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BASIC PROBLEMS OF THE METRIC FIXED POINT THEORY AND THE RELEVANCE OF A METRIC FIXED POINT THEOREM FOR A MULTIVALUED OPERATOR

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Dedicated to Simeon Reich on the occasion of his 65th anniversary

Abstract. In this paper, we will present some basic problems of the metric fixed point theory for multivalued operators. Following [A. Petruşel, I.A. Rus: The theory of a metric fixed point theorem for multivalued operators, Fixed Point Theory and its Applications, (L.J. Lin et al.-Eds.), Proc. 9th Internat. Conf. on Fixed Point Theory and its Applications, Yokohama Publ. 2010, 161-175], we define the relevance of a metrical fixed point theorem for a multivalued operator by the impact of this theorem on the basic problems of the theory. New results and some examples are also given.

1. Introduction

In the mathematical literature, there are various types of metrical fixed point theorems for multivalued operators (see for example the following books: I.A. Rus (1979) *Metrical Fixed Point Theorems*, Babeş-Bolyai University, Cluj-Napoca; K. Goebel and W.A. Kirk (1990): *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press; J.M. Ayerbe Toledano, T. Dominguez Benavides, G. López Acedo (1997): *Measures of Noncompactness in Metric Fixed Point Theory*, Birkhäuser, Basel; W.A. Kirk and B. Sims (Eds.) (2001): *Handbook of Metric Fixed Point Theory*, Kluwer Acad. Publ., Dordrecht; I.A. Rus, A. Petruşel, G. Petruşel (2008): *Fixed Point Theory*, Cluj University Press, Cluj-Napoca), as well as numerous articles such as: [16, 24, 25, 28, 41, 65, 97, 112, 127]), [1–3, 5, 11, 22, 31, 34, 42, 49, 57–59, 62, 65, 67, 68, 80, 81, 83, 94, 96, 105, 112, 118, 122, 128], ...

In this paper, we will present some basic problems of the metric fixed point theory for multivalued operators. Following [A. Petruşel, I.A. Rus: *The theory of a metric fixed point theorem for multivalued operators*, Fixed Point Theory and its Applications, (L.J. Lin et al.-Eds.) Yokohama Publ. 2010, 161-175] we define the relevance of a metrical fixed point theorem for a multivalued operator by the impact of this theorem on the basic problems of the theory. New results and some examples are also given.

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Key words and phrases. Metric space, generalized metric space, multivalued generalized contraction, Picard operator, fixed point, strict fixed point, multivalued weakly Picard operator, multivalued Picard operator, data dependence, well-posedness, limit shadowing, Ulam-Hyers stability, fractal operator, open problems.

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2. Preliminaries

Throughout this paper, the standard notations and terminologies in nonlinear analysis are used, see for example W.A. Kirk, B. Sims [53], I.A. Rus, A. Petruşel, G. Petruşel [111], A. Petruşel [76], I.A. Rus, A. Petruşel, A. Sîntămărian [112].

Let X be a nonempty set. Then we denote

$$\mathcal{P}(X) := \{Y \mid Y \text{ is a subset of } X\},\ P(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is nonempty}\}.$$

Let (X, d) be a metric space. Then we denote

$$P_b(X) := \{Y \in P(X) \mid Y \text{ is bounded }\},\ P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed}\},$$

$$P_{cp}(X) := \{Y \in P(X) \mid Y \text{ is compact}\},\ P_{op}(X) := \{Y \in P(X) \mid Y \text{ is open }\}.$$

Let $T : X \to P(X)$ be a multivalued operator. Then, the operator $\hat{T} : P(X) \to P(X)$ defined by

$$\hat{T}(Y) := \bigcup_{x \in Y} T(x),\ \text{for } Y \in P(X)$$

is called the fractal operator generated by T. It is known that if (X, d) is a metric spaces and $T : X \to P_{cp}(X)$, then the following conclusions hold:

(a) if T is upper semicontinuous, then $T(Y) \in P_{cp}(X)$, for every $Y \in P_{cp}(X)$;

(b) the continuity of T implies the continuity of $\hat{T} : P_{cp}(X) \to P_{cp}(X)$.

The set of all nonempty invariant subsets of T is denoted by $I(T)$, i.e.,

$$I(T) := \{Y \in P(X) | T(Y) \subset Y\}.$$

A sequence of successive approximations of T starting from $x \in X$ is a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in X with $x_0 = x,\ x_{n+1} \in T(x_n)$, for $n \in \mathbb{N}$.

If $T : Y \subseteq X \to P(X)$ then $F_T := \{x \in Y \mid x \in T(x)\}$ denotes the fixed point set of T, while $(SF)_T := \{x \in Y \mid \{x\} = T(x)\}$ is the strict fixed point set of T. By

$$Graph(T) := \{(x, y) \in Y \times X\ :\ y \in T(x)\}$$

we denote the graphic of the multivalued operator T.

If $T : X \to P(X)$, then $T^0 := 1_X,\ T^1 := T, \ldots, T^{n+1} = T \circ T^n,\ n \in \mathbb{N}$ denote the iterate operators of T.

By definition (see [66]), a periodic point for a multivalued operator $T : X \to P_{cp}(X)$ is an element $p \in X$ such that $p \in F_{T^m}$, for some integer $m \geq 1$, i.e., $p \in \hat{T}^m(\{p\})$ for some integer $m \geq 1$. In the same setting, a strict periodic point for T is an element $p \in X$ such that $p \in (SF)_{T^m}$, for some integer $m \geq 1$, i.e., $\{p\} = \hat{T}^m(\{p\})$ for some integer $m \geq 1$.

If $T : X \to P_{b,cl}(X)$, then $p \in X$ is a periodic point for T provided that there are finitely many elements $p_0 = p, p_1, \ldots, p_m$ in X such that $p_i \in T(p_{i-1})$, for each $i \in \{1, 2, \ldots, m\}$ and $p \in T(p_m)$.

Let (X, d) be a metric space. The following (generalized) functionals are used in the main sections of the paper.

The gap functional generated by d

$$(1)\quad D_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$D_d(A,B) = \begin{cases} \inf\{d(a,b)|\ a \in A,\ b \in B\}, & A \neq \emptyset \neq B \\ 0, & A = \emptyset = B \\ +\infty, & \text{otherwise} \end{cases}$$

The diameter generalized functional generated by d

$$(2)\ \ \delta_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\},$$

$$\delta_d(A,B) = \begin{cases} \sup\{d(a,b)|\ a \in A,\ b \in B\}, & A \neq \emptyset \neq B \\ 0, & \text{otherwise} \end{cases}$$

In particular, we denote $\delta_d(A) := \delta_d(A,A)$.

The excess generalized functional generated by d

$$(3)\ \ \rho_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$\rho_d(A,B) = \begin{cases} \sup\{D_d(a,B)|\ a \in A\}, & A \neq \emptyset \neq B \\ 0, & A = \emptyset \\ +\infty, & B = \emptyset \neq A \end{cases}$$

The Pompeiu-Hausdorff generalized functional generated by d

$$(4)\ \ H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$H_d(A,B) = \begin{cases} \max\{\rho_d(A,B), \rho_d(B,A)\}, & A \neq \emptyset \neq B \\ 0, & A = \emptyset = B \\ +\infty, & \text{othewise} \end{cases}$$

We will avoid the subscript d in the above notations when no confusion is possible.

For other details and basic results concerning the above notions see, for example, [15, 17, 45, 46, 48, 53, 66, 77, 111, 119, 128], ...

For basic notions and results on the theory of weakly Picard and Picard operators see [76, 82, 84, 87, 103, 111, 112].

Finally, a few words on comparison functions, concept which appears in some metrical conditions on multivalued operators.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. By definition (see [98] p. 41-42, [18] p. 41-42)

1) φ is called a comparison function if φ is increasing and $\varphi^n(t) \to 0$ as $n \to \infty$, for all $t \in \mathbb{R}_+$.

2) φ is called a strict comparison function if φ is a comparison function and $t - \varphi(t) \to \infty$ as $t \to \infty$.

3) φ is called a strong comparison function if φ is a comparison function and $\sum\limits_{n\in\mathbb{N}} \varphi^n(t) < +\infty$, for all $t \in \mathbb{R}_+$.

Notice that if φ is a comparison function, then $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$. Moreover, each iterate φ^k, $k \geq 1$, is a comparison function.

Let $\varphi : \mathbb{R}^5_+ \to \mathbb{R}_+$ and define $\psi_\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\psi_\varphi(t) := \varphi(t,t,t,t,t)$.

Consider on $\mathbb{R}^5_+$ the usual component-wise ordering $\preceq$. Then, by definition:

1) φ is called a comparison function if φ is increasing (i.e., $t, s \in (\mathbb{R}^5_+, \preceq)$ with $t \preceq s$ implies $\varphi(t) \leq \varphi(s)$) and ψ_φ is a comparison function.

2) φ is called a strict comparison function if φ is increasing and ψ_φ is a strict comparison function.

3) φ is called a strong comparison function if φ is increasing and ψ_φ is a strong comparison function.

3. Basic problems of the metric fixed point theory for multivalued operators

We start our considerations by presenting some general problems of the fixed point theory for multivalued operators.

If is not differently stated, we will suppose, through this paper, that (X, d) is a metric space and $T : X \to P(X)$ is a multivalued operator.

Problem 3.1. Which are the metric conditions on T which imply that $F_T \neq \emptyset$?

The fundamental results for this problem were given by: S.B. Nadler jr. (1967), J.T. Markin (1968), S.B. Nadler jr. (1969), H. Covitz and S.B. Nadler jr. (1970), C. Avramescu (1970), H. Schirmer (1970), R.E. Smithson (1971), S. Reich (1972), L.B. Ćirić (1974), T.C. Lin (1974), I.A. Rus (1975, 1978, 1991), R. Manka (1978), S. Czerwik (1980), J. Andres and L. Gorniewicz (2001), ...

As examples, we present some existence results for contraction type multivalued operators.

Nadler's Theorem. *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued k-contraction, i.e., $k \in [0, 1[$ and*

$$H_d(T(x), T(y)) \leq kd(x, y), \text{ for all } x, y \in X.$$

Then $F_T \neq \emptyset$.

Reich's Theorem. *Let (X, d) be a complete metric space and let $T : X \to P_{cl}(X)$ be a multivalued operator such that there exist $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that*

$$H_d\left(T\left(x\right), T\left(y\right)\right) \leq ad\left(x, y\right) + bD_d(x, T(x)) + cD_d(y, T(y)), \text{ for all } x, y \in X.$$

Then $F_T \neq \emptyset$.

Smithson's Theorem. *Let (X, d) be a compact metric space and $T : X \to P_{cl}(X)$ be a multivalued contractive operator, i.e.,*

$$H_d(T(x), T(y)) < d(x, y), \text{ for all } x, y \in X \text{ with } x \neq y.$$

Then $F_T \neq \emptyset$.

Lim's Theorem. *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space and $Y \in P_{b,cl,cv}(X)$. Let $T : Y \to P_{cp}(Y)$ be a multivalued nonexpansive operator, i.e.,*

$$H_{d_{\|\cdot\|}}(T(x), T(y)) \leq \|x - y\|, \text{ for all } x, y \in Y.$$

Then $F_T \neq \emptyset$.

Rus' Theorem. *Let (X, d) be a complete metric space and $T : X \to P_(X)$ be a multivalued operator with closed graph. We suppose that there exist $a, b \in \mathbb{R}_+$ with $a + b < 1$ such that*

$$H_d\left(T\left(x\right), T\left(y\right)\right) \leq ad\left(x, y\right) + bD(y, T(y)), \text{ for all } (x, y) \in Graph(T).$$

Then $F_T \neq \emptyset$.

References: [15,25,28,29,32,39–41,46–48,53,61,63,65,72,76,91,93,111,118,120, 124,130], ...

As a new result, we present the following.

Theorem 3.1.1 *Let* (X, d) *be a complete metric space and* $T : X \to P(X)$ *be a multivalued operator with closed graph. We suppose that there exists* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that:*

(i) φ *is strictly increasing;*
(ii) φ *is a strong comparison function;*
(iii) $H_d(T(x), T(y)) \leq \varphi(d(x, y))$, *for all* $(x, y) \in Graph(T)$.

Then $F_T \neq \emptyset$.

Proof. Let $x_0 \in X$ and $x_1 \in T(x_0)$. Then, by the properties of the functional H_d, there exists $q_1 > 1$ and $x_2 \in T(x_1)$ such that

$$d(x_1, x_2) \leq q_1 H(T(x_0), T(x_1)) \text{ and } q_1 \varphi(d(x_0, x_1)) < d(x_0, x_1).$$

Then, we get that

$$d(x_1, x_2) \leq q_1 H(T(x_0), T(x_1)) \leq q_1 \varphi(d(x_0, x_1)) < d(x_0, x_1).$$

From the above relation and as a consequence of the fact that φ is strictly increasing we obtain that

$$\varphi(d(x_1, x_2)) < \varphi(d(x_0, x_1)).$$

In a similar way, there exist $q_2 > 1$ and $x_3 \in T(x_2)$ such that

$$d(x_2, x_3) \leq q_2 H(T(x_1), T(x_2)) \leq q_2 \varphi(d(x_1, x_2)) < \varphi(d(x_0, x_1)).$$

Since φ is strictly increasing we obtain that

$$\varphi(d(x_2, x_3)) < \varphi^2(d(x_0, x_1)).$$

Thus, by induction, there exist $x_{n+1} \in T(x_n)$ (for each $n \in \mathbb{N}$) such that

$$d(x_{n+1}, x_{n+2}) \leq \varphi^n(d(x_0, x_1)), \text{ for each } n \in \mathbb{N}.$$

by (ii), it follows that the sequence $(x_n)_{\mathbb{N}}$ is Cauchy and, thus, convergent to an element $x^* \in X$. Since T has closed graph we immediately get that $x^* \in T(x^*)$. □

By a similar approach, we get a more general theorem, as follows.

Theorem 3.1.2 *Let* (X, d) *be a complete metric space and* $T : X \to P(X)$ *be a multivalued operator with closed graph. We suppose that there exists* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that:*

(i) φ *is strictly increasing;*
(ii) φ *is a strong comparison function;*
(iii)′ $D_d(y, T(y)) \leq \varphi(d(x, y))$, *for all* $(x, y) \in Graph(T)$.

Then $F_T \neq \emptyset$.

This result gives rise to the following open question.

Problem 3.1.3 Which generalized multivalued contractions satisfy the condition (iii)$'$?

Remark 3.1.4 (see [19,30,35], ...). Let $(X, \leq)$ be an ordered set with at least a maximal element $x^* \in X$. Let $T : X \to P(X)$ be a multivalued operator such that, for each $x \in X$ there exists $y \in T(x)$ such that $x \leq y$. Then $x^* \in F_T$.

This remark gives rise to the following question.

Problem 3.1.5 Given a complete metric space (X, d) and a multivalued operator $T : X \to P(X)$ find a partial order on X such that:

(i) $(X, d, \leq)$ is an ordered metric space;
(ii) $(X, \leq)$ has at least one maximal element;
(iii) for each $x \in X$ there is $y \in T(x)$ such that $x \leq y$.

For this problem, the following result in [30] is very useful.

Theorem 3.1.6 *Let* $(X, d, \leq)$ *be an ordered metric space. If, for each increasing sequence*

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots,$$

we have that

$$d(x_n, x_{n+1}) \to 0 \text{ as } n \to +\infty,$$

then there exists at least one maximal element in $(X, \leq)$.

For other considerations on this problem see [19,30,36,86,109,125], ...

Problem 3.2. Which are the metric conditions on T which imply that $(SF)_T \neq \emptyset$?

Problem 3.3. Which are the metric conditions on T which imply that $F_T = (SF)_T \neq \emptyset$?

Problem 3.4. Which are the metric conditions on T which imply that $(SF)_T = \{x^*\}$?

Problem 3.5. Which are the metric conditions on T which imply that $F_T = (SF)_T = \{x^*\}$?

Problem 3.6. In which metric conditions on T the following implication holds:

$$(SF)_T \neq \emptyset \implies F_T = (SF)_T = \{x^*\} \ ?$$

The basic results concerning the above problems were given by S. Reich (1972), Lj.B. Ćirić (1974), K. Iseki (1974), I.A. Rus (1975, 1978, 1997), H.W. Corley (1986), A. Sîntămărian (1997), etc.

For example, we have the following results.

Reich's Theorem. *Let (X,d) be a complete metric space and let $T : X \to P_b(X)$ be a multivalued operator such that there exist $a,b,c \in \mathbb{R}_+$ with $a+b+c<1$ such that*

$$\delta_d\left(T\left(x\right),T\left(y\right)\right) \leq ad\left(x,y\right) + b\delta_d(x,T(x)) + c\delta_d(y,T(y)), \text{ for all } x,y \in X.$$

Then $(SF)_T = \{x^*\}$.

Rus' Theorem. *Let (X,d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued k-contraction. If $(SF)_T \neq \emptyset$, then $F_T = (SF)_T = \{x^*\}$.*

For Problem 3.5 we have the following general result (see [100]).

Theorem 3.5.1 *Let (X,d) be a complete metric space, $T : X \to P_b(X)$ be a multivalued operator and $\varphi : \mathbb{R}^5_+ \to \mathbb{R}_+$. be a function. We suppose that:*

(i) *φ is increasing;*
(ii) *there exists $p>1$ such that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$\psi(t) := \varphi(t, pt, pt, t, t), \; t \in \mathbb{R}_+$$

is a strict comparison function;
(iii) *φ is continuous;*
(iv) *$\delta_d(T(x),T(y)) \leq \varphi(d(x,y), \delta_d(x,T(x)), \delta_d(y,T(y)), D_d(x,T(y)), D_d(y,T(x)))$, for all $x,y \in X$.*

Then $F_T = (SF)_T = \{x^*\}$.

References: [16, 19, 27, 30, 38, 63, 72, 76, 81, 91, 98–101, 107, 111, 114], ...

Problem 3.7. Which metric conditions imply that $F_T = F_{T^n} \neq \emptyset$ for all $n \in \mathbb{N}^*$?

Problem 3.8. Which metric conditions imply that $(SF)_T = (SF)_{T^n} \neq \emptyset$ for all $n \in \mathbb{N}^*$?

Problem 3.9. Which metric conditions imply that $\bigcap\limits_{n\in\mathbb{N}} T^n(X) = \{x^*\}$?

References: [41, 60, 61, 76, 81], ...

Problem 3.10. Which conditions assure that $T(F_T) = F_T$?

For the above problem we have the following result.

Theorem 3.10.1 *Let (X,d) be a metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. We suppose that there exists $\varphi : \mathbb{R}^4_+ \to \mathbb{R}_+$ such that:*

(i) *the function $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$\psi(t,s) := \varphi(0,t,0,s) \text{ is increasing;}$$

(ii) *$r \leq \varphi(0,r,0,r)$ implies $r=0$;*
(iii) *$H(T(x),T(y)) \leq \varphi(D(x,T(x)), D(y,T(y)), D(x,T(y)), D(y,T(x)))$, for all $x,y \in X$.*

Then $T(F_T) = F_T$.

Proof. Suppose that $F_T \neq \emptyset$. On the other hand, notice that

$$D(x, T(x)) \leq H(T(x), T(y)), \text{ for all } x \in F_T \text{ and all } y \in X$$

and

$$D(y, T(y)) \leq H(T(x), T(y)), \text{ for all } x \in X \text{ and all } y \in T(x).$$

From (iii) we get that $H(T(x), T(y)) < +\infty$, for all $x, y \in X$. Now, for $x \in F_T$ and $y \in T(x)$, using (i) and (iii), we obtain that

$$H(T(x), T(y)) \leq \varphi(0, H(T(x), T(y)), 0, H(T(x), T(y))).$$

Using (ii), we get that $H(T(x), T(y)) = 0$. Thus, $T(x) = T(y)$, for each $x \in F_T$ and $y \in T(x)$. This means that $y \in F_T$. □

Remark 3.10.2 For the case of multivalued operators $T : X \to P_{b,cl}(X)$ see [97].

References: [4, 12, 64, 97], ...

Problem 3.11. Which are the metric conditions on T implying that

(1) $F_T = (SF)_T = \{x^*\}$

(2) $T^n(x) \overset{H}{\to} \{x^*\}$ as $n \to +\infty$, uniformly with respect to $x \in X$.

By definition, an operator T satisfying the above two conditions is called a multivalued Picard operator (MP operator).

Concerning the above problem, we have the following result.

Theorem 3.11.1 *Let (X, d) be a complete metric space and $T : X \to P_{cp}(X)$ be a multivalued φ-contraction, i.e., $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and*

$$H(T(x), T(y)) \leq \varphi(d(x, y)), \text{ for all } x, y \in X.$$

If, additionally, $(SF)_T \neq \emptyset$, then T is a MP operator.

Proof. Let $x^* \in (SF)_T$ and let $y \in F_T$ be arbitrary chosen. Then

$$d(x^*, y) = H(T(x^*), y) \leq H(T(x^*), T(y)) \leq \varphi(d(x^*, y)).$$

By the properties of the comparison function φ we get that $d(x^*, y) = 0$ and thus $F_T = (SF)_T$. For the second conclusion, notice first that, for $Y_1, Y_2 \in P_{cp}(X)$, we have (see, for example, [29], [6], [81])

$$H(T(Y_1), T(Y_2)) \leq \varphi(Y_1, Y_2).$$

Then, we have:

$$H(x^*, T^n(x)) = H(T^n(x^*), T^n(x)) \leq \varphi^n(d(x^*, x)) \to 0 \text{ as } n \to +\infty, \text{ for each } x \in X.$$

□

References: [57, 76, 81, 84, 98, 111], ...

Problem 3.12. Which are metric conditions which imply the following conclusion:

for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

(1) $x_0 = x, \ x_1 = y$;

(2) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$;

(3) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

By definition, an operator $T : X \to P(X)$ satisfying the above conditions is called a multivalued weakly Picard operator (briefly MWP operator).

For a MWP operator $T : X \to P(X)$ we define the multivalued operator $T^\infty : Graph(T) \to P(F_T)$ by the formula $T^\infty(x, y) = \{\ z \in F_T \mid$ there exists a sequence of successive approximations of T starting from (x, y) that converges to $z\ \}$

Problem 3.13. If $T : X \to P(X)$ is a MP operator, in which conditions there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and for which $\psi(0) = 0$ such that

$$d(x, x^*) \le \psi(H_d(x, T(x))), \text{ for all } x \in X?$$

By definition, an operator T satisfying the above conditions is called a ψ-MP operator.

Problem 3.14. If $T : X \to P(X)$ is a MWP operator, in which conditions there exists a selection $t^\infty : Graph(T) \to X$ of T^∞ and there is an increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which is also continuous in 0 and with $\psi(0) = 0$ such that

$$d(x, t^\infty(x, y)) \le \psi(d(x, y)), \text{ for all } (x, y) \in Graph(T)?$$

By definition, an operator T satisfying the above conditions is called a ψ-MWP operator.

Notice first that the multivalued operators satisfying the assumptions from Nadler's Theorem, Reich's Theorem and Rus' Theorem from Problem 3.1 are MWP operators. (see [76]). For other examples, see [76], [81], [112]. For some applications of the MWP operator theory see Problem 3.17 and Problem 3.19.

References: [20, 33, 70, 76, 80–82, 84, 87, 111, 112], . . .

Problem 3.15. In which conditions on the multivalued operator $T : X \to P(X)$ for all sequences $(x_n)_{n\in\mathbb{N}}$ in X such that $x_{n+1} \in T(x_n)$ for $n \in \mathbb{N}$, we have that $T(x_n) \overset{H}{\to} F_T$ as $n \to +\infty$?

Problem 3.16. In which conditions a multivalued operator $T : X \to P(X)$ is a MWP operator and $T(y) = F_T$, for all $y \in F_T$?

An aswer to this problem is the following theorem.

Theorem 3.16.1 *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued Kannan type operator, i.e., there exists $\alpha \in [0, \frac{1}{2}[$ such that*

$$H(T(x), T(y)) \le \alpha(D(x, T(x)) + D(y, T(y))), \text{ for all } x, y \in X.$$

Then:

(a) *T is a $\frac{1-\alpha}{1-2\alpha}$-MWP operator;*
(b) *$T(y) = F_T$, for all $y \in F_T$.*

Proof. (a) Indeed, for any $(x_0, x_1) \in Graph(T)$ and for arbitrary $q \in]1, \frac{1-\alpha}{1-2\alpha}[$, we can choose $x_2 \in T(x_1)$ such that $d(x_1, x_2) \le qH(T(x_0), T(x_1)) \le q\alpha(D(x_0, T(x_0)) + D(x_1, T(x_1))) \le q\alpha(d(x_0, x_1) + d(x_1, x_2))$. Hence we get that $d(x_1, x_2) \le \frac{q\alpha}{1-q\alpha} d(x_0, x_1)$.

By induction, we can prove that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for T starting from (x_0, x_1) such that

$$d(x_n, x_{n+1}) \leq (\frac{q\alpha}{1-q\alpha})^n d(x_0, x_1), \text{ for each } n \in \mathbb{N}^*.$$

By a classical approach, we obtain that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy and, hence, it converges to $x^* \in X$. Moreover, x^* is a fixed point for T and $d(x_0, x^*) \leq \frac{1-\alpha}{1-2\alpha} d(x_0, x_1)$, proving that T is a $\frac{1-\alpha}{1-2\alpha}$-MWP operator.

(b) For the second conclusion, if $y \in F_T$ we will show that $T(y) = F_T$. Indeed, let $v \in T(y)$. Then $H(T(v), T(y)) \leq \alpha(D(v, T(v)) + D(y, T(y))) \leq \alpha H(T(y), T(v))$, which implies that $v \in F_T$. On the other hand, if we choose $w \in F_T$, then we get that $D(w, T(y)) \leq H(T(w), T(y)) \leq \alpha(D(w, T(w)) + D(y, T(y))) = 0$, proving that $w \in T(y)$. □

References: [41, 60, 61], ...

We will present now some data dependence problems for multivalued operators.

Problem 3.17. Let $T, S : X \to P(X)$ be two multivalued operators such that and F_T and F_S are nonempty. In which conditions there exists an increasing function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ which is also continuous in 0 and with $\theta(0) = 0$ such that the following implication holds:

$$\eta > 0 \text{ and } H(S(x), T(x)) \leq \eta, \text{ for each } x \in X \implies H(F_S, F_T) \leq \theta(\eta) \ ?$$

We present now an answer to this problem.

Theorem 3.17.1 *Let (X, d) be a metric space and $T, S : X \to P_{cl}(X)$ be two ψ-MWP operators. Let $\eta > 0$ such that $H(S(x), T(x)) \leq \eta$, for each $x \in X$. Suppose also that, for each $q > 1$, we have $\psi(q\eta) \leq q\psi(\eta)$. Then*

$$H(F_S, F_T) \leq \psi(\eta).$$

Proof. Let $x^* \in F_S$ be arbitary chosen. Then

$$d(x^*, t^\infty(x^*, y)) \leq \psi(d(x^*, y)), \text{ for each } y \in T(x^*).$$

Let $q > 1$ be arbitrary. Then, there exists $y^* \in T(x^*)$ such that $d(x^*, y^*) \leq qH(S(x^*), T(x^*))$. Thus

$$d(x^*, t^\infty(x^*, y^*)) \leq \psi(qH(S(x^*), T(x^*))) \leq \psi(q\eta).$$

By a similar procedure we can prove that, for each $u^* \in F_T$ there exists $v^* \in S(u^*)$ such that

$$d(u^*, s^\infty(u^*, v^*)) \leq \psi(q\eta).$$

Hence, the above two relations together imply that

$$H(F_S, F_T) \leq \psi(q\eta) \leq q\psi(\eta), \text{ for every } q > 1.$$

Letting $q \searrow 1$, we get the conclusion. □

References: [33, 40, 54, 58, 59, 84, 100, 111, 112, 123], ...

Problem 3.18. Let $T, T_n : X \to P(X)$ $(n \in \mathbb{N})$ be multivalued operators such that:

(a) $F_T \neq \emptyset$ and $F_{T_n} \neq \emptyset$, for all $n \in \mathbb{N}$;

(b) T_n converges uniformly to T.

Which are the metric conditions which imply that $H(F_{T_n}, F_T) \to 0$ as $n \to \infty$?

The following result is an answer for the above problem.

Theorem 3.18.1 *Let (X, d) be a metric space and $T_n, T : X \to P_{cl}(X)$ be ψ-MWP operators. Suppose that $T_n(x) \overset{H}{\to} T(x)$ as $n \to +\infty$, uniformly with respect to $x \in X$. Suppose also that, for each $q > 1$, we have $\psi(qt) \leq q\psi(t)$, for each $t \in \mathbb{R}_+$. Then $H(F_{T_n}, F_T) \to 0$ as $n \to +\infty$.*

Proof. Let $\varepsilon > 0$. Since $T_n(x) \overset{H}{\to} T(x)$ as $n \to +\infty$, uniformly with respect to each $x \in X$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\sup_{x \in X} H(T_n(x), T(x)) < \varepsilon, \text{ for each } n \geq N_\varepsilon.$$

Then, by Theorem 3.17.1 we get that $H(F_{T_n}, F_T) \leq \psi(\varepsilon)$, for each $n \geq N_\varepsilon$. Since ψ is continuous in 0 and $\psi(0) = 0$, we obtain that $F_{T_n} \overset{H}{\to} F_T$. □

References: [59, 76, 82, 123], ...

Problem 3.19. Let (X, d) be a metric space and $T : X \to P(X)$ be a multivalued operator. Let us consider the fixed point inclusion

$$x \in T(x), \ x \in X$$

and, for $\eta > 0$, consider the inequation

$$D(y, T(y)) \leq \eta, \ y \in X.$$

In which conditions there exists an increasing function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ which is also continuous in 0 with $\theta(0) = 0$ such that for each solution $y^* \in X$ of the above inequation there exists a solution $x^* \in X$ of the fixed point inclusion having the following property

$$d(x^*, y^*) \leq \theta(\eta) \text{ ?}$$

By definition, a fixed point inclusion which satisfies the above condition is called generalized Ulam-Hyers stable. If $\theta(t) = ct$ (with some $c > 0$), then the fixed point inclusion is called Ulam-Hyers stable.

Theorem 3.19.1 (see [105])**.** *Let (X, d) be a metric space and $T : X \to P_{cp}(X)$ be a multivalued ψ-weakly Picard operator. Then, the fixed point inclusion*

$$x \in T(x), \ x \in X$$

is generalized Ulam-Hyers stable.

Proof. Let $\varepsilon > 0$ and $y^* \in X$ be a ε-solution of (3), i.e., $D(y^*, T(y^*)) \leq \varepsilon$. By the compactness assumption on $T(y^*)$, there exists $u^* \in T(y^*)$ such that $d(y^*, u^*) \leq \varepsilon$.

Since T is a multivalued ψ-weakly Picard operator, for each $(x,y) \in Graph(T)$ we have

$$d(x, t^{\infty}(x,y)) \leq \psi(d(x,y)).$$

Thus, if we define $x^* := f^{\infty}(y^*, u^*) \in F_T$, we get

$$d(y^*, x^*) \leq \psi(d(y^*, u^*)) \leq \psi(\varepsilon).$$

□

References: [84, 102, 105], ...

Problem 3.20. Let $T : X \to P(X)$ be a multivalued operator such that $F_T = \{x^*\}$. In which conditions the following implication holds:

$$(x_n)_{n\in\mathbb{N}} \subset X \text{ with } D_d(x_n, T(x_n)) \to 0 \text{ as } n \to \infty \Longrightarrow (x_n)_{n\in\mathbb{N}} \to x^* \text{ as } n \to \infty.$$

By definition, if the above condition is satisfied, then we say that the fixed point inclusion (3.19) is well-posed with respect to D_d.

An abstract theorem on the above problem is the following.

Theorem 3.20.1 *Let (X,d) be a metric space and $T : X \to P_{cl}(X)$ be a multivalued operator such that $F_T = \{x^*\}$. Suppose that there exists an increasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ continuous in 0 with $\gamma(0) = 0$ such that*

$$d(x, x^*) \leq \gamma(D(x, T(x))), \text{ for each } x \in X.$$

Then the fixed point problem is well-posed for T with respect to D.

As a particular case, we have.

Theorem 3.20.2 *Let (X,d) be a complete metric space and $T : X \to P_{cl}(X)$ be a Ćirić type multivalued operator, i.e., there exists $\alpha \in]0,1[$ such that for each $x, y \in X$*

$$H(T(x),T(y)) \leq \alpha \max\left\{d(x,y), D(x,T(x)), D(y,T(y)), \frac{1}{2}(D(x,T(y))+D(y,T(x)))\right\}.$$

Suppose $(SF)_T \neq \emptyset$. Then the fixed point problem is well-posed with respect to D and with respect to H.

Problem 3.21. Let $T : X \to P(X)$ be a multivalued operator such that $(SF)_T = \{x^*\}$. In which conditions the following implication holds:

$$(x_n)_{n\in\mathbb{N}} \subset X \text{ with } H_d(x_n, T(x_n)) \to 0 \text{ as } n \to \infty \Longrightarrow (x_n)_{n\in\mathbb{N}} \to x^* \text{ as } n \to \infty.$$

By definition, if the above condition is satisfied, then we say that the fixed point inclusion (3.19) is well-posed with respect to H_d.

A general result concerning the above problem is the following.

Theorem 3.21.1 *Let (X,d) be a metric space and $T : X \to P_{cl}(X)$ be a multivalued operator such that $(SF)_T = \{x^*\}$. Suppose that there exists an increasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ continuous in 0 with $\gamma(0) = 0$ such that*

$$d(x, x^*) \leq \gamma(H(x, T(x))), \text{ for each } x \in X.$$

Then the fixed point problem is well-posed for T with respect to H.

As an example we have the following theorem

Theorem 3.21.2 *Let (X,d) be a complete metric space and $T : X \to P_b(X)$ be a multivalued (δ,α)-contraction of Ćirić type, i.e., $\alpha \in]0,1[$ and, for all $x,y \in X$, we have*

$$\delta(T(x),T(y)) \leq \alpha \max\Big\{d(x,y), \delta(x,T(x)), \delta(y,T(y)), \frac{1}{2}(D(x,T(y))+D(y,T(x)))\Big\}.$$

Then the fixed point problem is well-posed with respect to H.

Proof. By Ćirić [26] we have that $F_T = (SF)_T = \{x^*\}$. Let $x \in X$ be arbitary. Then

$$d(x,x^*) \leq \delta(x,T(x)) + \delta(T(x),T(x^*)) \leq$$
$$H(x,T(x)) + \alpha \max\Big\{d(x,x^*), \delta(x,T(x)), 0, \frac{1}{2}(D(x,T(x^*)) + D(x^*,T(x)))\Big\} \leq$$
$$H(x,T(x)) + \alpha \max\Big\{d(x,x^*), H(x,T(x)), \frac{1}{2}(2d(x,x^*) + H(x,T(x)))\Big\}.$$

Then

$$d(x,x^*) \leq \frac{1}{1-\alpha}H(x,T(x)), \text{ for all } x \in X.$$

The conclusion follows now by the above theorem for $\gamma(t) = \frac{1}{1-\alpha}t$. □

References: [26, 80, 83, 84, 104, 129],...

Problem 3.22. Let $T : X \to P(X)$ be a multivalued operator. In which conditions the following implication holds:

$(y_n)_{n\in\mathbb{N}} \subset X$ with $D_d(y_{n+1},T(y_n) \to 0$ as $n \to \infty \Longrightarrow$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of Picard iterations (i.e., $x_{n+1} \in T(x_n)$) such that $d(x_n,y_n) \to 0$ as $n \to \infty$.

By definition, if the above condition is satisfied, then we say that the fixed point inclusion (3.19) has the limit shadowing property with respect to d.

Theorem 3.22.1 *Let (X,d) be a complete metric space and let $T : X \to P_{cl}(X)$ be a multivalued k-contraction such that $(SF)_T \neq \emptyset$. Then T has the limit shadowing property.*

References: [84, 88, 89], ...

Let us present now some problems which are suggested by the above problems.

Problem 3.6 suggests the following open question.

Problem 3.23. Let X be a set with a structure and $T : X \to P(X)$ be a multivalued operator. In which conditions we have $(SF)_T \neq \emptyset$?

Concerning this problem, we can prove, for example, the following result. Notice first that if $A,B \subset X$ then $A \leq_s B$ means that for each $a \in A$ and every $b \in B$ we have that $a \leq b$.

Theorem 3.23.1. *Let $(X,d,\leq)$ be an ordered metric space such that d is a complete metric. Let $T : X \to P_{cl}(X)$ be a multivalued operator. We suppose that:*

(i) *T is progessive, that is $\{x\} \leq_s T(x)$, for each $x \in X$;*
(ii) *$(X,\leq)$ has at least a maximal element $x^* \in X$;*

(iii) *T is a φ-contraction.*

Then $F_T = (SF)_T = \{x^*\}$.

Proof. From (i) and (ii) we get that $x^* \in (SF)_T$ and so $(SF)_T \neq \emptyset$. This property together with the φ-contraction condition imply (using Theorem 3.11.1) that $F_T = (SF)_T = \{x^*\}$. □

In 1970, H. Schirmer presented the following open question.

Problem 3.24. Let $T : \mathbb{R}^n \to P_{cp,cv}(\mathbb{R}^n)$ be a contraction. In which conditions the fixed point set F_T is connected ?

A more general problem is the following.

Problem 3.25. Let (X, d) be a metric space and $T : X \to P(X)$ be a generalized contraction. The problem is to study some properties (such as compactness, convexity, absolute retract property) of the fixed point set F_T ?

Some partial answers, for the case of multivalued Reich contractions, are the following theorems.

Theorem 3.25.1 *Let (X, d) be a complete metric space and $T : X \to P_{cp}(X)$ be a multivalued Reich type operator, i.e., there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $\alpha + \beta + \gamma < 1$ such that*

$$H(T(x), T(y)) \leq \alpha d(x, y) + \beta D(x, T(x)) + \gamma D(y, T(y)), \text{ for all } x, y \in X.$$

Then, the fixed points set F_T is compact.

Let $X \in \mathcal{M}$, where $\mathcal{M}$ denotes the family of all metric spaces. Then X is called an absolute retract for metric spaces (briefly $X \in AR(\mathcal{M})$) if, for any $Y \in \mathcal{M}$ and any $Y_0 \in P_{cl}(X)$, every continuous function $f_0 : Y_0 \to X$ has a continuous extension over Y, that is $f : Y \to X$. Obviously, any absolute retract is arcwise connected.

Concerning the absolute retract property of the fixed point set of a multivalued Reich contraction we have the following result.

Theorem 3.25.2 *Let E be a Banach space, $X \in P_{cl,cv}(E)$ and $T : X \to P_{cl,cv}(X)$ be a multi-valued Reich type operator. Suppose that T is lower semi-continuous. Then $F_T \in AR(\mathcal{M})$.*

Another result with respect to the above problems is the following theorem.

Theorem 3.25.3 (M.C. Anisiu-O. Mark [13]) **.** *Let $T : \mathbb{R} \to P_{cp,cv}(\mathbb{R})$ be a multivalued operator. We suppose that there exists a strict comparison function $\varphi : \mathbb{R}^5 \to \mathbb{R}$ such that*

$$H(T(x), T(y)) \leq \varphi(d(x, y), D(x, T(x)), D(y, T(y)), D(x, T(y)), D(y, T(x))), \quad \text{for all } x, y \in X.$$

Then, the fixed points set F_T is compact and convex.

References: [13, 21, 37, 71, 73, 79, 95, 113, 115], ...

Problem 3.26. Let (X,d) be a metric space and $T : X \to P(X)$ be a multivalued operator. Characterize the subset $Y \subseteq X \times X$ for which the following implication holds:

$$\text{there is } \alpha \in [0,1[\text{ such that } H(T(x),T(y)) \leq \alpha d(x,y)$$
$$\text{for every } (x,y) \in Y \Rightarrow F_T \neq \emptyset.$$

By definition, an operator satisfying the above property is called a multivalued (Y,α)-contraction. A similar problem arises for multivalued generalized contractions.

Commentaries (see [110]). As well-known particular cases for Y we have:

(i) $Y := Graph(T)$. In this case, T is called a multivalued graphic α-contraction.

(ii) Let $Z \in P_b(X)$ and consider $Y := (X \setminus Z) \times (X \setminus Z)$. In this case, T is called a multivalued α-contraction outside a bounded set.

(iii) Let $(X,d,\leq)$ be an ordered metric space and consider $Y := \{(x,y) \in X \times X : x \leq y \text{ or } y \leq x\}$.

For fixed point results in the context of metric spaces endowed with a graph see [69].

References: [69, 110].

4. Other aspects of the metrical fixed point theory for multivalued operators

4.1. **Nonexpansive operators.** The basic results for Problem 3.1 in the case of multivalued nonexpansive operators were given by J.T. Markin (1968), N. Assad and W.A. Kirk (1972), S. Reich (1972), K. Goebel (1975), F.E. Browder (1976), T.C. Lim (1980), S. Massa (1983), ...

References: [32, 43–45, 53, 58, 93, 127, 128], ...

4.2. **Set-theoretic aspects.** For the set-theoretic aspects of the fixed point theory for multivalued operators see [101] and the references therein. We also mention the following variant of the Fryszkowski's Problem:

Problem 4.2.1 Let X be a nonempty set and $T : X \to P(X)$ be an operator such that

$$F_{T^n} = (SF)_{T^n} = \{x^*\}, \text{ for all } n \in \mathbb{N}^*.$$

Given $\alpha \in]0,1[$, in which conditions there exists a metric d on X such that:

(i) (X,d) is a complete metric space;

(ii) $T : (X,d) \to (P(X), H_d)$ is an α-contraction ?

For the Fryszkowski's Problem see [51], [52].

4.3. **Order-theoretic aspects.** For order-theoretic aspects of the fixed point theory for multivalued operators see [19, 30, 35, 50, 53, 111, 125], ...

4.4. **Impact of the convergence of iterative methods in fixed point inclusion theory.** Concerning the impact of the convergence of some iterative methods on the multivalued fixed point inclusions see [14, 18, 27, 42, 45, 53, 59, 76, 81, 85, 90, 108, 116, 117, 119], ...

4.5. **Non-self multivalued operators.** For the basic problems of the fixed point theory of non-self multivalued operators see [47, 53, 56, 75, 97, 106, 111], ...

4.6. **Multivalued fractals.** An important and interesting application of the metrical fixed point theory for multivalued operators is the (multi)fractal theory. For the basic results in this direction see [5–7, 9, 10, 23, 42, 65, 74, 79, 81, 111], ...

4.7. **Periodic points.** For some basic results concerning this topic we refer to Nadler [66]. For other results see [8, 55, 57, 101, 110, 121, 126], ...

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