

THE UNIFORM ASYMPTOTICAL REGULARITY OF FAMILIES OF MAPPINGS AND SOLUTIONS OF VARIATIONAL INEQUALITY PROBLEMS

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ABSTRACT. In this paper our aim is to introduce a new class of procedure, the *Uniformly Asymptotically Regular*-class of procedures (UAR-procedures), showing some examples of procedures as for finite family of mappings, as for infinite family of mappings.

Then by a UAR-procedure we prove the convergence of an implicit iterative method and of an explicit iterative method to the unique solution of a variational inequality problem on the set of common fixed points of a family of mappings, in the setting of uniformly smooth Banach spaces.

1. INTRODUCTION

In 1977, P.L. Lions [15], extends the well-known Halpern's method in [9] to a family of firmly nonexpansive mappings. In subsequent twenty-five years, the study on the approximation of common fixed points of mappings has known, a significant increase. In particular, many recent papers, have the main aim to approximate common fixed points that are also solutions of variational inequality problems (see, as an example [7]).

Mainly there are two kind of approach to this approximation problem. The first is to implement at a map at a time into the iterative method and the to show that this method (weakly or strongly) converges to a common fixed point of the starting family. Obviously, if the family of mappings is finite it requires to use the maps in a cyclically way. This approach can be attributed to Lions in [15]. For other references one can read [4, 13, 16, 26].

The second approach introduced (probably) by Kuhfittig in 1981 [14], introduces a procedure such that, starting by the family of mappings generates an auxiliary functions with some *good properties*.

Kuhfittig, given a finite family of nonexpansive mappings, uses the convex combinations of each map with the identity map (with fixed opportune coefficient) to obtain an auxiliary nonexpansive mapping such that its fixed points coincide with the common fixed points of the family. This idea has been generalized from Atsushiba and Takahashi in 1999 [1] using variable coefficients. For other references and procedures on the finite case one can read [5, 6, 11, 12, 17].

Also the case of an infinite family of mappings has been extensively studied. One of the most cited paper is due to Shimoji and Takahashi [20] in 2001, when, using

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the convex combinations of the map and the identity map, they construct a sequence of mappings that play a similar role to the auxiliary mappings.

A careful reading of these papers carry out the idea that many of the existing procedures has some *common and good properties* such that the convergence of the method does not depend by the particular construction but depend only by this properties.

In this paper our aim is to show that these common and good properties are satisfied by the *Uniformly Asymptotically Regular*-class of procedures.

Then by a UAR-procedure we prove the convergence of an implicit iterative method and of an explicit iterative method to the unique solution of a variational inequality problem on the set of common fixed points of a family of mappings .

We start with some relevant examples. The common setting for these procedures is a *strictly convex Banach space* X . We recall the following property

Lemma 1.1. *Let X be a strictly convex Banach space and $x, y \in X$. If*

$$\|x\| = \|y\| = \|tx + (1 - t)y\|,$$

for some $t \in (0, 1)$, then $x = y$.

Procedure 1.2 (Shimoji-Takahashi [20], 2001). *Let X be a strictly convex Banach space and $C \subset X$ closed and convex.*

Let $(T_n)_{n \in \mathbb{N}}$ be a sequence of nonexpansive mappings from C to C with $\bigcap_{n \in \mathbb{N}} \text{Fix}(T_n) \neq$

\emptyset . *Let $\Lambda := (\lambda_n)_{n \in \mathbb{N}} \subset (0, b] \subset (0, 1)$. Let consider the following construction:*

$$(1.1) \quad \begin{cases} U_{n,n+1} := I, \\ U_{n,n} := \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I, \\ \vdots \\ U_{n,k} := \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \\ \vdots \\ U_{n,2} := \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I, \\ W_n \equiv U_{n,1} := \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I. \end{cases}$$

Note that:

- *Lemma 3.1 in [20] assures that every mapping in the sequence $(W_n)_{n \in \mathbb{N}}$ is nonexpansive.*
- *Lemma 3.2 in [20] claims that:*

$$\|W_{n+1}x - W_nx\| \leq 2\|x - w\| \prod_{i=1}^{n+1} \lambda_i$$

holds, for every $w \in \bigcap_{n \in \mathbb{N}} \text{Fix}(T_n)$. If $B \subset C$ is any bounded subset of C , the restriction $\lambda_n \leq b < 1$ gives the uniform asymptotical regularity of W_n on B , i.e.:

$$\|W_{n+1}x - W_nx\| \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ uniformly in } x \in B.$$

- Again Lemma 3.2 assures that $Wx := \lim_{n \rightarrow \infty} U_{n,1}x$ is well defined and Lemma 3.3 claims that $Fix(W) = \bigcap_{n \in \mathbb{N}} Fix(T_n)$.

Procedure 1.3 (Kangtunyakarn [10], 2011). *Let X be a strictly convex Banach space and $C \subset X$.*

Let $(T_n)_{n \in \mathbb{N}}$ be a sequence of nonexpansive mappings from C to C with common fixed points.

Let $\Lambda := (\lambda_n)_{n \in \mathbb{N}} \subset (0, 1)$ with $\sum_n \lambda_n < \infty$.

Let consider the following procedure:

$$(1.2) \quad \begin{cases} U_{n,0} := I, \\ U_{n,1} := \lambda_1 T_1 U_{n,0} + (1 - \lambda_1) U_{n,0}, \\ \vdots \\ U_{n,k} := \lambda_k T_k U_{n,k-1} + (1 - \lambda_k) U_{n,k-1}, \\ \vdots \\ U_{n,n-1} := \lambda_{n-1} T_{n-1} U_{n,n-2} + (1 - \lambda_{n-1}) U_{n,n-2}, \\ W_n \equiv U_{n,n} := \lambda_n T_n U_{n,n-1} + (1 - \lambda_n) U_{n,n-1}. \end{cases}$$

Lemma 2.11 in [10] assures that there exists, for all $x \in C$,

$$Wx := \lim_{n \rightarrow \infty} W_n x$$

and Lemma 2.12 proves that $Fix(W) = \bigcap_{n \in \mathbb{N}} Fix(T_n)$.

Moreover in Lemma 2.11 the inequality

$$\|W_{n+1}x - W_n x\| \leq \lambda_{n+1} \|T_{n+1}W_n x - W_n x\|$$

is also proved. If $w \in \bigcap_{n \in \mathbb{N}} Fix(T_n)$, $W_n w = w$, for all n , thus

$$\|W_{n+1}x - W_n x\| \leq 2\lambda_{n+1} \|W_n x - w\| \leq 2\lambda_{n+1} \|x - w\|.$$

The uniform asymptotical regularity of $(W_n)_{n \in \mathbb{N}}$ easily follows if $x \in B$ where $B \subset C$ is a bounded set.

Remark 1.4. We recall that a mapping T is said k -strictly pseudocontractive mappings in the Browder-Petryshyn sense if, for all $x, y \in C$ and $j(x-y) \in J(x-y)$, the following holds:

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x - y\|^2 - k\|x - y - (Tx - Ty)\|^2.$$

The previous procedure can be suitably adapted for a family of k_i -strictly pseudocontractive mappings in the setting of q -uniformly smooth Banach spaces (for details one can refer to [3, 8]).

It is well-known, in fact, that an opportune convex combination of a k -strict pseudocontraction and the identity map, is a nonexpansive mapping (see [18]).

Procedure 1.5. Let us introduce the following procedure inspired to Bruck’s idea in [2]. Let $(T_n)_{n \in \mathbb{N}}$ an infinite family of nonexpansive mappings in a strictly convex Banach spaces X . Let $\Lambda := (\lambda_n)_{n \in \mathbb{N}} \subset (0, 1)$ and let us define:

$$(1.3) \quad \left\{ \begin{array}{l} U_{n,n+1} := I, \\ U_{n,n} := \lambda_n T_n + (1 - \lambda_n) U_{n,n+1}, \\ U_{n,n-1} := \lambda_{n-1} T_{n-1} + (1 - \lambda_n) U_{n,n}, \\ \vdots \\ U_{n,k} := \lambda_k T_k + (1 - \lambda_k) U_{n,k+1} \\ \vdots \\ U_{n,2} := \lambda_2 T_2 + (1 - \lambda_2) U_{n,3}, \\ W_n := U_{n,1} = \lambda_1 T_1 + (1 - \lambda_1) U_{n,2} \end{array} \right.$$

Lemma 1.6. Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $(T_i)_{i \in \mathbb{N}}$ be an infinite family of nonexpansive mappings of C into itself and let $(\lambda_i)_{i \in \mathbb{N}}$ be a real sequence such that $0 < a \leq \lambda_i < 1$ for every $i \in \mathbb{N}$. Then,

- (a) the sequence of nonexpansive mappings $(W_n)_{n \in \mathbb{N}}$ is uniformly asymptotically regular on the bounded subsets $B \subset C$.
- (b) for every $x \in C$ and $k \in \mathbb{N}$, there exists

$$(1.4) \quad \lim_{n \rightarrow \infty} U_{n,k}x.$$

Moreover, if $\bigcap_{i \in \mathbb{N}} \text{Fix}(T_i) \neq \emptyset$, then $\text{Fix}(W) = \bigcap_{i \in \mathbb{N}} \text{Fix}(T_i)$ where

$$Wx := \lim_{n \rightarrow \infty} W_nx.$$

Proof. First of all we prove that, for every $k \in \mathbb{N}$, the limit in (1.4) exists. Let $x \in C$ and $n > k$ we observe that

$$\begin{aligned} \|U_{n+1,k}x - U_{n,k}x\| &= \|(1 - \lambda_k)U_{n+1,k+1}x - (1 - \lambda_k)U_{n,k+1}x\| \\ &= (1 - \lambda_k)(1 - \lambda_{k+1})\|U_{n+1,k+2}x - U_{n,k+2}x\| \\ &\quad \dots \\ &= \prod_{i=k}^n (1 - \lambda_i)\|U_{n+1,n+1}x - U_{n,n+1}x\| \\ &= \prod_{i=k}^n (1 - \lambda_i)\|\lambda_{n+1}T_{n+1}x + (1 - \lambda_{n+1})x - x\| \\ &\leq \prod_{i=k}^n (1 - \lambda_i)\|T_{n+1}x - x\| \end{aligned}$$

Thus, if $w \in \bigcap_i \text{Fix}(T_i)$ then

$$\|U_{n+1,k}x - U_{n,k}x\| \leq 2\|w - x\| \prod_{i=k}^n (1 - \lambda_i)$$

Since $\lambda_i > a$ then $\|U_{n+1,k}x - U_{n,k}x\| \leq 2\|w - x\|(1 - a)^{n-k+1}$.

If $B \subset C$ is bounded, $x \in B$ and $k = 1$ we obtain that

$$\|W_{n+1}x - W_nx\| \rightarrow 0, \text{ as } n \rightarrow \infty,$$

i.e. $(W_n)_{n \in \mathbb{N}}$ is uniformly asymptotically regular on the bounded subset B in C .

Moreover, if $m > n > k$, we have

$$\begin{aligned} \|U_{m,k}x - U_{n,k}x\| &\leq \sum_{j=n}^{m-1} \|U_{n+1,k}x - U_{n,k}x\| \leq \sum_{j=n}^{m-1} 2\|x - w\|(1-a)^{j-k+1} \\ &= \frac{2\|x - w\|(1-a)^n}{a}. \end{aligned}$$

Hence the sequence $(U_{n,k}x)_{n \in \mathbb{N}}$ is a Cauchy sequence and its limit there exists. In particular for $k = 1$ we can define

$$(1.5) \quad Wx := \lim_{n \rightarrow \infty} U_{n,1}x = \lim_{n \rightarrow \infty} W_nx$$

Next step is to prove that $Fix(W) = \bigcap_{n \in \mathbb{N}} Fix(T_n)$.

If $w \in \bigcap_n Fix(T_n)$ then $U_{n,n}w = w$, for all fixed n . This implies, by (1.3), that $w = U_{n,n-1}w$.

Flowing down one obtains that $U_{n,1}w = w$. So $w \in Fix(W)$ passing to the limit for $n \rightarrow \infty$.

Viceversa, we prove that if $w \in Fix(W)$ then $w \in \bigcap_n Fix(T_n)$.

Let $w \in Fix(W)$ and $y \in \bigcap_n Fix(T_n)$.

We note that

$$\begin{aligned} \|W_nw - W_ny\| &\leq \lambda_1\|w - y\| + (1 - \lambda_1)\|U_{n,2}w - U_{n,2}y\| \\ &\leq (\lambda_1 + (1 - \lambda_1)\lambda_2)\|w - y\| + (1 - \lambda_1)(1 - \lambda_2)\|U_{n,3}w - U_{n,3}y\| \\ &\leq \dots \\ &\leq \left(\lambda_1 + (1 - \lambda_1)\lambda_2 + \dots + \prod_{i=1}^{n-2} (1 - \lambda_i)\lambda_{n-1} \right) \|w - y\| \\ &\quad + \prod_{i=1}^{n-1} (1 - \lambda_i) \|U_{n,n}w - U_{n,n}y\| \\ &\leq \left(\lambda_1 + (1 - \lambda_1)\lambda_2 + \dots + \prod_{i=1}^{n-2} (1 - \lambda_i)\lambda_{n-1} \right) \|w - y\| \\ &\quad + \prod_{i=1}^{n-1} (1 - \lambda_i) \|w - y\| = \|w - y\|. \end{aligned}$$

Denoting by $U_{\infty,k}x := \lim_{n \rightarrow \infty} U_{n,k}x$, one observes that

$$\begin{aligned} \|w - y\| &= \|Ww - Wy\| = \|\lambda_1(T_1w - T_1y) + (1 - \lambda_1)(U_{\infty,2}w - U_{\infty,2}y)\| \\ &\leq \lambda_1\|w - y\| + (1 - \lambda_1)\|U_{\infty,2}w - U_{\infty,2}y\| \leq \|w - y\|, \end{aligned}$$

i.e.

$$(1 - \lambda_1)\|w - x\| \leq (1 - \lambda_1)\|U_{\infty,2}w - U_{\infty,2}y\| \leq (1 - \lambda_1)\|w - y\|.$$

Hence $\|U_{\infty,2}w - U_{\infty,2}y\| = \|w - y\|$.

In a similar way,

$$\begin{aligned} \|w - y\| &= \|Ww - Wy\| = \|\lambda_1(T_1w - T_1y) + (1 - \lambda_1)(U_{\infty,2}w - U_{\infty,2}y)\| \\ &\leq \lambda_1\|T_1w - T_1y\| + (1 - \lambda_1)\|w - y\| \Rightarrow \|T_1w - T_1y\| = \|w - y\|. \end{aligned}$$

By Lemma 1.1, $U_{\infty,2}w - U_{\infty,2}y = T_1w - T_1y$, i.e.

$$U_{\infty,2}w - y = U_{\infty,2}w - U_{\infty,2}y = T_1w - T_1y = T_1w - y,$$

so we can conclude that

$$w = \lim_{n \rightarrow \infty} U_{n,1}w = \lambda_1 T_1w + (1 - \lambda_1)U_{\infty,2} = T_1w.$$

This means $w \in \text{Fix}(T_1)$.

Repeating this idea for a second step, we have

$$\begin{aligned} \|w - y\| &= \|U_{\infty,2}w - U_{\infty,2}y\| = \|\lambda_2(T_2w - T_2y) + (1 - \lambda_2)(U_{\infty,3}w - U_{\infty,3}y)\| \\ &\leq \lambda_2\|w - y\| + (1 - \lambda_2)\|U_{\infty,3}w - U_{\infty,3}y\|, \end{aligned}$$

i.e. $\|U_{\infty,3}w - U_{\infty,3}y\| = \|w - y\|$ and $\|T_2w - T_2y\| = \|w - y\|$. This implies that, by the strict convexity of the space,

$$U_{\infty,3}w - y = U_{\infty,3}w - U_{\infty,3}y = T_2w - T_2y = T_2w - y.$$

Then

$$w = \lim_{n \rightarrow \infty} U_{n,2}w = \lambda_2 T_2w + (1 - \lambda_2)U_{\infty,3} = T_2w (= U_{\infty,3}w),$$

i.e. $w \in \text{Fix}(T_2)$.

Iterating again one proves that $w \in \cap_n \text{Fix}(T_n)$. □

Next procedure uses a finite number of nonlinear mappings. This case is studied in many recent papers as those cited in [6].

Procedure 1.7. *Let us consider the following:*

Definition 1.8. Let X be a Banach space, $C \subset X$ closed and convex, $\mathfrak{T} := \{T_i\}_{i=1}^N$ be a finite family of mappings from C into itself.

Let $\Delta = \{1, \dots, L\} \subset \mathbb{N}$ be a finite index set (with L not necessarily equal to N) and let $\Theta := (\eta_i)_{i \in \Delta}$, $\tilde{\Theta} := (\tilde{\eta}_i)_{i \in \Delta} \in (0, 1)^L$.

A procedure lies in the *LDC-class of procedures* (Lipschitzian Dependence of the Coefficients class of procedures) if, starting from the family \mathfrak{T} and from admissible coefficients Θ , it constructs a mapping $V_{\mathfrak{T}, \Theta}$ satisfying the following

- (h1) $V_{\mathfrak{T}, \Theta}$ is nonexpansive and $\text{Fix}(V_{\mathfrak{T}, \Theta}) = F := \cap_{i \in \Delta} \text{Fix}(T_i)$ whenever F is nonempty;
- (h2) for every $B \subset C$ bounded there exists $M = M(B, \mathfrak{T}) \in \mathbb{R}$ such that

$$\|V_{\mathfrak{T}, \tilde{\Theta}}x - V_{\mathfrak{T}, \Theta}x\| \leq \sum_{i \in \Delta} M|\tilde{\eta}_i - \eta_i|, \quad \forall x \in B.$$

We will call $V_{\mathfrak{T}, \Theta}$ an auxiliary mapping generated by the *LDC-procedure*.

Let us consider:

$$\Lambda = (\Theta_n)_{n \in \mathbb{N}} = (\eta_{i,n})_{n \in \mathbb{N}, i \in \Delta}$$

with the constraint

$$\eta_{i,n} \rightarrow \eta_i \in (0, 1), \text{ for any fixed } i \in \Delta.$$

Condition (h1) assures that, for every n , $W_n = V_{\mathfrak{T}, \Theta_n}$ is nonexpansive. Condition (h2) assures that:

$$\|V_{\mathfrak{T}, \Theta_{n+1}}x - V_{\mathfrak{T}, \Theta_n}x\| \leq M \sum_{i \in \Delta} |\eta_{i,n+1} - \eta_{i,n}|, \quad \forall x \in B.$$

so, letting $n \rightarrow \infty$, we have the uniform asymptotical regularity of $(V_{\mathfrak{T}, \Theta_n})_{n \in \mathbb{N}}$ on B . Moreover, if $\Theta = (\eta_i)_{i \in \Delta}$, $Wx := V_{\mathfrak{T}, \Theta}x$ is also generated by an LCD-procedure and it preserves the common fixed points.

Let us consider a (not necessarily finite) family of mappings $\mathfrak{T} := \{T_i : C \rightarrow C : i \in I\}$ and let $\Lambda := (\lambda_n)_{n \in \mathbb{N}}$ be an opportune sequence of real numbers.

Definition 1.9. A procedure is said an *uniformly asymptotically regular procedure* (in the sequel UAR-procedure) if starting by a family \mathfrak{T} and by Λ

- (H1) it defines a sequence of nonexpansive mappings $W_n : C \rightarrow C$ uniformly asymptotically regular on bounded subsets of $B \subset C$.
- (H2) it is possible to define a *nonexpansive* mapping $V := V_{\mathfrak{T}, \Lambda} : C \rightarrow C$, with $Vx := \lim_{n \rightarrow \infty} W_n x$ such that if $\bigcap_{i \in I} \text{Fix}(T_i) \neq \emptyset$ then $\text{Fix}(V) = \bigcap_{i \in I} \text{Fix}(T_i)$.

2. PRELIMINARIES

Let X be a q -uniformly smooth Banach space i.e. there exists a constant $C_q > 0$ such that

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x) \rangle + C_q \|y\|^q,$$

for all $x, y \in X$ (see Corollary 1 in [23]).

Definition 2.1. An operator $D : X \rightarrow X$ is said to be β -strongly accretive if

$$\langle Dx - Dy, j(x - y) \rangle \geq \beta \|x - y\|^2,$$

for all $x, y \in X$, where $j : X \rightarrow X^*$ is the duality mapping on X .

Next Lemma can be easily proved.

Lemma 2.2. Let $D : X \rightarrow X$ be a β -strongly accretive and L -lipschitzian operator. Let $t \in (0, 1)$ and $0 < \rho < \min \left\{ \left(\frac{q\beta}{C_q L^q} \right)^{\frac{1}{q-1}}, 1 \right\}$. Then $(I - t\rho D) : X \rightarrow X$ is a contraction with coefficient $(1 - t\tau)$ where $\tau = \frac{q\beta\rho - C_q(\beta L)^q}{q}$.

To obtain our results we will use the following (well-known) Lemma proved in [25]:

Lemma 2.3. Assume $(a_n)_n$ is a sequence of nonnegative numbers for which,

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \geq 0,$$

where $(\gamma_n)_n$ is a sequence in $(0, 1)$ and δ_n is a sequence in \mathbb{R} such that,

- (1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
- (2) $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.4. [19] *Let $\{y_m\}$ be a bounded sequence contained in a separable subset K of a Banach space E . Then there is a subsequence $(y_{m_k})_k$ of $(y_m)_m$ such that $\lim_k \|y_{m_k} - z\|$ exists for all $z \in K$.*

Lemma 2.5. [19] *Let C be a closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm, and let $(y_m)_m$ be a sequence in C such that $h(z) = \lim_m \|y_m - z\|$ exists for all $z \in C$. If h attains its minimum over C at u , then*

$$\limsup_m \langle z - u, j(y_m - u) \rangle \leq 0$$

for all $z \in C$.

Theorem 2.6. [22] *Let E be a reflexive Banach space and let C be a closed convex subset of E . Let h be a proper convex lower semicontinuous function of C into $(-\infty, \infty]$ and suppose that $h(x_n) \rightarrow \infty$ as $\|x_n\| \rightarrow \infty$. Then, there exists $x_0 \in D(h)$ such that*

$$h(x_0) = \inf\{h(x) : x \in C\}.$$

Lemma 2.7. [24] *Let J be the normalized duality map of a Banach space E . Suppose E is smooth. Then for all $x, y \in E$, there holds the inequality,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle.$$

3. CONVERGENCE OF ITERATIVE METHODS

Theorem 3.1. *Let X be a q -uniformly smooth Banach space.*

Let \mathfrak{T} be a denumerable family of mappings defined on X with common fixed points set $F \neq \emptyset$.

Let $D : X \rightarrow X$ be a β -strongly accretive and L -lipschitzian operator.

Let $0 < \rho < \min \left\{ \left(\frac{q\beta}{C_q L^q} \right)^{\frac{1}{q-1}}, 1 \right\}$.

Let us consider an UAR-procedure for \mathfrak{T} with given $\Lambda = (\lambda_n)_{n \in \mathbb{N}}$.

Let us choose $(\mu_n)_{n \in \mathbb{N}} \subset (0, \mu)$ with $\mu < \frac{2\beta}{L^2}$ such that:

(A1) $\lim_{n \rightarrow \infty} \mu_n = 0$.

Let $(\alpha_n)_{n \in \mathbb{N}} \subset [0, \alpha] \subset [0, 1)$. Then the sequence generated the iteration

$$x_n = \alpha_n x_n + (1 - \alpha_n)(I - \mu_n \rho D)W_n x_n,$$

strongly converges to $x^ \in F$ that is the unique solution of the variational inequality*

(3.1) $\langle Dx^*, j(y - x^*) \rangle \geq 0, \quad \forall y \in F.$

Proof. First of all, let us denote with $B_n := (I - \mu_n \rho D)$. By Lemma 2.2:

$$\|B_n x - B_n y\| \leq (1 - \mu_n \tau) \|x - y\|.$$

In order to apply Lemmas 2.4, 2.5, 2.6 let us consider the following set:

$$\begin{aligned} K_0 &:= \{x^*\}, \\ \tilde{W}_\eta &= \bigcup_{i \in I} \{W_i(y) : y \in K_\eta\} \\ K_{\eta+1} &:= \text{co}(K_\eta \cup \tilde{W}_\eta \cup \{y - \mu_\eta D y : x \in K_\eta\}) \\ K &= \bigcup_{\eta \in \mathbb{N}} K_\eta, \end{aligned}$$

for which K is closed, convex and separable.

For any fixed $n \in \mathbb{N}$ the mapping:

$$Sx := \alpha_n x + (1 - \alpha_n) B_n W_n x$$

is such that:

$$\begin{aligned} \|Sx - Sy\| &\leq \alpha_n \|x - y\| + (1 - \alpha_n)(1 - \mu_n \tau) \|W_n x - W_n y\| \\ &\leq \alpha_n \|x - y\| + (1 - \alpha_n)(1 - \mu_n \tau) \|x - y\| \leq (1 - (1 - \alpha_n)\mu_n \tau) \|x - y\| \end{aligned}$$

i.e. S is a strict contraction from K to K then it has a unique fixed point. Hence our method is well defined.

Step 1. $(x_n)_n \in \mathbb{N}$ is bounded.

Let $p \in F$. Then, by Lemma 2.7,

$$\begin{aligned} \|x_n - p\|^2 &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|B_n W_n x_n - B_n p\|^2 \\ &\quad + 2(1 - \alpha_n) \langle B_n p - p, j(x_n - p) \rangle \\ (3.2) \quad &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)(1 - \mu_n \tau)^2 \|x_n - p\|^2 \\ &\quad - 2(1 - \alpha_n)\mu_n \rho \langle Dp, j(x_n - p) \rangle \\ &\leq \|x_n - p\|^2 + (1 - \alpha_n)(\mu_n^2 \tau^2 - 2\mu_n \tau) \|x_n - p\|^2 \\ &\quad + 2(1 - \alpha_n)\mu_n \rho \|Dp\| \|x_n - p\| \end{aligned}$$

from which:

$$(1 - \alpha_n)(2 - \mu_n \tau)\mu_n \tau \|x_n - p\|^2 \leq +2(1 - \alpha_n)\mu_n \rho \|Dp\| \|x_n - p\|$$

and so, definitively:

$$\|x_n - p\| \leq \frac{2\rho \|Dp\|}{(2 - \mu_n \tau)\tau} \leq \frac{2\rho \|Dp\|}{\tau}$$

and the claim is proved.

Step 2. We claim that:

$$\Gamma := \limsup_{n \rightarrow \infty} \langle -Dx^*, j(x_n - x^*) \rangle \leq 0$$

Let $(x_{n_k})_{k \in \mathbb{N}}$ such that:

- (a) $\limsup_{n \rightarrow \infty} \langle -Dx^*, j(x_n - x^*) \rangle = \lim_{k \rightarrow \infty} \langle -Dx^*, j(x_{n_k} - x^*) \rangle$
- (b) there exists $\lim_{k \rightarrow \infty} \|x_{n_k} - z\|$, for all $z \in K$.

Let us define $h : K \rightarrow \mathbb{R}$ as

$$h(z) = \lim_{k \rightarrow \infty} \|x_{n_k} - z\|.$$

The function h is well defined by (b), continuous and convex. Moreover since $h(z) \rightarrow \infty$ as $z \rightarrow \infty$, by Lemma 2.6, h reaches its minimum on K .

Let $M := \{y \in K : h(y) = \min_{z \in K} h(z)\}$. M is closed, convex (by the property of h) and bounded. We claim that $V : M \rightarrow M$ where V is defined by (H2) in UAR -procedure. Let $y_0 \in M$

$$\begin{aligned} \|x_{n_k} - Vy_0\| &= \|\alpha_{n_k}(x_{n_k} - Vy_0) + (1 - \alpha_{n_k})(B_{n_k}W_{n_k}x_{n_k} - Vy_0)\| \\ &\leq \alpha_{n_k}\|x_{n_k} - Vy_0\| + (1 - \alpha_{n_k})\|B_{n_k}W_{n_k}x_{n_k} - B_{n_k}W_{n_k}y_0\| \\ &\quad + (1 - \alpha_{n_k})\|B_{n_k}W_{n_k}y_0 - B_{n_k}Vy_0\| + (1 - \alpha_{n_k})\|B_{n_k}Vy_0 - Vy_0\| \\ &\leq \alpha_{n_k}\|x_{n_k} - Vy_0\| + (1 - \alpha_{n_k})\|W_{n_k}x_{n_k} - W_{n_k}y_0\| \\ &\quad + \|W_{n_k}y_0 - Vy_0\| + (1 - \alpha_{n_k})\mu_{n_k}\|DVy_0\| \\ &\leq \alpha_{n_k}\|x_{n_k} - Vy_0\| + (1 - \alpha_{n_k})\|x_{n_k} - y_0\| \\ &\quad + \|W_{n_k}y_0 - Vy_0\| + (1 - \alpha_{n_k})\mu_{n_k}\|DVy_0\| \end{aligned}$$

Passing to the limit on k , by (A1), we obtain that $h(Vy_0) \leq h(y_0)$ hence $Vy_0 \in M$ and $V : M \rightarrow M$.

Since V is nonexpansive on M then there exists $\tilde{p} \in F \cap M$. From Lemma 2.5,

$$\limsup_k \langle x - \tilde{p}, j(x_{n_k} - \tilde{p}) \rangle \leq 0$$

Since $(\tilde{p} - \mu_{n_k}D\tilde{p}) \in K$, for all index n_k , then:

$$(3.3) \quad \limsup_k \langle -D\tilde{p}, j(x_{n_k} - \tilde{p}) \rangle \leq 0$$

By (3.3) in Step 1. it results that:

$$\|x_{n_k} - \tilde{p}\|^2 \leq \frac{2\rho}{(2 - \mu_{n_k}\tau)\tau} \langle -D\tilde{p}, j(x_{n_k} - \tilde{p}) \rangle$$

i.e. $x_{n_k} \rightarrow \tilde{p}$. Moreover, since

$$\|x_n - W_n x_n\| \leq \alpha_n \|x_n - W_n x_n\| + (1 - \alpha_n)\mu_n \|DW_n x_n\|$$

then $W_{n_k} x_{n_k} \rightarrow \tilde{p}$.

Let us observe at first that:

$$x_n = \alpha_n x_n + (1 - \alpha_n)(I - \mu_n \rho D)W_n x_n \Rightarrow DW_n x_n = \frac{1}{\rho \mu_n}(W_n x_n - x_n)$$

Then, for any $w \in F$,

$$\begin{aligned} \langle DW_n x_n, j(x_n - w) \rangle &= \frac{1}{\rho \mu_n} \langle W_n x_n - x_n, j(x_n - w) \rangle \\ &= \frac{1}{\rho \mu_n} \langle W_n x_n - x_n - W_n w + w, j(x_n - w) \rangle \\ &= \frac{1}{\rho \mu_n} \langle (W_n - I)x_n - (W_n - I)w, j(x_n - w) \rangle \leq 0 \end{aligned}$$

since, for every n , W_n is an accretive operator.

Moreover, for every $w \in F$,

$$\begin{aligned} \langle D\tilde{p}, j(\tilde{p} - w) \rangle &= \langle D\tilde{p}, j(\tilde{p} - w) \rangle - \langle D\tilde{p}, j(x_{n_k} - w) \rangle \\ &\quad + \langle D\tilde{p}, j(x_{n_k} - w) \rangle - \langle DW_{n_k} x_{n_k}, j(x_{n_k} - w) \rangle \\ &\quad + \langle DW_{n_k} x_{n_k}, j(x_{n_k} - w) \rangle \end{aligned}$$

$$\begin{aligned} &\leq \langle D\tilde{p}, j(\tilde{p} - w) - j(x_{n_k} - w) \rangle \\ &\quad + \langle D\tilde{p} - DW_{n_k}x_{n_k}, j(x_{n_k} - w) \rangle \\ &\leq \langle D\tilde{p}, j(\tilde{p} - w) - j(x_{n_k} - w) \rangle \\ &\quad + L\|\tilde{p} - W_{n_k}x_{n_k}\|\|x_{n_k} - w\| \end{aligned}$$

Passing to the limit on $k \rightarrow \infty$, since j is norm to norm uniformly continuous,

$$\langle D\tilde{p}, j(\tilde{p} - w) \rangle \leq 0, \quad \forall w \in F$$

This implies that $\tilde{p} = x^*$ and then $\Gamma \leq 0$ by (3.3).

Step. 3. $x_n \rightarrow x^*$. From (3.3) it results that:

$$\|x_n - x^*\|^2 \leq \frac{2\rho}{(2 - \mu_n\tau)\tau} \langle -Dx^*, j(x_n - x^*) \rangle.$$

so by Step 2. and (A1) one obtains $x_n \rightarrow x^*$. □

Theorem 3.2. *Let X be a q -uniformly smooth Banach space and $x_0 \in X$.*

Let \mathfrak{T} be a family of mappings defined on X with common fixed points set $F \neq \emptyset$.

Let $D : X \rightarrow X$ be a β -strongly accretive and L -lipschitzian operator.

Let $w_0 \in F$ be a fixed element of F and let us indicate by $B(w_0, r)$ the ball centered

in w_0 and radius $r := \max \left\{ \|x_0 - w_0\|, \frac{\rho\|Dw_0\|}{\tau} \right\}$.

Let $0 < \rho < \min \left\{ \left(\frac{q\beta}{C_q L^q} \right)^{\frac{1}{q-1}}, 1 \right\}$.

Let us consider an UAR-procedure for \mathfrak{T} with given $\Lambda = (\lambda_n)_{n \in \mathbb{N}}$.

Let us choose $(\mu_n)_{n \in \mathbb{N}} \subset (0, \mu)$ with $\mu < \frac{2\beta}{L^2}$ such that:

$$(A1) \text{ for any } z \in B(w_0, r), \lim_{n \rightarrow \infty} \frac{\|W_n z - W_{n-1} z\|}{\mu_n} = 0.$$

$$(A2) \lim_{n \rightarrow \infty} \mu_n = 0, \sum_{n \in \mathbb{N}} \mu_n = \infty \text{ and } \lim_{n \rightarrow \infty} \frac{|\mu_{n-1} - \mu_n|}{\mu_n} = 0.$$

Let us choose $(\alpha_n)_{n \in \mathbb{N}} \subset [0, 1)$ such that:

$$(A3) \lim_{n \rightarrow \infty} \frac{|\alpha_{n-1} - \alpha_n|}{\mu_n} = 0 \text{ and } \lim_{n \rightarrow \infty} \frac{\alpha_n^2}{\mu_n} = 0.$$

Then the sequence generated by $x_0 \in X$ and by the iterations

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(I - \mu_n \rho D)W_n x_n$$

strongly converges to $x^ \in F$, that is the unique solution of the variational inequality*

$$(3.4) \quad \langle Dx^*, j(y - x^*) \rangle \geq 0, \quad \forall y \in F$$

Proof. As in previous result we will denote by $B_n = (I - \mu_n \rho D)$.

First of all we prove that $(x_n)_{n \in \mathbb{N}} \subset B(w_0, r)$. Since $w_0 \in F$ then

$$\begin{aligned} \|x_{n+1} - w_0\| &\leq \alpha_n \|x_n - w_0\| + (1 - \alpha_n) \|B_n W_n x_n - w_0\| \\ &\leq \alpha_n \|x_n - w_0\| + (1 - \alpha_n) \|B_n W_n x_n - w_0\| \\ &\leq \alpha_n \|x_n - w_0\| + (1 - \alpha_n) \|B_n W_n x_n - B_n w_0\| \\ &\quad + (1 - \alpha_n) \|B_n w_0 - w_0\| \end{aligned}$$

$$\begin{aligned} &\leq \alpha_n \|x_n - w_0\| + (1 - \alpha_n)(1 - \mu_n \tau) \|x_n - w_0\| \\ &\quad + (1 - \alpha_n) \|(I - \mu_n \rho D)w_0 - w_0\| \\ &= \alpha_n \|x_n - w_0\| + (1 - \alpha_n)(1 - \mu_n \tau) \|x_n - w_0\| \\ &\quad + (1 - \alpha_n) \rho \mu_n \|Dw_0\| \end{aligned}$$

Thus, by induction, it is not difficult to observe that:

$$\|x_n - w_0\| \leq \max \left\{ \|x_0 - w_0\|, \frac{\rho \|Dw_0\|}{\tau} \right\} = r$$

We prove that $(x_n)_{n \in \mathbb{N}}$ is asymptotically regular, i.e.

$$\|x_n - x_{n+1}\| \rightarrow 0,$$

as $n \rightarrow \infty$. Computing:

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \alpha_n \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|x_{n-1} - B_{n-1}W_{n-1}x_{n-1}\| \\ &\quad + (1 - \alpha_n) \|B_n W_n x_n - B_{n-1}W_{n-1}x_{n-1}\| \\ &\leq \alpha_n \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|x_{n-1} - B_{n-1}W_{n-1}x_{n-1}\| \\ &\quad + (1 - \alpha_n) \|B_n W_n x_n - B_n W_{n-1}x_{n-1}\| \\ &\quad + (1 - \alpha_n) \|B_n W_{n-1}x_{n-1} - B_{n-1}W_{n-1}x_{n-1}\| \\ &\leq \alpha_n \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|x_{n-1} - B_{n-1}W_{n-1}x_{n-1}\| \\ &\quad + (1 - \alpha_n)(1 - \mu_n \tau) \|W_n x_n - W_{n-1}x_{n-1}\| \\ &\quad + (1 - \alpha_n) \|B_n W_{n-1}x_{n-1} - B_{n-1}W_{n-1}x_{n-1}\| \\ &\leq \alpha_n \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|x_{n-1} - B_{n-1}W_{n-1}x_{n-1}\| \\ &\quad + (1 - \alpha_n)(1 - \mu_n \tau) \|W_n x_n - W_n x_{n-1}\| \\ &\quad + (1 - \alpha_n)(1 - \mu_n \tau) \|W_n x_{n-1} - W_{n-1}x_{n-1}\| \\ &\quad + (1 - \alpha_n) \|\mu_n - \mu_{n-1}\| \|DW_{n-1}x_{n-1}\| \\ &\leq \alpha_n \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|x_{n-1} - B_{n-1}W_{n-1}x_{n-1}\| \\ &\quad + (1 - \alpha_n)(1 - \mu_n \tau) \|x_n - x_{n-1}\| \\ &\quad + (1 - \alpha_n)(1 - \mu_n \tau) \|W_n x_{n-1} - W_{n-1}x_{n-1}\| \\ &\quad + |\mu_n - \mu_{n-1}| \|DW_{n-1}x_{n-1}\| \end{aligned}$$

The boundedness of $(x_n)_{n \in \mathbb{N}}$ guarantees that there exists a constant M such that:

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq [\alpha_n + (1 - \alpha_n)(1 - \mu_n \tau)] \|x_n - x_{n-1}\| + \|W_n x_{n-1} - W_{n-1}x_{n-1}\| \\ &\quad + M [|\alpha_n - \alpha_{n-1}| + |\mu_n - \mu_{n-1}|] \\ &= [1 + (1 - \alpha_n)\mu_n \tau] \|x_n - x_{n-1}\| + \|W_n x_{n-1} - W_{n-1}x_{n-1}\| \\ &\quad + M [|\alpha_n - \alpha_{n-1}| + |\mu_n - \mu_{n-1}|] \end{aligned}$$

By (A1), (A2) and (A3) and Lemma 2.3 the claim follows.

If $x^* \in F$ is the unique solution of (3.4) it results:

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)B_n W_n x_n - x^* \pm (1 - \alpha_n)B_n x^*\|^2 \\ &= \|\alpha_n(x_n - x^*) + (1 - \alpha_n)(B_n W_n x_n - B_n x^*) - (1 - \alpha_n)\mu_n \rho D x^*\|^2 \\ &\leq \|\alpha_n(x_n - x^*) + (1 - \alpha_n)(B_n W_n x_n - B_n x^*)\|^2 \end{aligned}$$

$$\begin{aligned}
& -2(1 - \alpha_n)\mu_n\rho\langle Dx^*, j(x_{n+1} - x^*) \rangle \\
\leq & \alpha_n\|x_n - x^*\|^2 + (1 - \alpha_n)(1 - \mu_n\tau)^2\|x_n - x^*\|^2 \\
& + 2(1 - \alpha_n)\mu_n\rho\langle -Dx^*, j(x_{n+1} - x^*) \rangle \\
= & [1 + (1 - \alpha_n)(\mu_n^2\tau^2 - 2\mu_n\tau)]\|x_n - x^*\|^2 \\
& + 2(1 - \alpha_n)\mu_n\rho\langle -Dx^*, j(x_{n+1} - x^*) \rangle \\
= & [1 - (1 - \alpha_n)\mu_n\tau(2 - \mu_n\tau)]\|x_n - x^*\|^2 \\
& + 2(1 - \alpha_n)\mu_n\rho\langle -Dx^*, j(x_{n+1} - x^*) \rangle
\end{aligned}$$

so, proving that:

$$\limsup_{n \rightarrow \infty} \langle -Dx^*, j(x_{n+1} - x^*) \rangle \leq 0$$

by Lemma 2.3 we will obtain that $x_n \rightarrow x^*$.

For this we use the convergence of the implicit method.

Let us consider a subsequence of $(x_n)_{n \in \mathbb{N}}$ such that:

$$\Gamma := \limsup_{n \rightarrow \infty} \langle -Dx^*, x_{n+1} - x^* \rangle = \lim_{k \rightarrow \infty} \langle -Dx^*, x_{n_k} - x^* \rangle$$

Let us consider the implicit method defined by:

$$y_m = \alpha_m y_m + (1 - \alpha_m) B_m V y_m$$

where V is defined in the AdC -procedure.

Let us observe that $y_m \rightarrow x^*$ by Theorem 3.1 and moreover:

$$\begin{aligned}
\|y_m - x_{n_k}\|^2 &= \|\alpha_m(y_m - x_{n_k}) + (1 - \alpha_m)(V y_m - x_{n_k}) - (1 - \alpha_m)\mu_m\rho DV y_m\|^2 \\
&\leq (1 - \alpha_m)^2 \|V y_m - x_{n_k}\|^2 \\
&\quad + 2\langle \alpha_m(y_m - x_{n_k}) - (1 - \alpha_m)\mu_m\rho DV y_m, j(y_m - x_{n_k}) \rangle \\
&\leq (1 - \alpha_m)^2 \|V y_m - x_{n_k}\|^2 + 2\alpha_m \|y_m - x_{n_k}\|^2 \\
&\quad - 2(1 - \alpha_m)\mu_m\rho \langle DV y_m, j(y_m - x_{n_k}) \rangle
\end{aligned}$$

Note that:

$$\begin{aligned}
\|V y_m - x_{n_k}\| &\leq \|V y_m - W_{n_k} y_m\| + \|W_{n_k} y_m - W_{n_k} x_{n_k}\| + \|W_{n_k} x_{n_k} - x_{n_k}\| \\
&\leq \|V y_m - W_{n_k} y_m\| + \|y_m - x_{n_k}\| + \|W_{n_k} x_{n_k} - x_{n_k+1}\| \\
&\quad + \|x_{n_k+1} - x_{n_k}\|
\end{aligned}$$

from which:

$$\limsup_{k \rightarrow \infty} \|V y_m - x_{n_k}\| \leq \limsup_{k \rightarrow \infty} \|y_m - x_{n_k}\|$$

by the asymptotical regularity of $(x_n)_{n \in \mathbb{N}}$.

Thus:

$$\begin{aligned}
\limsup_{k \rightarrow \infty} \|y_m - x_{n_k}\|^2 &\leq [(1 - \alpha_m)^2 + 2\alpha_m] \limsup_k \|y_m - x_{n_k}\| \\
&\quad - 2(1 - \alpha_m)\mu_m\rho \limsup_{k \rightarrow \infty} \langle DV y_m, j(y_m - x_{n_k}) \rangle
\end{aligned}$$

that implies:

$$\limsup_{k \rightarrow \infty} \langle DV y_m, j(y_m - x_{n_k}) \rangle \leq \frac{\alpha_m^2}{2(1 - \alpha_m)\mu_m\rho} \limsup_{k \rightarrow \infty} \|y_m - x_{n_k}\|$$

Now let us consider:

$$\begin{aligned} \langle Dx^*, j(x^* - x_{n_k}) \rangle &= \langle Dx^*, j(x^* - x_{n_k}) \rangle - \langle Dx^*, j(y_m - x_{n_k}) \rangle \\ &\quad \langle Dx^*, j(y_m - x_{n_k}) \rangle - \langle DVy_m, j(y_m - x_{n_k}) \rangle \\ &\quad + \langle DVy_m, j(y_m - x_{n_k}) \rangle \end{aligned}$$

Since j is norm to norm uniformly continuous and $y_m \rightarrow x^*$ there exists $\delta_m \rightarrow 0$ such that

$$|\langle Dx^*, j(x^* - x_{n_k}) \rangle - \langle j(y_m - x_{n_k}) \rangle| < \delta_m$$

Since D is a Lipschitzian operator:

$$\langle Dx^* - DVy_m, j(y_m - x_{n_k}) \rangle \leq L\|x^* - y_m\| \|x_{n_k} - y_m\|$$

Thus:

$$\begin{aligned} \limsup_{k \rightarrow \infty} \langle Dx^*, j(x^* - x_{n_k}) \rangle &\leq \delta_m + L\|x^* - y_m\| \limsup_{k \rightarrow \infty} \|x_{n_k} - y_m\| \\ &\quad + \frac{\alpha_m^2}{2(1 - \alpha_m)\mu_m\rho} \limsup_{k \rightarrow \infty} \|y_m - x_{n_k}\| \end{aligned}$$

Passing $m \rightarrow \infty$, since $\frac{\alpha_m^2}{\mu_m} \rightarrow 0$,

$$\limsup_{k \rightarrow \infty} \langle Dx^*, j(x^* - x_{n_k}) \rangle \leq 0$$

i.e. our last claim. □

Corollary 3.3. *Let X be a q -uniformly smooth Banach space.*

Let \mathfrak{T} be a one-parameter continuous semigroup of nonexpansive mappings defined on X with common fixed points set $F \neq \emptyset$.

Let $D : X \rightarrow X$ be a β -strongly accretive and L -lipschitzian operator.

Let $0 < \rho < \min \left\{ \left(\frac{q\beta}{C_q L^q} \right)^{\frac{1}{q-1}}, 1 \right\}$.

Let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda \in (0, 1)$.

Let $(\mu_n)_{n \in \mathbb{N}} \subset (0, \mu)$ with $\mu < \frac{2\beta}{L^2}$ such that:

$$(A1) \quad \lim_{n \rightarrow \infty} \mu_n = 0, \quad \sum_{n \in \mathbb{N}} \mu_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{|\mu_{n-1} - \mu_n|}{\mu_n} = 0.$$

$$(A2) \quad \lim_{n \rightarrow \infty} \frac{|\lambda_n - \lambda_{n-1}|}{\mu_n} = 0.$$

Let $(\alpha_n)_{n \in \mathbb{N}} \subset [0, 1)$ such that:

$$(A3) \quad \lim_{n \rightarrow \infty} \frac{|\alpha_{n-1} - \alpha_n|}{\mu_n} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\alpha_n^2}{\mu_n} = 0.$$

Then the sequence generated by $x_0 \in X$ and by the iteration

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(I - \mu_n \rho D)(\lambda_n T(1) + (1 - \lambda_n)T(\sqrt{2}))x_n$$

strongly converges to $x^ \in F$ that is the unique solution of the variational inequality*

$$(3.5) \quad \langle Dx^*, j(y - x^*) \rangle \geq 0, \quad \forall y \in F$$

Proof. It is enough to observe that $W_n x := \lambda_n T(1)x + (1 - \lambda_n)T(\sqrt{2})x$ is a nonexpansive mappings such that $Fix(W_n) = Fix(T(1)) \cap Fix(T(\sqrt{2})) = F$ (see Suzuki [21]). \square

Corollary 3.4. *Let X be a q -uniformly smooth Banach space.*

Let T be a nonexpansive mappings defined on X with fixed points set $Fix(T) \neq \emptyset$.

Let $D : X \rightarrow X$ be a β -strongly accretive and L -lipschitzian operator.

Let $0 < \rho < \min \left\{ \left(\frac{q\beta}{C_q L^q} \right)^{\frac{1}{q-1}}, 1 \right\}$.

Let $(\mu_n)_{n \in \mathbb{N}} \subset (0, \mu)$ with $\mu < \frac{2\beta}{L^2}$ such that:

$$(A1) \quad \lim_{n \rightarrow \infty} \mu_n = 0, \quad \sum_{n \in \mathbb{N}} \mu_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{|\mu_{n-1} - \mu_n|}{\mu_n} = 0.$$

Let $(\alpha_n)_{n \in \mathbb{N}} \subset [0, 1)$ such that:

$$(A2) \quad \lim_{n \rightarrow \infty} \frac{|\alpha_{n-1} - \alpha_n|}{\mu_n} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\alpha_n^2}{\mu_n} = 0.$$

Then the sequence generated by $x_0 \in X$ and by the iteration

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(I - \mu_n \rho D)T x_n$$

strongly converges to $x^ \in Fix(T)$ that is the unique solution of the variational inequality*

$$(3.6) \quad \langle Dx^*, j(y - x^*) \rangle \geq 0, \quad \forall y \in Fix(T)$$

REFERENCES

- [1] S. Atsushiba and W. Takahashi, *Strong convergence theorems for a finite family of nonexpansive mappings and applications*, Indian J. Math. **41** (1999), 435–453.
- [2] R. E. Bruck, *Properties of fixed point sets of nonexpansive mapping in Banach spaces*, Transactions of the Amer. Math. Soc. **179** (1973), 251–262.
- [3] G. Cai and C. S. Hu, *Strong convergence theorems of modified Ishikawa iterative process with errors for an infinite family of strict pseudo-contractions*, Nonlinear Analysis **71** (2009), 6044–6053.
- [4] R. Chen and Y. Song and H. Zhou, *Convergence theorems for implicit iteration process for a finite family of continuous pseudocontractive mappings*, J. Math. Anal. Appl. **314** (2006), 701–709.
- [5] V. Colao and G. Marino, *Common fixed points of strict pseudocontractions by iterative algorithms*, J. Math. Anal. Appl. **382** (2011), 631–644.
- [6] V. Colao, G. Marino and L. Muglia, *On some auxiliary mappings generated by nonexpansive and strictly pseudo-contractive mappings*, Applied Math Comp. **218** (2012), 6232–6241.
- [7] V. Colao and G. Marino and H.-K. Xu, *An iterative method for finding common solutions of equilibrium and fixed point problems*, J. Math. Anal. Appl. **344** (2008), 340–352.
- [8] Q.-L. Dong, S. He and F. Su, *Strong convergence of an iterative algorithm for an infinite family of strict pseudo-contractions in Banach spaces*, Applied Math. Comp. **216** (2010), 959–969.
- [9] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.
- [10] A. Kangtunyakarn, *Iterative algorithms for finding a common solution of system of the set of variational inclusion problems and the set of fixed point problems*, Fixed Point Theory and Appl. (2011), 2011:38, 16 pp.
- [11] A. Kangtunyakarn and S. Suantai, *Strong convergence of a new iterative scheme for a finite family of strict pseudo-contractions*, Comput. Math. Appl. **60** (2010), 680–694.

- [12] A. Kangtunyakarn and S. Suantai, *A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings*, *Nonlinear Anal.* **71** (2009), 4448–4460.
- [13] Y. Kimura and W. Takahashi, M. Toyoda, *Convergence to common fixed points of a finite family of nonexpansive mappings*, *Arch. Math. (Basel)* **84** (2005), 350–363.
- [14] P. K. F. Kuhfittig, *Common fixed points of nonexpansive mappings by iteration*, *Pacific J. Math.* **97** (1981), 137–139.
- [15] P. L. Lions, *Approximation de points fixes de contractions*, (French) *C. R. Acad. Sci. Paris Sér. A-B* **284** (1977), 1357–1359.
- [16] P.-E. Maingé, *Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces*, *J. Math. Anal. Appl.* **325** (2007), 469–479.
- [17] G. Marino, L. Muglia, Y. Yao *Viscosity methods for common solutions of equilibrium and variational inequality problems via multi-step iterative algorithms and common fixed points*, *Nonlinear Anal.* **75** (2012), 1787–1798.
- [18] M.-O. Osilike and A. Udomene, *Demiclosedness principle and convergence theorems for strictly pseudocontractive mappings of Browder-Petryshyn type*, *J. Math. Anal. Appl.* **256** (2001), 431–445.
- [19] S. Reich, *Product formulas, nonlinear semigroups, and accretive operators*, *J. Funct. Anal.* **36** (1980), 147–168.
- [20] K. Shimoji and W. Takahashi, *Strong convergence to common fixed points of infinite nonexpansive mappings and applications*, *Taiwanese J. Math.* **5** (2001), 387–404.
- [21] T. Suzuki, *The set of common fixed points of a one-parameter continuous semigroup of mappings is $F(T(1)) \cap F(T(\sqrt{2}))$* , *Proc. Amer. Math. Soc.* **134** (2005), 673–681.
- [22] W. Takahashi, *Nonlinear Functional Analysis, Fixed point theory and its applications*, Yokohama Publishers, Yokohama, 2000.
- [23] H.-K. Xu, *Inequalities in Banach spaces with applications*, *Nonlinear Anal.* **16** (1991), 1127–1138.
- [24] H.-K. Xu, *Iterative algorithms for nonlinear operators*, *J. London Math. Soc.* **66** (2002), 240–256.
- [25] H.-K. Xu, *An iterative approach to quadratic optimization*, *J. Optim. Theory Appl.* **116** (2003), 659–678.
- [26] H.-K. Xu and T. H. Kim, *Convergence of hybrid steepest-descent methods for variational inequalities*, *J. Optim. Theory Appl.* **119** (2003), 185–201. .

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