thus in this case, there exists a point $x \in A$ such that $Tx = x$. A good account of Metric fixed point theory can be found in [1, 5].

It is quite easy to see that a nonempty non-convex set A, even in a Hilbert space need not have normal structure. But the Browder-Göhde-Kirk theorem depends on the normal structure. Here the following question arises. Is it possible to extend the above theorem to a non-convex weakly compact set?

In this direction, the notion of $T-$ regular sets is introduced in [6] and the following result is proved, if A is a nonempty weakly compact $T-$regular set in a

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uniformly convex Banach space and T is a nonexpansive map then $Tx = x$, for some $x \in A$.

Motivated by the above results, we introduce a notion of cyclic $T-$regular sets and establish the following result. Suppose (A, B) is a nonempty non-convex weakly compact proximal pair in a Hilbert space.

If $A \cup B$ is a cyclic $T-$regular set and T is a relatively nonexpansive map, then there exists a point $x \in A \cup B$ such that $\|x - Tx\| = dist(A, B)$.

Also it is proved that if $A \cup B$ is a $T-$regular set and T is a relatively nonexpansive map which satisfies the condition (1.3), then there exists $(x, y) \in A \times B$ such that $Tx = x$, $Ty = y$, and $\|x - y\| = dist(A, B)$.

We have observed some facts about nonempty weakly compact convex proximal pairs in Hilbert spaces which enable us to introduce the notion of non-convex proximal parallel pairs. We prove the aforesaid theorems for non-convex proximal parallel pairs.

Let X be a Banach space and A and B be nonempty subsets of X. We use the following notations:

$$\begin{aligned} r_x(B) &= \sup\{\|x - y\| : y \in B\}, x \in A; \\ \delta(A, B) &= \sup\{r_x(B) : x \in A\}; \\ \delta(A) &= \sup\{r_x(A) : x \in A\}; \\ dist(A, B) &= \inf\{\|x - y\| : x \in A, y \in B\}. \end{aligned}$$

In section 2 we introduce the notion of cyclic $T-$ regular sets and give definitions related to this work. We discuss some results related to the Chebyshev radius. In section 3 we prove a result about proximal pairs which enables us to extend the concept of proximal parallel pairs to a non-convex proximal pair satisfying some conditions. Also, we show that a relatively nonexpansive map defined on a non-convex proximal pair has a best proximity point. We establish the existence of fixed points of a relatively nonexpansive map T defined on a non-convex proximal pair (A, B), if $A \cup B$ is a $T-$regular set and T satisfies the condition (1. 3). Moreover, in the above cases, we prove that the Kransnoel'skiĭ's iteration process yields a convergence result under suitable assumption.

We prefer to use the term proximal pair, see [2, Definition 1.1], which is labelled as proximinal pair in [4].

2. Preliminaries

Definition 2.1 ([2, 4])**.** Let A and B be nonempty subsets of a Banach space X. The pair (A, B) is said to be a proximal pair if for each $(x, y) \in A \times B$ there exists $(x_1, y_1) \in A \times B$ such that $\|x - y_1\| = dist(A, B) = \|y - x_1\|$.

In addition, if for each $(x, y) \in A \times B$, $(x_1, y_1) \in A \times B$ is a unique point such that $\|x - y_1\| = dist(A, B) = \|y - x_1\|$, then we say (A, B) is a sharp proximal pair.

Definition 2.2 ([4])**.** A pair (A, B) of nonempty subsets in a Banach space X is said to be a proximal parallel pair if

(i) (A, B) is a sharp proximal pair.

(ii) There exists a unique $h \in X$ such that $B = A + h$.

Remark 2.3. Let (A, B) be a nonempty convex proximal pair in a Banach space X. Let $x_0 \in A$ and $x'_0 \in B$ be such that $\|x_0 - x'_0\| = dist(A, B)$. In [4] it is shown that if X is a strictly convex Banach space, then $B = A + h$, where $h = x'_0 - x_0$. Further, if X is a Hilbert space, then it is quite easy to see that for every $x, y \in A$ or B, $x - y$ is orthogonal to h. That is $A - A (= B - B)$ is orthogonal to h.

Definition 2.4 ([2])**.** Let A and B be nonempty subsets of a Banach space X. A mapping $T : A \cup B \to X$ is said to be relatively nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for $x \in A$ and $y \in B$.

Definition 2.5 ([6])**.** Let A and B be nonempty subsets of a Banach space X. Let T be a self map on $A \cup B$ such that $T(A) \subseteq A$ and $T(B) \subseteq B$. The set $A \cup B$ is said to be $T-$regular if $\frac{x+Tx}{2} \in A$ for every $x \in A$ and $\frac{y+Ty}{2} \in B$ for every $y \in B$.

Remark 2.6. If we assume A and B are nonempty convex subsets in the above definition, then it is clear that $A \cup B$ is a $T-$regular.

We introduce the following concept.

Definition 2.7. Let A and B be nonempty subsets of a Banach space X and let $T : A \cup B \to A \cup B$ be a mapping. The set $A \cup B$ is said to be cyclic $T-$regular if

(1) $T(A) \subseteq B$ and $T(B) \subseteq A$.
(2) $\frac{x+Tx'}{2} \in A$, for every $x \in A$, where $x' \in B$ is such that $\|x - x'\| = dist(x, B)$.
(3) $\frac{y+Ty'}{2} \in B$, for every $y \in B$, where $y' \in A$ is such that $\|y - y'\| = dist(y, A)$.

Example 2.8. In the Euclidean space $\mathbb{R}^2$, let $A = \{(0,0), (0, \frac{1}{2}), (0,1)\}$ and $B = A + (1,0)$. Define $T : A \cup B \to A \cup B$ as follows, for $x \in A$, $T(x) = \begin{cases} (1,1) & \text{if } x = (0,0), \\ (1,0) & \text{if } x = (0,1), \\ (1, \frac{1}{2}) & \text{otherwise,} \end{cases}$

and for $x \in B$, $T(x) = \begin{cases} (0,1) & \text{if } x = (1,0), \\ (0,0) & \text{if } x = (1,1), \\ (0, \frac{1}{2}) & \text{otherwise.} \end{cases}$

Then T is a cyclic $T-$regular.

Remark 2.9. Let (A, B) be a nonempty convex proximal pair in a Banach space and $T : A \cup B \to A \cup B$ be a map satisfying: $T(A) \subseteq B$ and $T(B) \subseteq A$. Then $A \cup B$ is a cyclic $T-$regular.

The following fact is used in the proof of our main results.

Proposition 2.10 ([7])**.** *Let (A, B) be a bounded convex proximal parallel pair in a Hilbert space X over $\mathbb{R}$. Then for every $x \in A$,*

$$r_x(B) \;=\; r_{x+h}(A) = \sqrt{\|h\|^2 + (r_x(A))^2}.$$

Remark 2.11. Let X be a normed linear space, K be a nonempty bounded subset of X and $F = \overline{co}(K)$. Then for $x \in X$, $r_x(K) = r_x(F)$.

Proof. It suffices to show $r_x(F) \leq r_x(K)$. Suppose $y \in co(K)$, then $y = \Sigma_{i=1}^{l} \alpha_i x_i$, for $i = 1$ to l, $x_i \in K$, $\alpha_i \geq 0$ and $\Sigma_{i=1}^{l} \alpha_1 = 1$. Then $\|x - y\| \leq r_x(K)$. Hence $\|x - y_0\| \leq r_x(K)$, for all $y_0 \in F$. Thus $r_x(F) \leq r_x(K)$. □

The following result from [3] is used in the sequel.

Lemma 2.12 ([3])**.** *Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:*

(1) $\|x_n - y_n\|$ *converges to* $\text{dist}(A, B)$,
(2) $\|z_n - y_n\|$ *converges to* $\text{dist}(A, B)$.

Then $\|x_n - z_n\|$ converges to zero.

3. Main results

We throughout assume that X is a Hilbert space over $\mathbb{R}$.

Proposition 3.1. *Let (A, B) be a nonempty weakly compact convex proximal pair in a Hilbert space X. Then there exists a smallest closed subspace X_0 of X which satisfies the following:*

(1) $A \subset x + X_0$ *and* $B \subset x' + X_0$, *for every* $x \in A$, *where* $x' \in B$ *is such that* $\|x - x'\| = \text{dist}(A, B)$.
(2) $\text{dist}(x + X_0, x' + X_0) = \text{dist}(A, B)$.
(3) $(x + X_0, x' + X_0)$ *is a proximal pair.*

Proof. Suppose (A, B) is a proximal pair in a Hilbert space X over $\mathbb{R}$. Then there exists $h \in X$ such that $B = A + h$ and h is orthogonal to both $A - A$ and $B - B$. Note that $B - B = A - A$.

Let $X_0 = \overline{span}(A - A)$. Then X_0 is a closed subspace of X. It is easy to see that for every $x \in A$, $A \subset x + X_0$ and $B \subset x + h + X_0$. Fix $x \in A$. Let $H_1 = x + X_0$ and $H_2 = x + h + X_0$. Clearly $H_1 \cap H_2 = \emptyset$. For $z \in H_1 \cap H_2$. Then $\exists\ y_1, y_2 \in X_0$ such that $z = x + y_1$ and $z = x + h + y_2$. Then $x + y_1 = x + h + y_2$. This implies that $y_1 - y_2 = h \in X_0$. But h is orthogonal to X_0.

Now it is claimed that $dist(H_1, H_2) = dist(A, B)$.

$$\begin{aligned} dist(H_1, H_2) &= \inf\{\|x + y - (x + h + z)\| : y, z \in X_0\} \\ &= \inf\{\|y - h - z\| : y, z \in X_0\} \\ &= \inf\{\sqrt{\|y - z\|^2 + \|h\|^2} : y, z \in X_0\} \\ &= \|h\| \end{aligned}$$

Also it is clear that (H_1, H_2) is a proximal pair.

Suppose Y is another closed subspace of X satisfying the conclusions. Then $A - A \subset Y + Y = Y$ implies that $X_0 \subseteq Y$. □

Example 3.2. Consider the Hilbert space l_2. Let $A = \{e_n, 0 : n \geq 2\}$ and $B = \{e_{2n-1} + e_1, e_{2n} + e_2, e_1, e_2 : n \geq 2\}$. Then, it is easy to see that A and B are weakly compact subsets of l_2, and (A, B) is a proximal pair, but there is no closed subspace of l_2 satisfying the conclusion of the Proposition 3.1.

The previous example illustrates the fact that a non-convex proximal pair even in a Hilbert space need not be a proximal parallel pair. In the light of Proposition 3.1 we obtain a sufficient condition for a non-convex proximal pair to be a proximal parallel pair. The following result states that a non-convex proximal pair satisfying some conditions should be a proximal parallel pair. That is these proximal pairs possess all the properties satisfied by the convex proximal parallel pairs.

Proposition 3.3. *Let A and B be nonempty bounded subsets of a Hilbert space X and (A, B) be a proximal pair. Suppose there exists a closed subspace Y and points $x, y \in X$ such that $A \subset x + Y$, $B \subset y + Y$ and* $\operatorname{dist}(x + Y, y + Y) = \operatorname{dist}(A, B)$. *Then*

(1) *The pair $(K_1, K_2) = (\overline{co}(A), \overline{co}(B))$ is a proximal pair.*
(2) *For every $(x, y) \in A \times B$, there exists a unique $(x', y') \in A \times B$ such that $\|x - y'\| = \operatorname{dist}(A, B) = \|y - x'\|$.*
(3) *$B = A + h$, where $h \in X$ is such that $K_2 = K_1 + h$ and h is orthogonal to both $A - A$ and $B - B$.*

Then the pair (A, B) is said to be a non-convex proximal parallel pair.

Proof. Suppose (A, B) is a proximal pair in X, let $d := dist(A, B)$. Suppose there exists a closed subspace Y of X and points $x, y \in X$ satisfying:

(1) $A \subset x + Y$ and $B \subset y + Y$.
(2) $dist(x + Y, y + Y) = dist(A, B) = d$.

Clearly $K_1 := \overline{co}(A) \subseteq x + Y$ and $K_2 := \overline{co}(B) \subseteq y + Y$ and $dist(K_1, K_2) = d$.

Now let $F_1 := \{x \in K_1 : \exists\, y \in K_2 \text{ such that } \|x - y\| = d\}$ and $F_2 := \{y \in K_2 : \exists\, x \in K_1 \text{ such that } \|x - y\| = d\}$. It is clear that F_1 and F_2 are non empty weakly compact convex subsets of X satisfying:

(1) $(A, B) \subseteq (F_1, F_2)$
(2) (F_1, F_2) is a proximal pair and $dist(F_1, F_2) = d$.

Since $(A, B) \subseteq (F_1, F_2)$ and (F_1, F_2) is a weakly compact convex pair, hence $(K_1, K_2) \subseteq (F_1, F_2)$. Thus (K_1, K_2) is a proximal pair in X. Therefore there exists $h \in X$, which satisfies $K_2 = K_1 + h$ and the sets $K_1 - K_1$ and $K_2 - K_2$ are orthogonal to h.

Hence the proximal pair $(A, B) \subset (K_1, K_2)$ inherits the properties of the proximal pair (K_1, K_2). □

Remark 3.4. Let (A, B) be a non-convex proximal parallel pair in a Hilbert space X. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:

(1) $\|x_n - y_n\|$ converges to $dist(A, B)$,
(2) $\|z_n - y_n\|$ converges to $dist(A, B)$.

Then $\|x_n - z_n\|$ converges to zero.

Proof. Let $K_1 = \overline{co}(A)$ and $K_2 = \overline{co}(B)$. Then by the Proposition 3.3 (K_1, K_2) is a weakly compact convex proximal pair in X, and $dist(K_1, K_2) = dist(A, B)$. Now, $\{x_n\}$ and $\{z_n\}$ are sequences in K_1 and $\{y_n\}$ is a sequence in K_2 satisfying:

(1) $\|x_n - y_n\|$ converges to $dist(K_1, K_2)$,
(2) $\|z_n - y_n\|$ converges to $dist(K_1, K_2)$.

Hence by Lemma 2.12, $\|x_n - z_n\|$ converges to zero. □

The following result claims that Proposition 2.10 holds for non-convex proximal pairs.

Proposition 3.5. *Let (A, B) be a nonempty bounded proximal parallel pair in a Hilbert space X. Then for every $x \in A$,*

$$r_x(B) \;=\; r_{x+h}(A) = \sqrt{\|h\|^2 + (r_x(A))^2}.$$

Proof. For $x, y \in A$, as $(x-y) \perp h$, $\|x-(y+h)\|^2 = \|h\|^2 + \|x-y\|^2 = \|x+h-y\|^2$. Hence

$$\begin{aligned} r_x(B) &= \sup\{\|x-(y+h)\| : y \in A\} \\ &= \sqrt{\sup\{\|h\|^2 + \|x-y\|^2 : y \in A\}} \\ &= \sqrt{\|h\|^2 + (r_x(A))^2} \end{aligned}$$

Similarly $r_{x+h}(A) = \sqrt{\sup\{\|x+h-y\|^2 : y \in A\}} = \sqrt{\|h\|^2 + (r_x(A))^2}$. □

Remark 3.6. Suppose (K_1, K_2) is a nonempty weakly compact non-convex proximal parallel pair in a Hilbert space X. Then any proximal pair $(A, B) \subseteq (K_1, K_2)$ with $dist(A, B) = dist(K_1, K_2)$ inherits the properties of (K_1, K_2).

We hereafter assume that A and B are nonempty weakly compact subsets of X.

Lemma 3.7. *Let (A, B) be a non-convex proximal parallel pair in a Hilbert space X and let $T : A \cup B \to X$ be a relatively non-expansive mapping. Suppose $A \cup B$ is a cyclic $T-$regular set and the pair (A, B) does not contain any proper proximal pair which is cyclic $T-$regular. Then $(A, B) \subseteq (\overline{co}(T(B)), \overline{co}(T(A)))$.*

Proof. Suppose (A, B) is a non-convex proximal parallel pair in a Hilbert space X. Then there exists $h \in X$ such that $B = A + h$ and $A - A$ is orthogonal to h.

Let $d := dist(A, B)$, $K_1 := \overline{co}(T(B)) \cap A$ and $K_2 := \overline{co}(T(A)) \cap B$. Then K_1 and K_2 are weakly compact subsets of A and B, respectively.

Clearly $dist(K_1, K_2) \geq d$. Let $(x, x') \in A \times B$. Then $(Tx', Tx) \in T(B) \times T(A) \subseteq A \times B$ implies that $(Tx', Tx) \in K_1 \times K_2$. But if $\|x - x'\| = d$, then $\|Tx - Tx'\| \leq d$ and hence $dist(K_1, K_2) = d$.

It is claimed that (K_1, K_2) is a proximal pair. It suffices to prove that for every $x \in co(T(B)) \cap A$, there exits $y \in co(T(A)) \cap B$ such that $\|x - y\| = d$.

Let $x \in co(T(B)) \cap A$. Then $x \in A$ and $x = \Sigma_{i=1}^{n} \alpha_i T(y_i)$ where $y_i \in B$ and $\alpha_i \geq 0, \Sigma_{i=1}^{n} \alpha_i = 1$.

Now for $i = 1$ to n, there exists $y'_i \in A$ such that $\|y_i - y'_i\| = d$. Then $z = \Sigma_{i=1}^{n} \alpha_i T(y'_i) \in co(T(A))$ and $\|x - z\| = d$. But (A, B) is a proximal parallel pair. Thus $z \in co(T(A)) \cap B$. Hence $(K_1, K_2) \subseteq (A, B)$, is a proximal pair and clearly $(T(K_2), T(K_1)) \subseteq (K_1, K_2)$. This establishes the fact that $(K_1, K_2) = (A, B)$. That is $(A, B) \subseteq (\overline{co}(T(B)), \overline{co}(T(A)))$. □

The following result can be proved in a similar manner.

Lemma 3.8. *Let (A, B) be a non-convex proximal parallel pair in a Hilbert space X and let $T : A \cup B \to X$ be a relatively nonexpansive map satisfying $T(A) \subseteq A$ and $T(B) \subseteq B$. Further suppose $A \cup B$ is a $T-$regular set and the pair (A, B) does not contain any proper proximal pair which is $T-$regular. Then $(A, B) \subseteq (\overline{co}(T(A)), \overline{co}(T(B)))$.*

The next theorem establishes the fact that a relatively nonexpansive map defined on a non-convex proximal parallel pair has a best proximity point.

Theorem 3.9. *Let (A, B) be a non-convex proximal parallel pair in a Hilbert space X and let $T : A \cup B \to X$ be a relatively nonexpansive mapping. Suppose $A \cup B$ is a cyclic $T-$regular set. Then T has a best proximity point in $A \cup B$.*

Proof. Let $\mathfrak{F}$ be the set of all nonempty weakly closed subsets (K_1, K_2) of (A, B) satisfying: (i) (K_1, K_2) is a proximal pair and $dist(K_1, K_2) = d$, where $d = dist(A, B)$. (ii) $K_1 \cup K_2$ is cyclic $T-$regular.

Define a relation $\leq$ on $\mathfrak{F}$ as follows $(K_1, K_2) \leq (F_1, F_2)$ iff $(F_1, F_2) \subseteq (K_1, K_2)$. Then $\mathfrak{F}$ is a partially ordered set.

Suppose $\mathcal{T} \subseteq \mathfrak{F}$ is a totally ordered set. Then clearly $\mathcal{T}$ has finite intersection property and hence $(F_1, F_2) := \bigcap_{(K_1,K_2)\in\mathcal{T}}(K_1, K_2)$ is a nonempty weakly compact proximal pair. Also $(F_1, F_2) \in \mathfrak{F}$. By Zorn's lemma $\mathfrak{F}$ has a maximal element, say (K_1, K_2). As (K_1, K_2) is a proximal pair with $dist(K_1, K_2) = d$, by Proposition 3.3, (K_1, K_2) is a proximal parallel pair and $K_1 - K_1 (= K_2 - K_2)$ is orthogonal to h, where $h \in X$ such that $B = A + h$.

Now from the Lemma 3.7, $(K_1, K_2) \subseteq (\overline{co}(T(K_2)), \overline{co}(T(K_1)))$.

It is claimed that either $\delta(K_1, K_2) = dist(K_1, K_2)$ or there exists a point $x \in K_1 \cup K_2$ such that $\|Tx - x\| = dist(K_1, K_2)$.

Suppose neither of them are true. Then $\delta(K_1, K_2) > dist(K_1, K_2)$ and for every $x \in K_1 \cup K_2$, $\|x - Tx\| > d$.

Fix $x_0 \in K_1$. Since (K_1, K_2) is a proximal parallel pair in a Hilbert space X, hence by Proposition 3.5, $r_{x_0}(K_2) = r_{x_0+h}(K_1) = \sqrt{\|h\|^2 + r_{x_0}(K_1)^2} \leq \delta(K_1, K_2)$. Also $r_{T(x_0+h)}(K_2) \leq \delta(K_1, K_2)$. Now by the uniform convexity of X, $r_m(K_2) = \alpha\delta(K_1, K_2)$, for some $\alpha \in (0, 1)$, where $m = \frac{x_0+T(x_0+h)}{2} \in K_1$.

Let $R := (\frac{\alpha+1}{2})\delta(K_1, K_2)$. Define $M_1 := \{x \in K_1 : r_x(K_2) \leq R\}$ and $M_2 := \{y \in K_2 : r_y(K_1) \leq R\}$. Then $(m, m + h) \in M_1 \times M_2$ and by Proposition 3.5, (M_1, M_2) is a proximal pair. Also it is easy to see that $(M_1, M_2) \in \mathfrak{F}$. For $x \in M_1$, since $(K_1, K_2) \subseteq (\overline{co}(T(K_2)), \overline{co}(T(K_1)))$

$$\begin{aligned} r_{Tx}(K_1) &= \sup\{\|Tx - z\| : z \in K_1\} \\ &\leq \sup\{\|x - y\| : y \in K_2\} = r_x(K_2) \leq R \end{aligned}$$

Thus $(T(M_2), T(M_1)) \subseteq (M_1, M_2)$ and this also implies that $M_1 \cup M_2$ is a cyclic $T-$regular set. Therefore $(M_1, M_2) \in \mathfrak{F}$ and hence $(K_1, K_2) = (M_1, M_2)$. This forces that $\alpha = 1$.

Hence either $\delta(K_1, K_2) = dist(K_1, K_2)$ or there exists a point $x \in K_1 \cup K_2$ such that $\|Tx - x\| = dist(K_1, K_2)$. □

In view of Remark 2.9 we get the following result which is Theorem 2.1 in [2].

Corollary 3.10. *Let (A, B) be a weakly compact convex proximal pair in a Hilbert space X and let $T : A \cup B \to A \cup B$ be a relatively nonexpansive map which satisfies $T(A) \subseteq B$ and $T(B) \subseteq A$. Then T has a best proximity point in $A \cup B$.*

The following example illustrates the above theorem.

Example 3.11. Consider the Hilbert space l_2. Let $A = \{0, e_n, \frac{e_n+e_{n+1}}{2} : n \geq 2\}$ and $B = A + e_1$. Then A and B are weakly compact subsets of l_2. Also (A, B) is a proximal parallel pair.

Define $T : A\cup B \to A\cup B$ as follows: for $x \in A$, $T(x) = \begin{cases} e_{n+1}+e_1 & \text{if } x = e_n, \\ x + e_1 & \text{otherwise,} \end{cases}$

and for $y \in B$, $T(y) = \begin{cases} e_{n+1} & \text{if } y = e_n + e_1, \\ y - e_1 & \text{otherwise.} \end{cases}$

Then $A \cup B$ is a cyclic $T-$regular set and T is a relatively nonexpansive map on $A \cup B$. Hence by the Theorem 3.9, T has a best proximity point in $A \cup B$. Note that 0 is a best proximity point of T.

The next result shows that the Kransnoel'skiĭ's iteration process yields a convergence result, if the pair (A, B) and T are as in Theorem 3.9. We adopt the proof techniques from [2].

Theorem 3.12. *Let (A, B) be a nonempty weakly compact proximal parallel pair in a Hilbert space X. Let $T : A \cup B \to X$ be a relatively nonexpansive map such that $A\cup B$ is a cyclic $T-$regular set. Let $(x_0, y_0) \in A \times B$ be such that $\|x_0 - y_0\| =$ dist(A, B). Define $x_n = \frac{x_{n-1}+T(x'_{n-1})}{2}$ and $y_n = \frac{y_{n-1}+T(y'_{n-1})}{2}$, for $n \in \mathbb{N}$. Then $\|x_n - T(x'_n)\|$ and $\|y_n - T(y'_n)\|$ converge to zero, where x' denotes the unique best approximant to $x \in A \cup B$.*

If $\overline{T(B)}$ is a compact set, then $\{x_n\}$ converges to a and $\{y_n\}$ converges to a', where $a \in A$ is such that $\|a - Ta\| = \text{dist}(A, B)$.

Proof. Consider the sequences $\{x_n\} \subseteq A$ and $\{y_n\} \subseteq B$. By Proposition 3.3, there exists $h \in X$ such that $B = A + h$ and h is orthogonal to both $A - A$ and $B - B$.

It is enough to prove that $\|x_n - T(x'_n)\|$ converges to zero.

By Theorem 3.9, there exists $z \in B$ such that $\|z-Tz\| = d$, where $d = dist(A, B)$. Note that $Tz = z'$ is also a best proximity point of T. Now

$$\begin{aligned} \|x_n - z\| &\leq \frac{1}{2}\{\|x_{n-1} + T(x'_{n-1}) - z - T(z')\|\} \longrightarrow (*) \\ &\leq \frac{1}{2}\{\|x_{n-1} - z\| + \|x'_{n-1} - z'\|\} \end{aligned}$$

But for all n, $\|x_n - z\| = \|x'_n - z'\|$. Hence $\{\|x_n - z\|\}$ is a non-increasing sequence. Let $r = \lim \|x_n - z\|$.

As $\|T(x'_n)-z\| \leq \|x'_n-z'\|$, hence $\liminf \|T(x'_n)-z\| \leq r$, $\limsup \|T(x'_n)-z\| \leq r$. Now from eqn.$(*)$, $\liminf \|T(x'_n) - z\| \geq r$. Hence $\lim \|T(x'_n) - z\| = r$.

Suppose there exists $\epsilon_0 > 0$ and $\{n_k\} \subseteq \mathbb{N}$ such that $\|x_{n_k} - T(x'_{n_k})\| \geq \epsilon_0$.

Choose $\gamma \in (0, 1)$ and ϵ such that $\epsilon_0/\gamma > r$ and $0 < \epsilon < min\{\frac{\epsilon_0}{\gamma} - r, \frac{r\delta(\gamma)}{1-\delta(\gamma)}\}$. As the modulus of convexity function $\delta(.)$ is strictly increasing in the Hilbert space X, $0 < \delta(\gamma) < \delta(\frac{\epsilon_0}{r+\epsilon})$. Also from the choice of ϵ, we have $[1 - \delta(\frac{\epsilon_0}{r+\epsilon})](r + \epsilon) < r$.

As $\|x_n - z\|$ and $\|T(x'_n) - z\|$ converges to r, choose $N \in \mathbb{N}$ such that $\|x_n - z\|$, $\|T(x'_n) - z\| \leq r + \epsilon$, for $n \geq N$. Now for $n_k \geq N$,

$$\begin{aligned} \|z - x_{n_k+1}\| &= \|z - \frac{x_{n_k} + T(x'_{n_k})}{2}\| \\ &\leq (1 - \delta(\frac{\epsilon_0}{r + \epsilon}))(r + \epsilon) \end{aligned}$$

This gives a contradiction. Hence $\|x_n - T(x'_n)\| \to 0$.

Suppose $\overline{T(B)}$ is a compact set. Then $\{T(x'_n)\}$ has a subsequence $\{T(x'_{n_k})\}$ which converges to $a \in \overline{T(B)}$. Hence x_{n_k} converges to a, thus $\lim \|x_{n_k} - a'\| = d$ and $\lim \|T(x'_{n_k}) - a'\| = d$.

Now,

$$d \leq \|T(x'_{n_k}) - Ta\| \leq \|x'_{n_k} - a\| = \|x_{n_k} - a'\|.$$

This implies that $\lim \|T(x'_{n_k}) - Ta\| = d$, and hence by Remark 3.4 $Ta = a'$. Since $\{\|x_n - a'\|\}$ is non-increasing, $\lim \|x_n - a'\| = d$. Hence by Remark 3.4, x_n converges to a. Now, note that

$$d \leq \|x_1 - y_1\| \leq \tfrac{1}{2}\|x_0 + T(x'_0) - y_0 - T(y'_0)\| = d$$

By induction hypothesis $\|x_n - y_n\| = d$, for all $n \in \mathbb{N}$. Thus $\lim \|x_n - y_n\| = d$. Since $\lim \|x_n - a'\| = d$, hence from Remark 3.4 $\lim \|y_n - a'\| = 0$. □

The following theorem proves that a relatively nonexpansive map T defined on $A \cup B$ has fixed points in A and B.

Theorem 3.13. *Let (A, B) be a non-convex proximal parallel pair in a Hilbert space X. Let $T : A \cup B \to X$ be a relatively nonexpansive map satisfying $T(A) \subseteq A$ and $T(B) \subseteq B$. Further suppose $A \cup B$ is a $T-$regular set. Then there exists $(x, y) \in A \times B$ such that $Tx = x$, $Ty = y$ and $\|x - y\| = \text{dist}(A, B)$.*

Proof. Let $\mathfrak{F}$ be the set of all nonempty weakly closed subsets (K_1, K_2) of (A, B) satisfying: (i) (K_1, K_2) is a proximal pair and $dist(K_1, K_2) = d$, where $d = dist(A, B)$. (ii) $K_1 \cup K_2$ is $T-$regular and $T(K_i) \subseteq K_i$, $i = 1, 2$.

Define a relation $\leq$ on $\mathfrak{F}$ as follows $(K_1, K_2) \leq (F_1, F_2)$ iff $(F_1, F_2) \subseteq (K_1, K_2)$. Then $\mathfrak{F}$ is a partially ordered set. It is easy to see that every totally ordered subset $\mathcal{T}$ of $\mathfrak{F}$ has an upper bound. Hence by Zorn's lemma $\mathfrak{F}$ has a maximal element say (K_1, K_2).

As (K_1, K_2) is a proximal pair with $dist(K_1, K_2) = d$, by Proposition 3.6, (K_1, K_2) is a proximal parallel pair and $K_1 - K_1 (= K_2 - K_2)$ is orthogonal to h, where $h \in X$ such that $B = A + h$.

Now from the Lemma 3.8, $(K_1, K_2) \subseteq (\overline{co}(TK_1), \overline{co}(TK_2))$. It is claimed that K_1 and K_2 are singleton sets, that is $\delta(K_1, K_2) = dist(K_1, K_2)$.

Suppose $\delta(K_1, K_2) > dist(K_1, K_2)$. As K_1 and K_2 are weakly compact sets, there exists $x_0 \in K_1$ such that $r_{x_0}(K_2) = \inf_{x \in K_1} r_x(K_2) < \delta(K_1, K_2)$. Let $r_{x_0}(K_2) = \alpha\delta(K_1, K_2)$, $\alpha \in (0, 1)$ and $R := (\frac{\alpha+1}{2})\delta(K_1, K_2)$.

Define $M_1 := \{x \in K_1 : r_x(K_2) \leq R\}$ and $M_2 := \{y \in K_2 : r_y(K_1) \leq R\}$. Then $(x_0, x_0 + h) \in M_1 \times M_2$ and by Proposition 3.5, (M_1, M_2) is a proximal pair. It is claimed that $(M_1, M_2) \in \mathfrak{F}$. Let $x \in M_1$. Since $(K_1, K_2) \subseteq (\overline{co}(T(K_1)), \overline{co}(T(K_2)))$

$$\begin{aligned} r_{Tx}(K_2) &= \sup\{\|Tx - z\| : z \in K_2\} \\ &\leq \sup\{\|x - y\| : y \in K_2\} = r_x(K_2) \leq R \end{aligned}$$

Thus $(T(M_1), T(M_2)) \subseteq (M_1, M_2)$ and it is clear that $M_1 \cup M_2$ is a $T-$regular set. Hence $(M_1, M_2) \in \mathfrak{F}$.

Now the maximality of (K_1, K_2) implies that $(K_1, K_2) = (M_1, M_2)$. This forces that $\alpha = 1$, and hence $\delta(K_1, K_2) = dist(K_1, K_2)$. Thus K_1 and K_2 are singleton sets and $Tx = x$, for every $x \in K_1 \cup K_2$. □

The next result shows that the Kransnoel'skiĭ's iteration process yields a convergence result, if the pair (A, B) and T are as in Theorem 3.13.

Theorem 3.14. *Let (A, B) be a nonempty weakly compact proximal parallel pair in a Hilbert space X. Suppose $T : A \cup B \to A \cup B$ is a relatively nonexpansive map satisfying $T(A) \subseteq A$ and $T(B) \subseteq B$ and $A \cup B$ is a $T-$regular set. Let $(x_0, y_0) \in A \times B$ be such that $\|x_0 - y_0\| = \operatorname{dist}(A, B)$. Define $x_{n+1} = \frac{x_n + Tx_n}{2}$ and $y_{n+1} = \frac{y_n + Ty_n}{2}$, for $n = 0, 1, 2, \ldots$. Then $\|x_n - Tx_n\|$ and $\|y_n - Ty_n\|$ converge to zero.*

If $\overline{T(A)}$ is a compact set, then $\{x_n\}$ converges to a and $\{y_n\}$ converges to b, where $a \in A$ and $b \in B$ are fixed points of T such that $\|a - b\| = \operatorname{dist}(A, B)$.

Proof. A similar proof can be given as that of Theorem 3.12. □

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