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COMPARISON BETWEEN SEVERAL CONDITIONS IMPLYING WEAK NORMAL STRUCTURE AND THE WEAK FIXED POINT PROPERTY

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Abstract. In 1996, T. Domínguez-Benavides associated to each Banach space X the family of coefficients $\{R(a,X) : a \geq 0\}$, and he proved the two following results:
(I) X has weak normal structure whenever $R(0,X) < 1$.
(II) X has the weak fixed point property for nonexpansive mappings whenever $R(a,X) < 1+a$ for some $a > 0$.

Later, in 2008, S. Saejung proved that F-convex Banach spaces have uniform normal structure. In this paper, we prove that any F-convex Banach space X verifies $R(0,X) < 1$. This shows that the result by Saejung can be derived from (I).

Also in 2008, P.N. Dowling, B. Randrianantoanina and B. Turret identified several sufficient conditions for the weak fixed point property for nonexpansive mappings, including E-convexity. In this paper we prove that if a Banach space X verifies any of these conditions, in particular if X is a E-convex, then $R(a,X) < 1+a$ for some $a > 0$. Consequently, the fixed point results by Dowling, Randrianantoanina and Turret can be obtained as a byproduct of (II).

1. Introduction

Let C be a subset of a Banach space $(X, \|\cdot\|)$. A mapping $T\colon C \to X$ is called *nonexpansive* whenever $\|T(x)-T(y)\| \leq \|x-y\|$ for all $x, y \in C$. A Banach space X is said to have the *fixed point property for nonexpansive mappings* (FPP for short), if for each nonempty bounded closed convex subset C of X, every nonexpansive mapping of C into itself has a fixed point. And a Banach space X is said to have the *weak fixed point property for nonexpansive mappings* (wFPP for short) if the class of sets C in the definition above is restricted to the class of nonempty weakly compact convex sets.

One of the central and elusive goals in Metric Fixed Point Theory for nonexpansive mappings is to fully characterize those Banach spaces which have wFPP or FPP. This problem remains open, although several partial affirmative answers have been given since then. In fact, the classical theory produced a plethora of geometric/topological properties of Banach spaces which are sufficient to ensure the spaces have wFPP (and hence FPP in reflexive spaces). These properties are, in general, hard to check in a given space, and several relationships between them remained partially hidden for many years.

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One of the first results in this direction was given by W. Kirk in 1965 [9], who proved that Banach spaces with weak normal structure have the wFPP. Since then, weak normal structure plays an important role in the theory, and many properties of Banach spaces implying weak normal structure have been studied.

In 1980, W.L. Bynum [2] defined the *weakly convergent sequence coefficient* of a Banach space X, $WCS(X)$, and proved that a Banach space X has weak normal structure whenever $WCS(X) > 1$. We shall say that X satisfies *Bynum's condition* if $WCS(X) > 1$.

T. Domínguez-Benavides [3] in 1996 associated to each Banach space X a family of coefficients, $\{R(a, X)|a \geq 0\}$, and proved that a Banach X has wFPP whenever $R(a, X) < 1 + a$ for some $a \geq 0$. It turns out that $R(0, X) = 1/WCS(X)$, and therefore Bynum's condition $WCS(X) > 1$ is equivalent to $R(0, X) < 1$. On the other hand it is easy to see that $R(0, X) < 1$ implies $R(a, X) < 1 + a$ for all $a > 0$. We shall say that X satisfies the *Domínguez-Benavides' condition* whenever $R(a, X) < 1 + a$ for some $a > 0$.

In 2008, S. Saejung [13] proved that F-convex Banach spaces have uniform normal structure; but nothing was said about the coefficient $WCS(X)$ of an F-convex Banach space X. In this paper, we prove that if a Banach space X is F-convex then $R(0, X) < 1$, in other words, Bynum's condition is more general than F-convexity. This fact let us deduce the result by Saejung from the result by Bynum.

In 2006 [7] it was affirmatively solved the longstanding open question if uniformly nonsquare Banach spaces enjoyed the FPP. Indeed, it was realized that if X is a uniformly nonsquare Banach space, then X satisfies the Domínguez-Benavides condition, and consequently X has the FPP by virtue of the fixed point result in [3]. This result revealed the largeness of the family of Banach spaces X enjoying the Domínguez-Benavides' condition.

Two years later, P.N. Dowling, B. Randrianantoanina and B. Turett [5] identified several conditions upon a Banach space implying the wFPP. In particular, they improved in part the previous result by proving that E-convex Banach spaces also have the FPP. E-convex Banach spaces are superreflexive, and the class of E-convex Banach spaces properly contains the class of uniformly nonsquare Banach spaces. Notice that in [5] nothing was said about the relationship between E-convexity and the coefficients $R(a, X)$. In this paper, we prove that Domínguez-Benavides' condition is more general than any of the sufficient conditions for the wFPP identified in [5]. In particular, we deduce that any E-convex Banach space satisfies Domínguez-Benavides' condition. Moreover, we give an example of a Banach space X which is not E-convex but satisfying Domínguez-Benavides' condition.

The paper is organized as follows: In the following section we fix the notation and recall the definitions and results from [3], [13] and [5]. In the third section, we characterize Bynum's and Domínguez-Benavides' conditions via properties of the dual space X^*. In Section 4, we use these characterizations to obtain sufficient conditions for both Bynum's and Domínguez-Benavides' conditions, involving the separation measure of noncompactness γ of certain bounded subsets of of the dual space X^*. Next, we use the results in Section 4, to study the relationship between

Bynum's condition and F-convexity (Section 5), and to study the relationship between Domínguez-Benavides' condition and other sufficient conditions for wFPP in [5], including weak* uniform Kadec-Klee property and E-convexity (Section 6).

2. Notation and preliminaries

Throughout this paper we will use the standard notation in Banach space geometry. In particular, given a Banach space X, $B_X := \{x \in X : \|x\| \leq 1\}$, $S_X := \{x \in X : \|x\| = 1\}$, the convergence of a sequence (x_n) in X to $x \in X$ in the weak topology will be denoted as $x_n \overset{w}{\to} x$, and the convergence of a sequence (f_n) in X^* to f in the weak* topology as $f_n \overset{w^*}{\to} f$.

For a nonempty bounded subset C of a Banach space X, we will denote

$$\operatorname{diam}(C) := \sup\{\|x - y\| : x, y \in C\} \quad \text{and} \quad \operatorname{rad}(C) := \inf_{x \in C} \sup\{\|x - y\| : y \in C\}.$$

It is said that a Banach space X has *normal structure* [respectively *weak normal structure*] if every bounded closed [resp. weakly compact] convex subset C of X with $\operatorname{diam}(C) > 0$, verifies

$$\operatorname{rad}(C) < \operatorname{diam}(C).$$

And a Banach space X is said to have *uniform normal structure* if there exists $\rho \in (0, 1)$ such that, for all bounded closed convex subsets C of X with $\operatorname{diam}(C) > 0$, the inequality

$$\operatorname{rad}(C) < \rho \operatorname{diam}(C)$$

holds.

For a bounded sequence (x_n) in a Banach space X we will use the notations

$$\operatorname{diam}_{\mathrm{a}}\big[(x_n)\big] := \limsup_{k \to \infty} \sup_{n,m \geq k} \|x_n - x_m\|,$$

and

$$\mathrm{r}_{\mathrm{a}}\big[(x_n)\big] := \inf\{\limsup_{n \to \infty} \|x_n - y\| \;:\; y \in \overline{\mathrm{co}}\{x_n \;:\; n \in \mathbb{N}\}\}.$$

The *weakly convergent sequence coefficient* of a Banach space X was introduced by Bynum in [2] and is defined as

$$WCS(X) := \inf\left\{ \frac{\operatorname{diam}_{\mathrm{a}}\big[(x_n)\big]}{\mathrm{r}_{\mathrm{a}}\big[(x_n)\big]} \;:\; (x_n) \text{ is a weakly convergent sequence which is not norm convergent} \right\}.$$

It is clear that $1 \leq WCS(X) \leq 2$. We shall say that X satisfies *Bynum's condition* if $WCS(X) > 1$.

Theorem 2.1 ([2]). *Any Banach space satisfying Bynum's condition has weak normal structure.*

Given a Banach space X, $\mathcal{M}_X$ will stand for the set of all sequences (x_n) in B_X with $x_n \overset{w}{\to} 0$ and $D[(x_n)] \le 1$, where

$$\mathrm{D}\big[(x_n)\big] := \limsup_n \left(\limsup_m \|x_n - x_m\| \right).$$

Given a Banach space X and any $a \ge 0$, the coefficient $R(a, X)$ is defined by

$$R(a, X) := \sup\big\{ \liminf \|x_n + ax\| : x \in B_X, (x_n) \in \mathcal{M}_X \big\}.$$

It is clear that $\max\{a, 1\} \le R(a, X) \le 1 + a$. The main result in [3] is the following.

Theorem 2.2 (Theorem 2.2 in [3])**.** *Let X be a Banach space.*
If $R(a, X) < 1 + a$ for some $a \ge 0$, then X has the wFPP.

Theorem 2.3 (see Remarks in page 841 of [3])**.** *For any Banach space X*

$$R(0, X) = \frac{1}{WCS(X)}.$$

As a consequence of the previous result, a Banach space X satisfies Bynum's condition if and only if $R(0, X) < 1$. We shall say that a Banach space X satisfies the *Domínguez-Benavides' condition* whenever $R(a, X) < 1 + a$ for some $a > 0$. It is easy to see that, if $R(0, X) < 1$, then $R(a, X) < 1 + a$ for all $a > 0$, in other words, Bynum's condition implies Domínguez-Benavides' condition.

The notion of P-convexity was introduced by Kottman in [10]. Given $\eta \in (0, 2)$, a subset A of a Banach space X is said to be *η-separated* if the distance between any two distinct points of A is at least η. A Banach space X is *P-convex* if there exist $\delta > 0$ and $n \in \mathbb{N}$ such that B_X contains no $(2-\delta)$-separated subset of cardinality n. It is known that P-convex Banach spaces are superreflexive. Kottman also identified in [10] the dual notion of P-convexity and called it *F-convexity*; that is, a Banach space X is *F-convex* if and only if its dual space X^* is P-convex.

Theorem 2.4 (Theorem 3 in [13])**.** *If a Banach space X is F-convex, then X has uniform normal structure.*

In Section 5 (Theorem 5.1), we will prove that F-convexity implies Bynum's condition, showing then that the above result can be derived from Theorem 2.1.

Following [5], we will say that a dual Banach space X^* has the *weak* uniform Kadec-Klee property* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, if (f_n) is a sequence in B_{X^*} with $\mathrm{sep}\big[(f_n)\big] > \varepsilon$ and $f_n \overset{w^*}{\to} f$, then $\|f\| \le 1 - \delta$.

Theorem 2.5 (Theorem 3 in [5])**.** *Let X be a Banach space. If B_{X^*} is weak* sequentially compact and X^* has the weak* uniform Kadec-Klee property, then X has the wFPP.*

In Section 6 (Theorem 6.1), we will prove that a Banach space X satisfies Domínguez-Benavides' condition whenever B_{X^*} is weak* sequentially compact and X^* enjoys a property which is fullfilled if X^* has the weak* uniform Kadec-Klee property. This shows that Theorem 2.5 is a particular case of Theorem 2.2.

In [11] Naidu and Sastry defined the notion of *O-convexity.* Given $\eta \in (0, 2)$, a subset A of a Banach space X is said to be *symmetrically η-separated* if the distance between any two distinct points of $A \cup (-A)$ is at least η. A Banach space X is *O-convex* if there exist $\delta > 0$ and $n \in \mathbb{N}$ such that B_X contains no symmetrically $(2-\delta)$-separated subset of cardinality n. It is known that O-convex Banach spaces are superreflexive. Naidu and Sastry also identified in [11] the dual notion of O-convexity and named it *E-convexity*; that is, a Banach space X is *E-convex* if and only if its dual space X^* is O-convex.

Theorem 2.6 (Theorem 6 in [5])**.** *If a Banach space X is E-convex, then X has the FPP.*

In Section 6 (Theorem 6.4), we will prove that any E-convex Banach space satisfies Domínguez-Benavides' condition. Consequently Theorem 2.6 is also consequence of Theorem 2.2.

3. Characterization of Bynum's and Domínguez-Benavides' conditions

Let X be a Banach space. For each bounded subset A of the dual space X^*, we define

$$\mu(A) := \sup\left\{\liminf_{n\to\infty} f_n(x_n) \ : \ (x_n) \in \mathcal{M}_X \text{ and } (f_n) \text{ is a sequence in } A\right\}.$$

It is evident from the definitions that $R(0,X) = \mu(B_{X^*})$, so by virtue of Theorem 2.3

$$\mu(B_{X^*}) = \frac{1}{WCS(X)},$$

and we obtain the following characterization of Bynum's condition.

Theorem 3.1. *A Banach space X satisfies Bynum's condition if and only if*

$$\mu(B_{X^*}) < 1.$$

Our next goal is to characterize Domínguez-Benavides' condition in terms of the measure μ of the slices defined as follows: For each x in the unit ball of a Banach space X and each $\delta \geq 0$, we put

$$S(x,\delta) := \{f \in B_{X^*} \ : \ f(x) \geq 1-\delta\}.$$

The following two propositions are the key results in order to prove the characterization of Domínguez-Benavides' condition next given in Theorem 3.4.

Proposition 3.2. *Let X be a Banach space. For any $a \geq 0$ and any $\delta \geq 0$*

$$R(a,X) \leq \max\left\{\sup_{x\in B_X} \mu\big(S(x,\delta)\big) + a \ , \ \mu(B_{X^*}) + a(1-\delta)\right\}.$$

Proof. Let $a \geq 0$ and $\delta \geq 0$. According to the definition of $R(a,X)$, we have to prove that for any sequence (x_n) in $\mathcal{M}_X$ and any $x \in B_X$

$$\liminf_{n\to\infty} \|x_n + ax\| \leq \max\left\{\mu(B_{X^*}) + a(1-\delta), \sup_{x\in B_X} \mu\big(S(x,\delta)\big) + a\right\}.$$

So let $(x_n) \in \mathcal{M}_X$ and $x \in B_X$. Consider, for each $n \in \mathbb{N}$, $f_n \in S_{X^*}$ such that $f_n(x_n + ax) = \|x_n + ax\|$. Since our aim is to find an upper bound for

$\liminf_{n\to\infty} f_n(x_n+ax) = \liminf_{n\to\infty} \|x_n+ax\|$, we can assume, passing to subsequences if necessary, that both $\lim_{n\to\infty} f_n(x_n)$ and $\lim_{n\to\infty} f_n(x)$ exist and

$$\lim_{n\to\infty} \|x_n+ax\| = \lim_{n\to\infty} f_n(x_n) + a \lim_{n\to\infty} f_n(x).$$

Let us distinguish two cases: $\lim_{n\to\infty} f_n(x) > 1-\delta$ and $\lim_{n\to\infty} f_n(x) \le 1-\delta$.
Case 1: If $\lim_{n\to\infty} f_n(x) > 1-\delta$, then we can choose n_0 such that, for all $n \ge n_0$, $f_n \in S(x,\delta)$. Then, according to the definition of μ,

$$\lim_{n\to\infty} f_n(x_n) \le \mu\big(S(x,\delta)\big),$$

so

$$\lim_{n\to\infty} \|x_n+ax\| = \lim_{n\to\infty} f_n(x_n) + a \lim_{n\to\infty} f_n(x) \le \mu\big(S(x,\delta)\big) + a.$$

Case 2: If $\lim_{n\to\infty} f_n(x) \le 1-\delta$, then

$$\lim_{n\to\infty} \|x_n+ax\| = \lim_{n\to\infty} f_n(x_n) + a \lim_{n\to\infty} f_n(x) \le \mu(B_{X^*}) + a(1-\delta).$$

□

Proposition 3.3. *Let X be a Banach space. For any $a \ge 0$ and $\delta \ge 0$*

$$\sup_{x\in B_X} \mu\big(S(x,\delta)\big) + a(1-\delta) \le R(a,X).$$

Proof. Let $a \ge 0$ and $\delta \ge 0$. Let $x \in B_X$, (f_n) in $S(x,\delta)$ and $(x_n) \in \mathcal{M}_X$. For each $n \in \mathbb{N}$, we have

$$f_n(x_n) + a(1-\delta) \le f_n(x_n) + af_n(x) = f_n(x_n+ax) \le \|x_n+ax\|.$$

Therefore

$$\liminf_{n\to\infty} f_n(x_n) + a(1-\delta) \le \liminf_{n\to\infty} \|x_n+ax\| \le R(a,X).$$

From the arbitrariness of x, (f_n) and (x_n), we conclude

$$\sup_{x\in B_X} \mu\big(S(x,\delta)\big) + a(1-\delta) \le R(a,X).$$

□

Theorem 3.4. *A Banach space X satisfies Domínguez-Benavides condition if and only if*

$$\inf_{\delta>0} \sup_{x\in B_X} \mu\big(S(x,\delta)\big) < 1. \tag{3.1}$$

Proof. If X satisfies Domínguez-Benavides condition, then there exists $a > 0$ such that $R(a,X) < 1+a$. Let $\delta_0 = \dfrac{1+a-R(a,X)}{2a}$. By Proposition 3.3 we have that

$$\inf_{\delta>0} \sup_{x\in B_X} \mu\big(S(x,\delta_0)\big) \le \sup_{x\in B_X} \mu\big(S(x,\delta_0)\big) \le R(a,X) - a(1-\delta_0) < 1.$$

Conversely: If inequality (3.1) holds, then there exists $\delta > 0$ such that

$$\sup_{x\in B_X} \mu\big(S(x,\delta)\big) < 1.$$

Let $a_0 = \dfrac{1 - \sup_{x\in B_X} \mu\big(S(x,\delta)\big)}{2\delta}$. Applying Proposition 3.2 we get

$$\inf_{a>0}(R(a,X) - a) \le R(a_0, X) - a_0 \le \sup_{x\in B_X} \mu\big(S(x,\delta)\big) < 1.$$

□

4. Sufficient conditions for Bynum's and Domínguez-Benavides' conditions

For a bounded sequence (x_n) in a Banach space X let

$$\operatorname{sep}\big[(x_n)\big] := \inf_{n\neq m} \|x_n - x_m\|.$$

The *separation measure of noncompactness* of a bounded subset A of Banach space X is given by

$$\gamma(A) := \sup\big\{\operatorname{sep}\big[(x_n)\big] \;:\; (x_n) \text{ is a sequence in } A\big\}.$$

In order to study the relationship between $\gamma(A)$ and $\mu(A)$ for a bounded subset A of a dual space, we need the following technical lemma.

Lemma 4.1. *Let X be a Banach space. Suppose (x_n) is a weakly null sequence in X and (f_n) is a bounded sequence in X^*.*

Then there exists an increasing sequence of nonnegative integers (k_n) such that,

$$\liminf_{p\to\infty}\big(\inf_{n\neq m} \|f_{k_{p+n}} - f_{k_{p+m}}\|\big) \ge \frac{2\liminf_{n\to\infty} f_n(x_n)}{\mathrm{D}\big[(x_n)\big]}.$$

Proof. Let us define $Y = \overline{\cup_{n\in\mathbb{N}}\{\sum_{i=1}^{n} \alpha_i x_i \;:\; \alpha_1,\dots,\alpha_n \in \mathbb{R}\}}$. Since Y is a separable Banach space, B_{Y^*} is $\sigma(Y,Y^*)$-sequentially compact and we can then assume, passing to subsequences if necessary, the existence of $f \in Y^*$ such that, for all $y \in Y$

$$\lim_{n\to\infty} f_n(y) = f(y),$$

so, in particular, for all $m \in \mathbb{N}$

$$\lim_{n\to\infty} (f_n - f)(x_m) = 0. \tag{4.1}$$

On the other hand, (x_n) converges to 0 in both topologies $\sigma(X,X^*) =: \omega$ and $\sigma(Y,Y^*) = \omega_{|Y}$ and therefore

$$\lim_{n\to\infty} f(x_n) = 0, \tag{4.2}$$

and, for all $m \in \mathbb{N}$

$$\lim_{n\to\infty} f_m(x_n) = 0. \tag{4.3}$$

Let us put

$$D := \mathrm{D}\big[(x_n)\big] \quad \text{and} \quad \eta := \liminf_{n\to\infty} f_n(x_n).$$

By (4.2) and the definitions of D and η, we can find an index $k_1 \in \mathbb{N}$ such that $|f(x_{k_1})| < 1$,

$$\limsup_{m\to\infty} \|x_{k_1} - x_m\| < D + 1, \tag{4.4}$$

and $f_{k_1}(x_{k_1}) > \eta - 1$.

By (4.1) and (4.3) we have in particular

$$\lim_{n\to\infty}(f_n - f)(x_{k_1}) = \lim_{n\to\infty} f_{k_1}(x_n) = 0. \tag{4.5}$$

Taking into account (4.2), (4.4), (4.5), and the definitions of D and η, we can assure the existence of an index $k_2 > k_1$ for which

- $|f(x_{k_2})| < 1/2$
- $\|x_{k_1} - x_{k_2}\| < D + 1$
- $|(f_{k_2} - f)(x_{k_1})| < 1/2$
- $|f_{k_1}(x_{k_2})| < 1/2$
- $\limsup_{m\to\infty} \|x_{k_2} - x_m\| < D + 1/2$
- $f_{k_2}(x_{k_2}) > \eta - 1/2$

Suppose defined $k_1 < k_2 < \cdots < k_n$ ($n \geq 2$) such that, for any $i \in \{1, \ldots, n\}$ and any $j \in \{1, \ldots, i-1\}$, the following inequalities hold

(a) $|f(x_{k_i})| < 1/i$
(b) $\|x_{k_j} - x_{k_i}\| < D + 1/j$
(c) $|(f_{k_i} - f)(x_{k_j})| < 1/i$
(d) $|f_{k_j}(x_{k_i})| < 1/i$
(e) $\limsup_{m\to\infty} \|x_{k_i} - x_m\| < D + 1/i$
(f) $f_{k_i}(x_{k_i}) > \eta - 1/i$

By (4.1) and (4.3) we have in particular that, for each $j \in \{1, \ldots, n\}$

$$\lim_{n\to\infty}(f_n - f)(x_{k_j}) = \lim_{n\to\infty} f_{k_j}(x_n) = 0. \tag{4.6}$$

Taking into account (4.2), item (e), (4.6), and definitions of D and η, we can assure the existence of an index $k_{n+1} > k_n$ for which

- $|f(x_{k_{n+1}})| < 1/(n+1)$
- $\|x_{k_j} - x_{k_{n+1}}\| < D + 1/j$ for each $j = 1, \ldots, n$
- $|(f_{k_{n+1}} - f)(x_{k_j})| < 1/(n+1)$ for each $j = 1, \ldots, n$
- $|f_{k_j}(x_{k_{n+1}})| < 1/(n+1)$ for each $j = 1, \ldots, n$
- $\limsup_{m\to\infty} \|x_{k_{n+1}} - x_m\| < D + 1/(n+1)$
- $f_{k_{n+1}}(x_{k_{n+1}}) > \eta - 1/(n+1)$

We conclude, by induction, that there exists an strictly increasing sequence of natural numbers (k_n) such that, for any $i, j \in \mathbb{N}$ with $i > j$ inequalities (a) to (f) hold.

Let $i, j \in \mathbb{N}$ with $i > j$. By items (a) and (c)

$$|f_{k_i}(x_{k_j})| \leq |f(x_{k_j})| + |(f_{k_i} - f)(x_{k_j})| \leq \frac{1}{j} + \frac{1}{i}. \tag{4.7}$$

Combining items (d) and (f) and inequality (4.7), we obtain

$$\begin{aligned}(f_{k_i} - f_{k_j})(x_{k_i} - x_{k_j}) &= f_{k_i}(x_{k_i}) + f_{k_j}(x_{k_j}) - \big(f_{k_i}(x_{k_j}) + f_{k_j}(x_{k_i})\big)\\ &\geq f_{k_i}(x_{k_i}) + f_{k_j}(x_{k_j}) - \big(|f_{k_i}(x_{k_j})| + |f_{k_j}(x_{k_i})|\big)\\ &\geq 2\eta - \frac{3}{i} - \frac{2}{j}. \qquad (4.8)\end{aligned}$$

Taking into account (4.8) and (b)

$$\|f_{k_i} - f_{k_j}\| \geq (f_{k_i} - f_{k_j})\left(\frac{x_{k_i} - x_{k_j}}{\|x_{k_i} - x_{k_j}\|}\right)$$
$$\geq \frac{2\eta - 3/i - 2/j}{D + 1/j}.$$

We have proved that, for any $i, j \in \mathbb{N}$ with $i > j$

$$\|f_{k_i} - f_{k_j}\| \geq \frac{2\eta - 3/i - 2/j}{D + 1/j}, \tag{4.9}$$

and consequently, we have that for any $p, n, m \in \mathbb{N}$ with $n \neq m$,

$$\|f_{k_{p+n}} - f_{k_{p+m}}\| \geq \frac{2\eta - 3/(p + \max\{n, m\}) - 2/(p + \min\{n, m\})}{D + 1/(p + \min\{n, m\})} \geq \frac{2\eta - 5/p}{D + 1/p}.$$

We finally conclude that, for any $p \in \mathbb{N}$

$$\inf_{n \neq m} \|f_{k_{p+n}} - f_{k_{p+m}}\| \geq \frac{2\eta - 5/p}{D + 1/p},$$

so

$$\liminf_{p\to\infty} \inf_{n \neq m} \|f_{k_{p+n}} - f_{k_{p+m}}\| \geq \frac{2\eta}{D}$$

and the proof is finished.

□

Proposition 4.2. *Let X be a Banach space. For any bounded subset A of the dual space X^**

$$\gamma(A) \geq 2\mu(A).$$

Proof. Let A be a bounded subset of X^*. Consider a sequence (f_n) in A and $(x_n) \in \mathcal{M}_X$.

By Lemma 4.1, there exists an increasing sequence of nonnegative integers (k_n) such that,

$$\liminf_{p\to\infty}\big(\inf_{n \neq m} \|f_{k_{p+n}} - f_{k_{p+m}}\|\big) \geq 2 \liminf_{n\to\infty} f_n(x_n).$$

From the definition of $\gamma(A)$, for any fixed $p \in \mathbb{N}$ we have

$$\gamma(A) \geq \inf_{n \neq m} \|f_{k_{p+n}} - f_{k_{p+m}}\|,$$

so

$$\gamma(A) \geq \liminf_{p\to\infty}\big(\inf_{n \neq m} \|f_{k_{p+n}} - f_{k_{p+m}}\|\big) \geq 2 \liminf_{n\to\infty} f_n(x_n).$$

By the arbitrariness of (f_n) and (x_n), we conclude $\gamma(A) \leq 2\mu(A)$, as desired. □

We are now in a position to determine sufficient conditions for both Bynum's and Domínguez-Benavides' conditions, in terms of the measure of separation γ of some subsets of the unit ball of the dual space.

Theorem 4.3. *A Banach space X satisfies Bynum's condition whenever*

$$\gamma(B_{X^*}) < 2.$$

Proof. If $\gamma(B_{X^*}) < 2$, by Proposition 4.2

$$\mu(B_{X^*}) \leq \frac{\gamma(B_{X^*})}{2} < 1,$$

so X satisfies Bynum's condition by Theorem 3.1. □

Theorem 4.4. *A Banach space X satisfies Domínguez-Benavides' condition whenever*

$$\inf_{\delta>0} \sup_{x \in B_X} \gamma\big(S(x,\delta)\big) < 2. \tag{4.10}$$

Proof. By Proposition 4.2, for each $\delta > 0$ and $x \in B_X$,

$$\mu\big(S(x,\delta)\big) \leq \frac{\gamma\big(S(x,\delta)\big)}{2}.$$

Therefore, if inequality (4.10) holds,

$$\inf_{\delta>0} \sup_{x \in B_X} \mu\big(S(x,\delta)\big) < 1$$

and X satisfies Domínguez-Benavides' condition by virtue of Theorem 3.4. □

5. Bynum's condition and F-convexity

Theorem 5.1. *Any F-convex Banach space verifies Bynum's condition.*

Proof. Let X be a F-convex Banach space. Then X^* is *P-convex* and consequently $\gamma(B_{X^*}) < 2$. So X satisfies Bynum's condition by virtue of Theorem 4.3. □

In the light of the previous result, Theorem 2.4 can be easily derived from Theorem 2.1 as follows: As noted in [13], since F-convex Banach spaces are superreflexive and any ultrapower of an F-convex Banach space is also F-convex, to establish the result it suffices to prove that F-convex Banach spaces have weak normal structure. According to Theorem 5.1 X verifies Bynum's condition and therefore X has weak normal structure by Theorem 2.1.

6. Domínguez-Benavides' condition and other sufficient conditions for the wFPP

In this section we will show that Domínguez-Benavides' condition is more general than any of the sufficient conditions for the wFPP given in [5]. In particular we will prove that E-convex Banach spaces satisfy Domínguez-Benavides' condition.

Theorem 6.1. *Let X be a Banach space such that B_{X^*} is weak* sequentially compact. Suppose that there exist $\delta > 0$ such that if (f_n) is a sequence in B_{X^*} with $\operatorname{sep}\big[(f_n)\big] > 2-\delta$ and $f_n \overset{w^*}{\to} f$, then $\|f\| \leq 1-\delta$. Then X satisfies Domínguez-Benavides' condition, and consequently X has the wFPP.*

Proof. Assume

$$\sup_{x \in B_X} \gamma\big(S(x,\delta/2)\big) > 2-\delta.$$

Then there exists $x \in B_X$ such that $\gamma(S(x,\delta/2)) > 2-\delta$, and consequently we can find a sequence (f_n) in $S(x,\delta/2)$ with $\operatorname{sep}\big[(f_n)\big] > 2-\delta$. Since B_{X^*} is weak* sequentially compact, there exists a subsequence (g_n) of (f_n) which is weak* convergent, say $g_n \overset{w^*}{\to} g$. Provided that $\operatorname{sep}\big[(g_n)\big] \geq \operatorname{sep}\big[(f_n)\big] > 2-\delta$, it must be $\|g\| \leq 1-\delta$, so

$$\liminf_{n\to\infty} f_n(x) \leq \liminf_{n\to\infty} g_n(x) = g(x) \leq \|g\| \leq 1-\delta.$$

But this is a contradiction, because $f_n \in S(x,\delta/2)$ for all $n \in \mathbb{N}$. Therefore the initial assumption is false and

$$\inf_{\delta>0}\sup_{x\in B_X} \gamma\big(S(x,\delta)\big) \leq \sup_{x\in B_X} \gamma\big(S(x,\delta/2)\big) \leq 2-\delta < 2,$$

so X satisfies Domínguez-Benavides condition by virtue of Theorem 4.4. □

As an immediate consequence of the previous theorem we obtain the following result, which shows that Theorem 2.5 can be derived from Theorem 2.2.

Theorem 6.2. *Let X be a Banach space such that B_{X^*} is weak* sequentially compact and X^* has the weak* uniform Kadec-Klee. Then X satisfies Domínguez-Benavides' condition.*

Our next aim is to prove that Domínguez-Benavides' condition is strictly more general than E-convexity.

Theorem 6.3. *Let X be a Banach space. Suppose that there exist $\delta > 0$ such that for any sequence (f_n) in B_{X^*} the set $\{f_n : n \in \mathbb{N}\}$ fails to be symmetrically $(2-\delta)$-separated. Then X satisfies Domínguez-Benavides condition.*

Proof. Assume

$$\sup_{x\in B_X} \gamma\big(S(x,\delta/2)\big) > 2-\delta.$$

We can then find $x \in B_X$ and (f_n) in B_{X^*} such that $f_n(x) \geq 1-\delta/2$ for any $n \in \mathbb{N}$ and $\operatorname{sep}\big[(f_n)\big] > 2-\delta$.

Let us see that the set $\{f_n : n \in \mathbb{N}\}$ is symmetrically $(2-\delta)$-separated, that is

$$\inf_{n\neq m} \min\big\{\|f_n - f_m\|, \|f_n + f_m\|\big\} \geq 2-\delta.$$

For any $n, m \in \mathbb{N}$ with $n \neq m$, we have

$$\|f_n - f_m\| \geq \operatorname{sep}\big[(f_n)\big] > 2-\delta,$$

and, on the other hand,

$$\|f_n + f_m\| \geq (f_n + f_m)(x) = f_n(x) + f_m(x) \geq 2(1-\delta/2) = 2-\delta.$$

Since this contradicts the hypothesis, the initial assumption must be false and

$$\inf_{\delta>0}\sup_{x\in B_X} \gamma\big(S(x,\delta)\big) \leq \sup_{x\in B_X} \gamma\big(S(x,\delta/2)\big) \leq 2-\delta < 2.$$

So X satisfies Domínguez-Benavides condition by Proposition 4.4. □

Theorem 6.4. *Let X be a Banach space. Any E-convex Banach space satisfies Domínguez-Benavides condition.*

Proof. If X be a E-convex, then X^* is *O-convex*, and consequently there exist $n \in \mathbb{N}$ and $\delta > 0$ such that any subset of B_{X^*} of cardinality n fails to be symmetrically $(2-\delta)$-separated. So X satisfies Domínguez-Benavides condition by virtue of Theorem 6.3. □

According to the previous result, Domínguez Benavides' condition is more general than E-convexity. In order to show that it is indeed strictly more general, we provide in Example 6.5 a family of Banach spaces which fail to be E-convex but verify Domínguez Benavides' condition. Before presenting the example we need to recall the following notion: A Banach space X is said to have the WORTH property if

$$\limsup \|x_n + x\| = \limsup \|x_n - x\|$$

for all weakly null sequences (x_n) in X and all $x \in X$.

Example 6.5. For $\beta \geq 1$, let E_β be the space ℓ_2 equivalently renormed with the James norm

$$|x|_\beta := \max\{\|x\|_2, \beta\|x\|_\infty\}.$$

Some known properties of these spaces are the following:

(a) For any $\beta \geq 1$, E_β has the WORTH property (see [14]).
(b) E_β has normal structure if and only if $1 \leq \beta < \sqrt{2}$ (see [1]).
(c) For any $\beta \geq \sqrt{2}$, E_β satifies Domínguez Benavides' condition. Indeed, it was shown in [4] that

$$\sup_{a>0} \frac{1+a}{R(a, E_\beta)} = 1 + \frac{1}{\sqrt{2}}.$$

Let us know argue that, for $\beta \geq 2$, the space E_β fails to be E-convex: From a result due to S. Saejung ([13], Theorem 5) every E-convex Banach space X with the WORTH property has normal structure. So, taking into account (a) and (b), we would have a contradiction if E_β was E-convex.

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