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APPROXIMATION OF FIXED POINTS AND ERGODIC SEQUENCES FOR SEMIGROUPS OF NON-EXPANSIVE MAPPINGS

ANTHONY TO-MING LAU

Dedicated to Professor Simeon Reich on the occasion of his 65^{th} *birthday with admiration and respect.*

Abstract. When $\mathcal{S} = \{T_s : s \in S\}$ is a representation of an amenable semigroup S as non-expansive mappings on a closed convex subset C of a Banach space, then there is a strong relation between approximation of fixed point in C and strongly invariant sequences (or nets) of means in $\ell^1(S)$. It is the purpose of this paper to survey some recent results in this direction as well as some related results on the measure algebra and the Fourier Stieltjes algebra of a locally compact group. Some open problems are posted.

1. Introduction

Let E be a real Banach space with the topological dual E^* and let C be a closed convex subset of E. Then, a mapping T of C into itself is called nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$ for each $x, y \in C$. In 1967, Browder [3] introduced the following iterative scheme for finding a fixed point of a nonexpansive mapping T on C :

Let x be an arbitrary point in C and let $\{\alpha_n\}$ be a sequence in $(0, 1)$. For each n, let

$$T_n(z) = \alpha_n x + (1 - \alpha_n)T(z), \quad z \in C.$$

Then T_n is a strict contraction on C and hence has a unique fixed point x_n in C by the Banach contractive fixed point theorem, i.e.

$$x_n = \alpha_n x + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \dots . \tag{1.1}$$

He then studied strong convergence of this sequence to a fixed point of T in C. This result for nonexpansive mappings was extended to strong convergence theorems for accretive operators in a Banach space by Reich [29] (see [28]) and Takahashi and Ueda [39]. Reich [30] also studied the following iterative scheme for nonexpansive mappings: $x = x_1 \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \dots , \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ (see the original work of Halpern [10]). Wittmann [40] showed that the sequence generated by (1.2) in a Hilbert space converges strongly to the point of $F(T)$ which is nearest to x if $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. In [35], Shioji and Takahashi extended

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this result to Banach spaces. Shimizu and Takahashi ([33] and [34]) introduced the first iterative schemes for finding common fixed points of families of nonexpansive mappings and obtained convergence theorems for the families. In [36], Shioji and Takahashi established strong convergence theorems of the types (1.1) and (1.2) for families of mappings in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm by using the theory of means of abstract semigroups (see [4] and [5]).

When $\mathcal{S} = \{T_s : s \in S\}$ is a representation of an amenable semigroup S as non-expansive mappings on a closed convex subset C of a Banach space, then there is a strong relation between approximation of fixed point in C and strongly invariant sequences (or nets) of means in $\ell^1(S)$. It is the purpose of this paper to survey some recent results in this direction as well as some related results on the measure algebra and the Fourier Stieltjes algebra of a locally compact group. Some open problems are posted.

2. Preliminaries and some notations

A semigroup S is called *amenable* if there is a linear functional m on $\ell^\infty(S)$, the Banach space of all bounded real-valued functions on S with supremum norm such that:

(i) $m(f) \geq 0$ for all $f \in \ell^\infty(S), \quad f \geq 0$.
(ii) $m(1) = 1$
(iii) $m(\ell_\alpha f) = m(r_a f) = m(f)$ for all $f \in \ell^\infty(S)$ where $(\ell_a f)(t) = f(at)$ and $(r_a f)(t) = f(ta)$ for all $t \in S$.

A linear functional m on $\ell_\infty(S)$ satisfying (i) and (ii) is called a *mean*; m satisfying (i), (ii), (iii) is called an *invariant mean.* As well known, any commutative semigroup is amenable. A finite semigroup S is amenable if and only if S has a (unique) minimal ideal. However, the free group (or semigroup) on two generators is not amenable (see [5], [16], [17]).

Let D be a subset of B where B is a subset of a Banach space E and let P be a retraction of B onto D, that is, $Px = x$ for each $x \in D$. Then P is said to be *sunny* [38] if for each $x \in B$ and $t \geq 0$ with $Px + t(x - Px) \in B$,

$$P\big(Px + t(x - Px)\big) = Px.$$

A subset D of B is said to be a *sunny nonexpansive retract* of B if there exists a sunny nonexpansive retraction P of B onto D. We know that if E is smooth and P is a retraction of B onto D, then P is sunny and nonexpansive if and only if for each $x \in B$ and $z \in D$,

$$\langle x - Px,\, J(z - Px)\rangle \leq 0.$$

For more details, see [38].

3. Ergodic sequences

Let S be a semigroup and $\ell^1(S)$ denote the Banach space of all $f : S \to \mathbb{R}$ such that $\|f\|_1 = \sum |f(x)| < \infty$. Let $\big(\ell^1(S)\big)_1^+ =$ all $\theta \in \ell^1(S)$ such that $\theta \geq 0$ and

$\|\theta\|_1 = 1$ (countable means). There is a natural convolution on $\ell^1(S)$:

$$(\theta_1 * \theta_2)(s) = \sum\{\theta(s_1)\theta(s_2);\ s_1 s_2 = s\}.$$

Then $\big(\ell^1(S), *\big)$ is a Banach algebra, i.e.

$$\|\theta_1 * \theta_2\| \le \|\theta_1\|\,\|\theta_2\|.$$

Let H be a Hilbert space over the real, C be a closed convex subset of H, and $\mathcal{S} = \{T_s; s \in S\}$ be a representation of S as non-expansive mappings from C into C such $F(\mathcal{S}) \neq \emptyset$.

Let $x \in C$. For each $y \in H$, consider the bounded real-valued function on S $s \mapsto \langle T_s x, y\rangle$, Let θ be a *mean* on $\ell^\infty(S)$, define

$$\begin{aligned}\langle T_\theta(x), y\rangle &= \theta_s\big(\langle T_s x, y\rangle\big)\\ &= \sum\{\langle T_s x, y\rangle\theta(s);\ s \in S\} \quad \text{if} \quad \theta \in \big(\ell^1(S)\big)_1^+ .\end{aligned}$$

Then T_θ is a non-expansive mapping from $C \to C$ (see [32]).

Call a *sequence (net)* $\{\theta_n\}$ of means on S an *ergodic sequence (net) for non-expansive mappings* if for any representation $\mathcal{S} = \{T_s;\ s \in S\}$ of S as non-expansive mappings on a closed convex subset C of a Hilbert space into C such that $F(\mathcal{S}) \neq \emptyset$, then for each $x \in C$, the sequence (net) $T_{\theta_n}(x)$ converges weakly to a fixed point of $\mathcal{S}$.

A net of means $\{\mu_\alpha\}$ on $\ell^\infty(S)$ is called "asymptotically invariant" if

$$\begin{aligned}&\lim_\alpha\big(\mu_\alpha(\ell_s f) - \mu_\alpha(f)\big) = 0 \quad \text{and}\\ &\lim_\alpha\big(\mu_\alpha(r_s f) - \mu_\alpha(f)\big) = 0 \quad \text{for all} \quad s \in S.\end{aligned}$$

The net is said to be "strongly asymptotically invariant" if

$$\lim_\alpha\|\ell_s^+ u_\alpha - u_\alpha\| \quad \text{and} \quad \lim_\alpha\|r_s^+ u_\alpha - \mu_\alpha\| = 0 \quad \text{for all} \quad s \in S.$$

One-sided asymptotically invariant or strongly asymptotically invariant is defined similarly.

Theorem 3.1 (Rodé [32]). *Let S be an amenable semigroup. Then any "asymptotically invariant net" of means is an ergodic net for non-expansive mappings.*

Remark 3.2. (1) Every invariant mean on $\ell^\infty(S)$ is asymptotically invariant.

(2) Let S be commutative. If m is an invariant mean on $\ell^\infty(S)$, then there is a net $\theta_\alpha \in \big(\ell^1(S)\big)_1^+$ such that θ_α has *finite* support i.e. $\theta_\alpha = \sum_{i=1}^n \lambda_i \delta_{s_i}$ (convex combination) such that $\theta_\alpha \xrightarrow{w^*} m$. In particular the net $\{\theta_\alpha\}$ is asymptotically invariant. Hence $\{\theta_\alpha\}$ is an ergodic *net* of finite means on S for non-expansive mappings.

Example 3.3 ([32]). $S = (\{0, 1, 2, \dots\}, +)$

$$\theta_n = \frac{1}{n}\sum_{k=0}^{n-1}\delta_k\,,$$

then$\{\theta_n\}$ is an asymptotically invariant sequence of finite means on S. Consequently, $\{\theta_n\}$ is an ergodic *sequence* of finite means on S for non-expansive mappings.

The following answers an open problem in [18] for countable semigroup:

Theorem 3.4. *Let S be an amenable countable semigroup. Then there exists a strongly asymptotically invariant sequence of finite means. In particular, there is an ergodic sequence of finite means for non-expansive mappings.*

Proof. This follows from Theorem 1 and Proposition 3.6 in [13]. For the sake of completeness, we include a proof.

Since S is countable, the Banach space $\ell^1(S)$ is norm separable. Let $X = \{f_1, f_2, \dots\}$ be a countable bounded subset of $\ell^1(S)$ such that the closed subalgebra generated by X is $\ell^1(S)$. We may assume that $f_n \geq 0$ and $\|f_n\| = 1$. By Proposition 3.3 [13], there is a net $\{u_\alpha\}$ of elements in $\ell^1(S)$ consisting of finite *means* such that $u_\alpha \geq 0, \quad \|u_\alpha\| = 1$. Choose $\{u_{\alpha_n}, n \in \mathbb{N}\}$, inductively to satisfy:

$$\|f_i * u_{\gamma_n} - u_{\gamma_n}\| \leq \frac{1}{n} \quad \text{and} \quad \|u_{\gamma_n} * f_i - u_{\gamma_n}\| \leq \frac{1}{n}$$

for $1 \leq i \leq n$, and

$$\|u_{\gamma_i} * u_{\gamma_n} - u_{\gamma_n}\| \leq \frac{1}{n} \quad \text{and} \quad \|u_{\gamma_n} * u_{\gamma_i} - u_{\gamma_n}\| \leq \frac{1}{n}$$

for $1 \leq i \leq n$.

We claim that $\{u_{\gamma_n}\}$ is an asymptotic sequence of finite means for S. Take any k elements $a_1, \dots, a_k$ from $\{f_1, f_2, \dots\} \cup \{u_{\gamma_1}, u_{\gamma_2}, \dots\}$, and let $\varepsilon > 0$. Choose N so large that $kC^{k-1}/N < \varepsilon$ and that it $n > N$, then f_n and $u_{\gamma_n} \notin \{a_1, \dots, a_n\}$. Now for $n > N$, let

$$C_n = \|a_1 * a_2 * \cdots * a_k * u_{\gamma_n} - u_{\gamma_n}\|.$$

Then

$$\begin{aligned} C_n &\leq \sum_{j=1}^{k} \|a_1 * \cdots * a_j * u_{\gamma_n} - a_1 \dots a_{j-1} * u_{\gamma_n}\| \\ &\leq \sum_{j=1}^{k} \|a_1\| \dots \|a_{j-1}\| \, \|a_j * u_{\gamma_n} - u_{\gamma_n}\| \\ &\leq \sum_{j=1}^{k} C^{j-1} C^{k-j} \, \frac{1}{n} = kC^{k-1} \, \frac{1}{n} < \varepsilon. \end{aligned}$$

A parallel calculation also works for $u_{\gamma_n} * a_1 * \cdots * a_k$. It follows that $\{u_{\gamma_n}\}$ is an asymptotic sequence of finite means for $\ell^1(S)$, and hence an ergodic sequence by Rode's Theorem above. □

Example 3.5 ([26])**.** The bicyclic semigroup is the semigroup generated by a unit e and two more elements p and q subject to the relation $pq = e$. We denote it by $S_1 = \langle e, p, q \,|\, pq = e \rangle$.

For any $\varepsilon > 0$ and a finite set

$$F = \{q^{m_i} p^{n_i} : m_i \geq 0, \, n_i \geq 0, \, i = 1, 2, \dots, \ell\} \subset S_1 \, ,$$

let

$$m = \max\{n_i, m_i - n_i : i = 1, 2, \ldots, \ell\} \quad \text{and} \quad k = 2mt,$$

where $t > 1/\varepsilon$ is an integer. Then setting

$$A = \{q^m, q^{m+1}, \ldots, q^{m+k}\} \subset S_1$$

we have that for any $s = q^{m_i} p^{n_i} \in F$,

$$sA = \{q^{m+m_i-n_i}, q^{m+m_i-n_i+1}, \ldots, q^{m+m_i-n_i+k}\}.$$

So $|A| = k+1$, $|A \sim sA| \leq m$ and $|sA \sim A| \leq m$. Define $\Phi_{F,\varepsilon} = \frac{1}{|A|}\chi_A$, where for a subset E, χ_E denotes the characteristic function of E. Then

$$\begin{aligned} \|s * \Phi_{F,\varepsilon} - \Phi_{F,\varepsilon}\|_1 &= \frac{1}{|A|}\|\chi_{sA} - \chi_A\|_1 \\ &= \frac{1}{|A|}(|A \sim sA| + |sA \sim A|) \\ &\leq \frac{2m}{k+1} < \varepsilon \end{aligned}$$

for $s \in F$. Let $\Lambda = \{(F, \varepsilon) : F \subset S_1 \text{ is finite}, \varepsilon > 0\}$ with the usual partial order

$$\alpha_1 = (F_1, \varepsilon_1) \geq \alpha_2 = (F_2, \varepsilon_2) \quad \text{iff} \quad F_1 \supseteq F_2 \quad \text{and} \quad \varepsilon_1 \leq \varepsilon_2 .$$

Then $(\Phi_\alpha)_{\alpha \in \Lambda} \subset \ell^1(S_1)$ satisfies $\|\Phi_\alpha\| = 1$ and

$$\|s * \Phi_\alpha - \Phi_\alpha\|_1 \xrightarrow{\alpha} 0 \quad (s \in S_1).$$

In particular, S_1 is amenable, and $\{\Phi_\alpha\}$ is strongly asymptotically invariant net. Since S_1 is countable, it follows from the proof of Theorem 3.4 that there is a strongly asymptotically invariant sequence $\{\Phi_{\alpha_n}\}$ extracted from $\{\Phi_\alpha\}$. In particular, $\{\Phi_{\alpha_n}\}$ is an ergodic sequence for S_1.

Remark 3.6. The semigroup generated by a unit e and three more elements a, b and c subject to the relations $ab = ac = e$ is denoted by $S_2 = \langle e, a, b, c \,|\, ab = e\rangle$; and the semigroup generated by a unit e and four more elements a, b, c, d subject to the relations $ac = bd = e$ is denoted by $S_{1,1} = \langle e, a, b, c, d \,|\, ac = e, bd = e\rangle$. S_2 and $S_{1,1}$ will be called partially bicyclic semigroups. Duncan and Namioka showed in [6] that S_1 is an amenable semigroup by revealing the maximal group homomorphic image of S_1. Here we can prove the same result directly by constructing a left and a right invariant mean on $\ell^\infty(S_1)$. However, as shown in [26], the partially bicyclic semigroups S_2 and $S_{1,1}$ are not left amenable. $S_{1,1}$ is not even right amenable but S_2 is right amenable (i.e. $\ell_\infty(S)$ has a right invariant mean). In particular, S_2 has a strongly asymptotically right invariant sequence.

The argument is similar to that for S_1. Let F be a finite set of S_2. We can write

$$F = \{f_1 a^{m_1}, f_2 a^{m_2}, \ldots, f_n a^{m_n}\},$$

where each $f_i \in \langle e, b, c\rangle$, and $m_i \geq 0$, $i = 1, 2, \ldots, n$. Denote the length of any element $f \in \langle e, b, c\rangle$ by $\ell(f)$. Given $\varepsilon > 0$, take

$$m = \max\{\ell(f_i), m_i - \ell(f_i) | i = 1, 2, \ldots, n\} \quad \text{and} \quad k = 2mt,$$

where $t > 1/\varepsilon$ is an integer. Define

$$A = \{a^m, a^{m+1}, \dots, a^{m+k}\}.$$

Then for $f_i a^{m_i} \in F$, we have

$$A \cdot f_i a^{m_i} = \{a^{m+m_i-\ell(f_i)}, a^{m+m_i-\ell(f_i)+1}, \dots, a^{m+m_i-\ell(f_i)+k}\}.$$

Thus $|A\Delta(a \cdot f_i a^{m_i})| \leq 2m$. Define $\Phi_{F,\varepsilon} = \frac{1}{|A|}\chi_A$. We then have

$$\|\Phi_{F,\varepsilon} \cdot s - \Phi_{F,\varepsilon}\|_1 = \frac{1}{|A|}\,|A\Delta A \cdot s| \leq \frac{2m}{k+1} \leq \varepsilon$$

for all $s \in F$. The argument in the proof of Theorem 3.4 showed that we can extract a strongly asymptotically right invariant sequence for S_2 .

The following problem is still open.

Open problem 1: Given an amenable semigroup S, when does there exist an ergodic sequence of countable (or finite) means on S for non-expansive mappings? In particular, is this true for all amenable semigroups?

4. Approximation of fixed points

A well-known result of Day [4] (see also [5]) asserts that if S is an amenable semigroup, then whenever $\mathcal{S} = \{T_s; s \in S\}$ is a bounded representation of S as bounded linear operators on a Banach space E, there exists a net of finite averages A_α of $\mathcal{S}$ (i.e. for each $x \in E$, $A_\alpha(x)$ is in the convex hull of $\{T_s(x);\ s \in S\}$) such that $\lim_\alpha \|A_\alpha(T_s - I)(x)\| = 0$ and $\lim_\alpha \|(T_s - I)A_\alpha(x)\| = 0$, for each $x \in E$.

In this case, if $E(\mathcal{S}) = F(\mathcal{S}) + D(\mathcal{S})$, where $F(\mathcal{S})$ is the fixed point set of $\mathcal{S}$, and $D(\mathcal{S})$ is the closed linear span of $\{T_s x - x;\ s \in S \text{ and } x \in E\}$, then there is a projection P onto $F(\mathcal{S})$ along $D(\mathcal{S})$ and $PT_s = T_s P = P$ for all $s \in S$. Furthermore, if $x \in E(S)$, then $P(x)$ is the unique common fixed point on $\overline{\text{co}}\,\{T_s x;\ s \in S\}$, where $\overline{\text{co}}\,A$ is the closed convex hull of A.

The first nonlinear ergodic theorem for nonexpansive maps was established in 1975 by Baillon [1]: Let C be a closed convex subset of a Hilbert space and T a nonexpansive mapping of C into itself. If the fixed point set $F(T)$ of T is non-empty, then for each $x \in C$, the Cesàrio means

$$S_n(x) = \frac{1}{n}\sum_{k=1}^{n-1} T^k x$$

converges weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and $Px \in \overline{\text{co}}\,\{T^n x;\ n = 1, 2, \dots\}$ for each $x \in C$. In [37], Takahashi proved the following fundamental result relating amenability of a semigroup and ergodic theorems for non-expansive mappings:

Theorem 4.1 ([37])**.** *Let S be an amenable semigroup, C be a non-empty closed convex subset of a Hilbert space H, and $\mathcal{S} = \{T_s;\ s \in S\}$ be a representation of S as non-expansive mappings from C into C. Assume that $F(\mathcal{S})$ = fixed point set of $\mathcal{S}$ is non-empty. Then there is a non-expansive retraction P of C onto $F(\mathcal{S})$ such* *that $T_sP = PT_s = P$ for every $s \in S$, and $Px \in \overline{co}\{T_sx;\, s \in S\}$ for every $s \in C$, where $\mathrm{co}\, A$ is the closed convex full of A.*

Takahasi's result was extended to uniformly convex Banach space with a Fréchet differentiable norm when S is commutative by Hirano, Kido and Takahaski [11]. However, it has been an open problem for some time, whether Takahaski's result can be fully extended to such Banach spaces for amenable semigroups. This problem was answered by Lau, Shioji and Takahaski in [22]:

Theorem 4.2 ([22]). *Let C be a closed convex subset of a uniformly convex Banach space E, let S be an amenable semigroup, let $\mathcal{S} = \{T_t;\, t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Then there exists a net $\{A_\alpha\}$ of finite averages of $\mathcal{S}$ such that for each $t \in S$ and for each bounded subset B of C, $\lim_\alpha \|A_\alpha T_t x - A_\alpha x\| = 0$ and $\lim_\alpha \|T_t A_\alpha x - A_\alpha x\| = 0$ uniformly for each $x \in B$.*

Remark 4.3. The proof of Theorem 4.2 depends on Theorem 3.4. In the case S is countable, then $\{A_\alpha\}$ can be taken to be a sequence.

The following theorem is proved in [21] (see also [11], [12], [24], and [35]).

Theorem 4.4. *Let S be an amenable semigroup with identity and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as nonexpansive mappings from a compact convex subset C of a strictly convex and smooth Banach space E into C, let $\{\mu_n\}$ be a strongly sequence of means on $\ell^\infty(S)$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. Let $x_1 = x \in C$ and let $\{x_n\}$ be the sequence defined by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T(\mu_n)x_n\,, \quad n = 1, 2, \ldots .$$

Then $\{x_n\}$ converse strongly to Px, where P denotes the unique sunny nonexpansive retraction of C onto $F(\mathcal{S})$.

Let S be a semigroup with identity. S is called *left reversible* if any two right ideals in S have nonvoid intersection, i.e. $aS \cap bS \neq \emptyset$ for $a, b \in S$. In this case, we can define a partial ordering " $\leq$ " on S by $a \leq b$ if and only if $aS \supset bS$. The class of left reversible semigroups includes all groups and commutative semigroups. If a semigroup S is left amenable, then S is left reversible. But the converse is false.

Remark 4.5. Theorem 4.4 remains true for left reversible semigroups and subspaces X with a strongly left regular sequence of means.

Open problem 2: Does Theorem 4.4 remain valid when C is a weakly compact convex subset of a strictly convex and smooth Banach space E?

5. Ergodic sequences in $M(G)$ and $B(G)$

Let G be a locally compact group and π be a continuous unitary representation of G on a Hilbert space H, i.e. π is a homomorphism from G into the group of unitary operator of the Hilbert space H such that for each $\xi,\ n \in H$, the function $x \to \langle \pi(x)\xi, n\rangle,\ \ x \in G$ is continuous.

Let H_f denote the fixed point set of π in H, i.e.

$$H_f = \{\xi \in H;\, \pi(x)\xi = \xi \quad \text{for all} \quad x \in G\}.$$

A sequence $\{\mu_n\}$ of probability measures on G is called a *strongly* (resp. *weakly*) *ergodic* sequence if for every representation π of G on a Hilbert space H and for every $\xi \in H$, $\{\pi(\mu_n)\xi\}$ converges in norm (resp. weakly) to a member of H_f. When G is abelian or compact, or G is a [Moore]-group (i.e. every irreducible representation of G is finite dimensional), then every weakly ergodic sequence is strongly ergodic. However, this is not true in general (see [27, Proposition 1 and Proposition 5]).

In [2], Blum and Eisenberg proved that if G is a locally compact abelian group, and $\{\mu_n\}$ is a sequence of probability measures on G, then the following are equivalent:

(i) $\{\mu_n\}$ is strongly ergodic.
(ii) $\widehat{\mu}_n(\gamma) \to 0$ for all $\gamma \in \widehat{G}\backslash\{1\}$.
(iii) $\{\mu_n\}$ converges weakly to the Haar measure on the Bohr compactification of G.

More recently Milnes and Paterson [27] obtained the following generalization of Blum and Eisenberg's result for general locally compact groups.

Let G be a locally compact group with a fixed left Haar measure and $M(G)$ be the Banach algebra of regular Borel measures on G with the total variation norm.

Theorem 5.1 (Milnes and Paterson [27]). *Let G be a second countable locally compact group. Then the following statements about a sequence $\{\mu_n\}$ of probability measures in $M(G)$ are equivalent:*

(i) *$\{\mu_n\}$ is a weakly ergodic sequence.*
(ii) *$\pi(\mu_n) \to 0$ in the weak operator topology for every $\pi \in \widehat{G}\backslash\{1\}$.*
(iii) *$\widehat{\mu}_n$ converges to the unique invariant mean on $B_I(G)$, the closure in $C(G)$ of the linear span of the set of coefficient functions of the irreducible representations of G.*

(Here $\widehat{G}$ denotes the set of irreducible continuous representations of G which is the same as the dual group of G when G is abelian.)

Let $P(G)$ denote the subset of $C(G)$ consisting of all continuous positive definite functions on G, and let $B(G)$ be its linear span. Then $B(G)$ (the *Fourier-Stieltjes algebra* of G) can be identified with the dual of $C^*(G)$, and $P(G)$ is precisely the set of positive linear functionals on $C^*(G)$.

Let $\mathcal{B}(L^2(G))$ be the algebra of bounded linear operators from $L^2(G)$ into $L^2(G)$ and let $VN(G)$ denote the weak operator topology closure of the linear span of $\{\rho(a) : a \in G\}$, where $\rho(a)f(x) = f(a^{-1}x)$, $x \in G$, $f \in L^2(G)$, in $\mathcal{B}(L^2(G))$. Let $A(G)$ denote the subalgebra of $C_0(G)$ (continuous complex-valued functions vanishing at infinity), consisting of all functions of the form $h * \widetilde{k}$ where $h, k \in L^2(G)$ and $\widetilde{k}(x) = \overline{k(x^{-1})}$, $x \in G$. Then each $\phi = h * \widetilde{k}$ in $A(G)$ can be regarded as an ultraweakly continuous functional on $VN(G)$ defined by

$$\phi(T) = \langle Th, k\rangle \quad \text{for each} \quad T \in VN(G).$$

Furthermore, as shown by Eymard in [7, pp. 210, Theorem 3.10], each ultraweakly continuous functional on $VN(G)$ is of this form. Also $A(G)$ with pointwise multiplication and the norm $\|\phi\| = \sup\{|\phi(T)|\}$, where the supremum runs through all $T \in VN(G)$ with $\|T\| \le 1$, is a semisimple commutative Banach algebra with spectrum G; $A(G)$ is called the *Fourier algebra* of G and it is an ideal of $B(G)$. When G is abelian, $A(G) \cong L^1(\widehat{G})$, $B(G) \cong M(\widehat{G})$ and $VN(G) \cong L^\infty(\widehat{G})$.

There is a natural action of $A(G)$ on $VN(G)$ given by $\langle \phi \cdot T, \gamma \rangle = \langle T, \phi \cdot \gamma \rangle$ for each ϕ, $\gamma \in A(G)$ and each $T \in VN(G)$. A linear functional m on $VN(G)$ is called a *topological invariant mean* if

(i) $T \geq 0$ implies $\langle m, T \rangle \geq 0$,
(ii) $\langle m, I \rangle = 1$ where $I = \rho(e)$ denotes the identity operator, and
(iii) $\langle m, \phi \cdot T \rangle = \phi(e) \langle m, T \rangle$ for $\phi \in A(G)$.

As known, $VN(G)$ always has a topological invariant mean. However $VN(G)$ has a unique topological invariant mean if and only if G is discrete (see [31, Theorem 1], [19], and [27, Corollary 4.11].

Let $C^*_\delta(G)$ denote the norm closure of the linear span of $\{\rho(a);\, a \in G\}$. Let $B_\delta(G)$ denote the linear span of $P_\delta(G)$, where $P_\delta(G)$ is the pointwise closure of $A(G) \cap P(G)$. Then $B_\delta(G)$ can be identified with $C^*_\delta(G)^*$ by the map $\pi(\phi)(f) = \sum\{\phi(t)f(t),\, t \in G\}$ for each $f \in \ell_1(G)$ and $\phi \in B_\delta(G)$ (see [7]). Furthermore $B_\delta(G)$ with pointwise multiplication and dual norm is a commutative Banach algebra. If m is topological invariant mean on $VN(G)$, then $m' =$ restriction of m to $C^*_\delta(G)$, is also a topological invariant mean on $C^*_\delta(G)$. Furthermore, if m'' is another topological invariant mean on $C^*_\delta(G)$, then $m' = m''$, by commutativity of $B_\delta(G)$. If G is amenable, then $B(G) \subseteq B_\delta(G)$. In particular, each $\phi \in B(G)$ corresponds to a continuous linear functional on $C^*_\delta(G)$ defined by $\langle \phi, \rho(a) \rangle = \phi(a)$, $a \in G$. Also if G is abelian, then $C^*_\delta(G) \cong AP(\widehat{G})$, the space of continuous almost periodic functions on $\widehat{G}$ (see [14]).

A sequence $\{\phi_n\}$ in $A(G) \cap P_1(G)$ is called *strongly* (respectively *weakly*) *ergodic* if whenever $\{T, H\}$ is a $*$-representation of $A(G)$, $\xi \in H$, the sequence $T(\phi_n)\xi$ converges in the norm (resp. weak) topology to a member of the fixed point set:

$$H_f = \{\xi \in H;\, T(\phi)\xi = \xi \quad \text{for all} \quad \phi \in A(G) \cap P_1(G)\}.$$

Theorem 5.2 ([20]). *Let G be a locally compact group. The following are equivalent for a sequence $\{\phi_n\}$ in $A(G) \cap P_1(G)$:*

(i) *$\{\phi_n\}$ is strongly ergodic.*
(ii) *$\{\phi_n\}$ is weakly ergodic.*
(iii) *For each $g \in G$, $g \neq e$, $\phi_n(g) \to 0$.*
(iv) *For each $T \in C^*_\delta(G)$, $\langle \phi_n, T \rangle \to \langle m, T \rangle$, where m is the unique topological invariant mean on $C^*_\delta(G)$.*

Theorem 5.3 ([20]). *Let G be an amenable locally compact group. The following are equivalent for a sequence $\{\phi_n\}$ in $P_1(G)$:*

(i) *$\{\phi_n\}$ is strongly ergodic.*
(ii) *$\{\phi_n\}$ is weakly ergodic.*
(iii) *For each $g \in G$, $g \neq e$, $\phi_n(g) \to 0$.*
(iv) *For each $T \in C^*_\delta(G)$, $\langle \phi_n, T \rangle \to \langle m, T \rangle$, where m is the unique topological invariant mean on $C^*_\delta(G)$.*

Remark 5.4. Both Theorem 5.2 and 5.3 has recently been extended by S. Guex [8] and [9]) for $A_p(G)$, $1 < p < \infty$, the Figa-Talamanca algebra generalizing

$A_2(G) = A(G)$ as well as the study of left amenability of the class of F-algebras ([15]) by means of some ergodic and fixed point properties.

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Anthony To-Ming Lau
Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1
E-mail address: tlau@math.ualberta.ca