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ON COMMON FIXED POINTS OF SEMIGROUPS OF MAPPINGS NONEXPANSIVE WITH RESPECT TO CONVEX FUNCTION MODULARS

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In tribute to Simeon Reich on his 65th birthday

Abstract. We prove that the set of all common fixed points for a strongly continuous nonexpansive semigroup of nonlinear mappings acting in modular function spaces can be represented as a set of fixed points of a single nonexpansive mapping. This representation is then used to prove convergence of several iterative methods for construction of common fixed points of semigroups of nonlinear mappings.

1. Introduction

The purpose of this paper is to prove that the set of all common fixed points for a strongly continuous nonexpansive semigroup of nonlinear mappings acting in modular function spaces can be represented as a set of fixed points of a single nonexpansive mapping, where nonexpansiveness is understood in the modular sense. Modular function spaces are natural generalizations of both function and sequence variants of many important, from applications perspective, spaces like Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii spaces and many others, see [27] for an extensive list of examples and special cases.

The fixed point theory in modular function spaces originated in the 1990 seminal paper by Khamsi, Kozlowski and Reich [23]. In that paper, the authors showed that there exist mappings which are ρ-nonexpansive but are not norm-nonexpansive. They demonstrated that for a mapping T to be norm nonexpansive in a modular function space L_ρ, a stronger than ρ-nonexpansiveness assumption is needed: $\rho(\lambda(T(x) - T(x)) \leq \rho(\lambda(x - y))$ for any $\lambda \geq 0$. From this perspective, the fixed point theory in modular function spaces should be considered as complementary to the fixed point theory in normed spaces and in metric spaces. It is worthwhile to mention that from the perspective of applications, modular type conditions are typically more easily verified than their metric or norm counterparts. For earlier and recent results of fixed point theory in modular function spaces, refer e.g. to [4, 6–8, 17, 18, 21, 22, 24, 30, 31].

Let us recall that a family $\{T_t\}_{t\geq 0}$ of mappings forms a semigroup if $T_0(x) = x$, $T_{s+t} = T_s(T_t(x))$. Such a situation is quite typical in mathematics and applications. For instance, in the theory of dynamical systems, the modular function space L_ρ

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would define the state space and the mapping $(t,x) \to T_t(x)$ would represent the evolution function of a dynamical system. The question about the existence of common fixed points, and about the structure of the set of common fixed points, can be interpreted as a question whether there exist points that are fixed during the state space transformation T_t at any given point of time t, and if yes - what the structure of a set of such points may look like. In the setting of this paper, the state space may be an infinite dimensional. Therefore, it is natural to apply these result not only to deterministic dynamical systems but also to stochastic dynamical systems.

An existence of common fixed points of ρ-nonexpansive semigroups was demonstrated in 2011, [30]. However, a structure of the set of common fixed points can be a priori very complicated and therefore it can be difficult to apply any methods of construction of such common fixed points, which is of a major importance for applications. In the current paper we show that in the case of a strongly continuous nonexpansive semigroup, the set of its common fixed points can be actually represented by a set of common fixed points of a single, suitably chosen, nonexpansive mapping. The idea of such representation is known in Banach spaces, see e.g. the 2006 paper by Suzuki [45] and references therein. However the case of ρ-nonexpansive mappings acting in modular function spaces have not been investigated prior to the current paper. It is worthwhile to mention that we use only convexity of the function modular ρ as it does not need to have any triangle inequality or homogeneity properties. This shows the strength of the convexity assumptions because convexity of ρ suffices to prove both the existences and the representation of a set of common fixed points.

In the section "Convergence theorems" we use such a representation to show how several iterative methods can be used for construction of common fixed points of continuous nonexpansive semigroups. The idea of using such processes in this context can be traced back to the seminal 1950s - 1970s papers by Mann [37], Krasnosel'skii [35], Ishikawa [12], Reich [40,41], and others. See also an extensive body of work from the 1980s and 1990s [1,3,10,13,39,42–44,47–50], and more recent research from the current century [4,9,11,14,15,20,28,29,32–34,36,38,46] and the works referred there.

2. Preliminaries

Let us introduce basic notions related to modular function spaces and related notation which will be used in this paper. For further details we refer the reader to preliminary sections of the recent articles [4,21,22] or to the survey article [31]; see also [25–27] for the standard framework of modular function spaces.

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$ we will denote the space of all extended measurable functions, i.e. all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By 1_A we denote the characteristic function of the set A.

Definition 2.1. Let $\rho : \mathcal{M}_\infty \to [0,\infty]$ be a notrivial, convex and even function. We say that ρ is a regular convex function pseudomodular if:

(i) $\rho(0) = 0$;
(ii) ρ is monotone, i.e. $|f(\omega)| \le |g(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f) \le \rho(g)$, where $f, g \in \mathcal{M}_\infty$;
(iii) ρ is orthogonally subadditive, i.e. $\rho(f1_{A\cup B}) \le \rho(f1_A)+\rho(f1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathcal{M}$;
(iv) ρ has the Fatou property, i.e. $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;
(v) ρ is order continuous in $\mathcal{E}$, i.e. $g_n \in \mathcal{E}$ and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

Similarly, as in the case of measure spaces, we say that a set $A \in \Sigma$ is ρ-null if $\rho(g1_A) = 0$ for every $g \in \mathcal{E}$. We say that a property holds ρ-almost everywhere if the exceptional set is ρ-null. As usual we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind we define $\mathcal{M} = \{f \in \mathcal{M}_\infty; |f(\omega)| < \infty\ \rho - a.e\}$, where each element is actually an equivalence class of functions equal ρ-a.e. rather than an individual function.

Definition 2.2. We say that a regular function pseudomodular ρ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0\ \rho - a.e.$. The class of all nonzero regular convex function modulars defined on Ω will be denoted by $\Re$.

Definition 2.3 ([25–27])**.** Let ρ be a convex function modular. A modular function space is the vector space $L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0\ as\ \lambda \to 0\}$.

The following notions will be used throughout the paper.

Definition 2.4. Let $\rho \in \Re$.

(a) We say that $\{f_n\}$ is ρ-convergent to f and write $f_n \to f\ (\rho)$ if and only if $\rho(f_n - f) \to 0$.
(b) A sequence $\{f_n\}$ where $f_n \in L_\rho$ is called ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as $n, m \to \infty$.
(c) A set $B \subset L_\rho$ is called ρ-closed if for any sequence of $f_n \in B$, the convergence $f_n \to f\ (\rho)$ implies that f belongs to B.
(d) A set $B \subset L_\rho$ is called ρ-bounded if $\sup\{\rho(f - g); f \in B, g \in B\} < \infty$.
(e) A set $B \subset L_\rho$ is called strongly ρ-bounded if there exists $\beta > 1$ such that $M_\beta(B) = \sup\{\rho(\beta(f - g)); f \in B, g \in B\} < \infty$.

Since ρ fails in general the triangle identity, many of the known properties of limit may not extend to ρ-convergence. For example, ρ-convergence does not necessarily imply ρ-Cauchy condition. However, it is important to remember that the ρ-limit is unique when it exists. The following proposition brings together few facts that will be often used in the proofs of our results.

Proposition 2.5. *Let $\rho \in \Re$.*

(i) *L_ρ is ρ-complete.*
(ii) *ρ-balls $B_\rho(x, r) = \{y \in L_\rho; \rho(x - y) \le r\}$ are ρ-closed and ρ-a.e. closed.*
(iii) *If $\rho(\alpha f_n) \to 0$ for an $\alpha > 0$ then there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that $g_n \to 0\ \rho - a.e.$*

(iv) $\rho(f) \leq \liminf \rho(f_n)$ *whenever* $f_n \to f$ $\rho-$*a.e. (Note: this property is equivalent to the Fatou Property).*

We will also need the definition of the Δ_2-property of a function modular, see e.g. [4, 27].

Definition 2.6. Let $\rho \in \Re$. We say that ρ has the Δ_2-property if

$$\sup_n \rho(2f_n, D_k) \to 0$$

whenever $D_k \downarrow \emptyset$ and $\sup_n \rho(f_n, D_k) \to 0$.

The modular equivalents of uniform convexity were introduced in [22].

Definition 2.7. Let $\rho \in \Re$. We define the following uniform convexity type properties of the function modular ρ:

(i) Let $r > 0, \varepsilon > 0$. Define

$$D(r,\varepsilon) = \{(f,g) : f,g \in L_\rho, \rho(f) \leq r, \rho(g) \leq r, \rho(f-g) \geq \varepsilon r\}.$$

Let

$$\delta(r,\varepsilon) = \inf\Big\{1 - \frac{1}{r}\,\rho\Big(\frac{f+g}{2}\Big) : (f,g) \in D(r,\varepsilon)\Big\}, if D(r,\varepsilon) \neq \emptyset,$$

and $\delta(r,\varepsilon) = 1$ *if* $D(r,\varepsilon) = \emptyset$. We say that ρ satisfies (UC) if $r > 0$ and $\varepsilon > 0$ implies $\delta(r,\varepsilon) > 0$. Note, that for every $r > 0$, $D(r,\varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough.

(ii) We say that ρ satisfies (UUC) if for every $s \geq 0, \varepsilon > 0$ there exists

$$\eta(s,\varepsilon) > 0$$

depending on s and ε such that

$$\delta(r,\varepsilon) > \eta(s,\varepsilon) > 0 \ for \ r > s.$$

Let us also introduce modular definitions of Lipschitzian and nonexpansive mappings, and associated definitions of semigroups of nonlinear mappings.

Definition 2.8 ([30])**.** Let $\rho \in \Re$ and let $C \subset L_\rho$ be nonempty and ρ-closed. A mapping $T : C \to C$ is called a ρ-Lipschitzian if there exists a constant $L > 0$ such that

$$\rho(T(f) - T(g)) \leq L\rho(f-g) \ for \ any \ f,g \in L_\rho.$$

T is called a ρ-nonexpansive mapping if $L = 1$.

For any mapping T by $F(T)$ we denote the set of all fixed points of T.

Definition 2.9 ([30])**.** A one-parameter family $\mathcal{F} = \{T_t : t \geq 0\}$ of mappings from C into itself is said to be a ρ-Lipschitzian (resp. ρ-nonexpansive) semigroup on C if $\mathcal{F}$ satisfies the following conditions:

(i) $T_0(x) = x$ for $x \in C$;
(ii) $T_{t+s}(x) = T_t(T_s(x))$ for $x \in C$ and $t, s \geq 0$;
(iii) for each $t \geq 0$, T_t is ρ-Lipschitzian (resp. ρ-nonexpansive).

Definition 2.10. A semigroup $\mathcal{F} = \{T_t : t \geq 0\}$ is called strongly continuous if for every $z \in C$, the following function

$$\Lambda_z(t) = \rho\Big(T_t(z)) - z\Big) \tag{2.1}$$

is continuous at every $t \in [0, \infty)$.

Definition 2.11. A semigroup $\mathcal{F} = \{T_t : t \geq 0\}$ is called continuous if for every $z \in C$, the mapping $t \longmapsto T_t(z)$ is ρ-continuous at every $t \in [0, \infty)$, i.e. $\rho\Big(T_{t_n}(z) - T_t(z)\Big) \to 0$ as $t_n \to t$.

By $F(\mathcal{F})$ we will denote the set of common fixed points of the semigroup $\mathcal{F}$.

Let us finish this section with the existence theorem for semigroups of nonexpansive mappings acting in modular function spaces.

Theorem 2.12 ([30])**.** *Assume $\rho \in \Re$ is (UUC). Let C be a ρ-closed ρ-bounded convex nonempty subset. Let $\mathcal{F}$ be a nonexpansive semigroup on C. Then the set $F(\mathcal{F})$ of common fixed points is nonempty, ρ-closed and convex.*

3. Representation theorems

In this section we will use the following notation: let $0 < \alpha < \beta$, and $\alpha \leq s \leq \beta$. Define inductively a sequence $\{A_n(s)\}$ of subsets of $[\alpha, \beta]$ by

$$A_1(s) = \{s\},\ A_{n+1}(s) = \bigcup_{t \in A_n(s)} \{|\alpha - t|, |\beta - t|\}\ for\ n \in \mathbb{N}. \tag{3.1}$$

Set

$$A(s) = \bigcup_{n=1}^{\infty} A_n(s). \tag{3.2}$$

We will use the following two results concerning real numbers.

Lemma 3.1 ([45])**.** *If α/β is an irrational number then for every $s \in [\alpha, \beta]$, the closure of $A(s)$ is equal to $[0, \beta]$.*

Lemma 3.2 ([45])**.** *If α/β is a rational number then for every $s \in [\alpha, \beta]$, the set $A(s)$ is finite.*

We are now ready to prove our main theorem about representation of a set of common fixed points by a set of fixed points of one nonexpansive mapping.

Theorem 3.3. *Let $\mathcal{F} = \{T_t : t \geq 0\}$ be a strongly continuous nonexpansive semigroup on a ρ-bounded subset C of a modular function space L_ρ, where $\rho \in \Re$. Let α and β be positive real numbers such that α/β is an irrational number. Let $\lambda \in (0, 1)$ be arbitrary. Then*

$$F(\mathcal{F}) = F\Big(\lambda T_\alpha + (1 - \lambda)T_\beta\Big). \tag{3.3}$$

Proof. Note first that the mapping $\lambda T_\alpha + (1-\lambda)T_\beta$ is nonexpansive with respect to the convex modular ρ. Without loosing generality we can assume that $\alpha < \beta$. It suffices to prove

$$F\Big(\lambda T_\alpha + (1-\lambda)T_\beta\Big) \subset F(\mathcal{F}), \tag{3.4}$$

as the opposite direction inclusion is trivial. Also, this inclusion is trivial when $F(\lambda T_\alpha + (1-\lambda)T_\beta) = \emptyset$. We can assume therefore that there exists $w \in C$ such that

$$\lambda T_\alpha(w) + (1-\lambda)T_\beta(w) = w. \tag{3.5}$$

Since $\mathcal{F}$ is a strongly continuous semigroup it follows that the function Λ_w is a continuous real-valued function on the interval $[0,\beta]$ and hence it attains its maximum at a number $\tau \in [0,\beta]$ which means that

$$\rho\Big(T_\tau(w) - w\Big) = \max\Big\{\rho\Big(T_t(w) - w\Big) : t \in [0,\beta]\Big\}. \tag{3.6}$$

Since $\tau \in A(\tau) \subset [0,\beta]$, we have

$$\rho\Big(T_\tau(w) - w\Big) = max\Big\{\rho\Big(T_s(w) - w\Big) : s \in A(\tau)\Big\}. \tag{3.7}$$

Let us prove by induction that for every $n \in \mathbb{N}$ and any $s \in A_n(\tau)$,

$$\rho\Big(T_\tau(w) - w\Big) = \rho\Big(T_s(w) - w\Big), \tag{3.8}$$

Since $A_1(\tau) = \{\tau\}$, (3.8) is true for $n = 1$. Let us make an inductive assumption that (3.8) holds for $n = k$. Fix arbitrary $t \in A_k(\tau)$. By the inductive assumption we have

$$\rho\Big(T_\tau(w) - w\Big) = \rho\Big(T_t(w) - w\Big). \tag{3.9}$$

Substituting (3.5) into the right hand side of (3.9) and using the convexity of ρ, and then using nonexpansiveness, we obtain the following

$$\begin{aligned} \rho\Big(T_\tau(w) - w\Big) &\le \lambda\rho\Big(T_t(w) - T_\alpha w\Big) + (1-\lambda)\rho\Big(T_t(w) - T_\beta w\Big) \\ &\le \lambda\rho\Big(T_{|\alpha-t|}(w) - w\Big) + (1-\lambda)\rho\Big(T_{|\beta-t|}(w) - w\Big) \\ &\le \rho\Big(T_\tau(w) - w\Big), \end{aligned} \tag{3.10}$$

where the last inequality comes from the fact that $|\alpha - t|$ and $|\beta - t|$ belong to $A_{k+1}(\tau) \subset A(\tau)$ and from (3.7). From (3.10) it follows easily that

$$\rho\Big(T_\tau(w) - w\Big) = \rho\Big(T_{|\alpha-t|}(w)\Big) + \rho\Big(T_{|\beta-t|}(w)\Big). \tag{3.11}$$

By arbitrariness of $t \in A_k(\tau)$ we conclude that (3.8) holds for $k+1$ and hence by induction it also holds for all natural n. From (3.8) it follows that for any $s \in A(\tau)$,

$$\rho\Big(T_\tau(w) - w\Big) = \rho\Big(T_s(w) - w\Big). \tag{3.12}$$

By Lemma 3.1 $A(\tau)$ is dense in $[0,\beta]$. Hence, by continuity of Λ_w we deduce from (3.12) that

$$\rho\Big(T_\tau(w) - w\Big) = \rho\Big(T_s(w) - w\Big) \tag{3.13}$$

for every $s \in [0,\beta]$. Substituting $s = 0$ and remembering that $T_0(w) = w$ we have

$$\rho\Big(T_\tau(w) - w\Big) = 0, \tag{3.14}$$

and consequently $\rho\Big(T_s(w) - w\Big) = 0$ for every $s \in [0,\beta]$ implying $T_s(w) = w$ for any $s \in [0,\beta]$. Let t be any positive real number, hence $t = n\beta + s$ for a $n \in \mathbb{N} \cup \{0\}$ and $s \in [0,\beta]$. Therefore

$$T_t(w) = T_\beta^n \circ T_s(w) = T_\beta^n(w) = w, \tag{3.15}$$

which means that $w \in F(\mathcal{F})$, as claimed. □

Please note that we did not assume in Theorem 3.3 that the common fixed points actually exist. This theorem, however, reduces the question of existence of common fixed points to the question of existence of fixed points for each ρ-nonexpansive mapping. We have therefore the following result which corresponds to the Banach space results from [2, 45].

Theorem 3.4. *Let $\mathcal{F} = \{T_t : t \geq 0\}$ be a strongly continuous nonexpansive semigroup on a ρ-closed, ρ-bounded and convex subset C of a modular function space L_ρ, where $\rho \in \Re$. Assume that every ρ-nonexpansive mapping on C has a fixed point. Then the set of common fixed points of $\mathcal{F}$ is nonempty.*

Remark 3.5. Please note that our Theorem 3.4 combined with Theorem 4.1 in [22] gives us an alternative proof of Theorem 2.12.

As we saw the strong continuity assumption of the semigroup was of critical importance. Let us give an important example when this condition is satisfied but first let us recall the definition of the uniform continuity of the function modular ρ in the sense of the following definition (see e.g. [21]).

Definition 3.6. We say that $\rho \in \Re$ is uniformly continuous if to every $\varepsilon > 0$ and $L > 0$, there exists $\delta > 0$ such that

$$|\rho(x) - \rho(x+h)| \leq \varepsilon, \tag{3.16}$$

provided $\rho(h) < \delta$ and $\rho(x) \leq L$.

Let us mention that the uniform continuity holds for a large class of function modulars. For instance, it can be proved that in Orlicz spaces over a finite atomless measure [5] or in sequence Orlicz spaces [16] the uniform continuity of the Orlicz modular is equivalent to the Δ_2-type condition on the Orlicz function.

It is easy to see that if a semigroup $\mathcal{F}$ is continuous and the modular ρ is uniformly continuous then $\mathcal{F}$ is strongly continuous. Consequently, we have the following result.

Theorem 3.7. *Let $\mathcal{F} = \{T_t : t \geq 0\}$ be a continuous nonexpansive semigroup on a subset C of a modular function space L_ρ, where $\rho \in \Re$ is uniformly continuous. Let α and β be positive real numbers such that α/β is an irrational number. Let $\lambda \in (0,1)$ be arbitrary. Then*

$$F(\mathcal{F}) = F\Big(\lambda T_\alpha + (1-\lambda)T_\beta\Big). \tag{3.17}$$

A natural question arises what representation of common fixed point sets can be achieved without assuming that the semigroup $\mathcal{F}$ is strongly continuous. It turns out that we can indeed characterize finite intersections of sets of fixed points.

Theorem 3.8. *Let $\mathcal{F} = \{T_t : t \geq 0\}$ be a nonexpansive semigroup on a subset C of a modular function space L_ρ, where $\rho \in \Re$. Let α and β be positive real numbers such that α/β is a rational number. Let $\lambda \in (0,1)$ be arbitrary. Then*

$$F(T_\alpha) \cap F(T_\beta) = F\Big(\lambda T_\alpha + (1-\lambda)T_\beta\Big). \tag{3.18}$$

Proof. Assume that $\alpha < \beta$. It suffices to prove that

$$F\Big(\lambda T_\alpha + (1-\lambda)T_\beta\Big) \subset F(T_\alpha) \cap F(T_\beta). \tag{3.19}$$

Fix an arbitrary $w \in F\Big(\lambda T_\alpha + (1-\lambda)T_\beta\Big)$. By Lemma 3.2, the set $A(0)$ is finite. Therefore, there exists $\tau \in A(0)$ such that

$$\rho\Big(T_\tau(w) - w\Big) = \max\Big\{\rho\Big(T_s(w) - w\Big) : s \in A(0)\Big\}. \tag{3.20}$$

Arguing like in the proof of Theorem 3.3 we conclude that

$$\rho\Big(T_\tau(w) - w\Big) = \rho\Big(T_s(w) - w\Big), \tag{3.21}$$

for every $s \in A(0)$. Since $0 \in A(0)$ and $T_0(w) = w$, it follows that

$$\rho\Big(T_\tau(w) - w\Big) = 0. \tag{3.22}$$

Using this and the fact that both $\alpha \in A(0)$ and $\beta \in A(0)$, we have

$$\rho\Big(T_\alpha(w) - w\Big) = \rho\Big(T_\beta(w) - w\Big) = \rho\Big(T_\tau(w) - w\Big) = 0, \tag{3.23}$$

which implies that

$$T_\alpha(w) = T_\beta(w) = w. \tag{3.24}$$

The proof is complete. □

4. Convergence theorems

The results of the previous section reduce, in some cases, a complex task of finding the set of all common fixed points of a semigroup of mappings to a simpler one of finding a set of fixed points of just one nonexpansive mapping: $\lambda T_\alpha + (1-\lambda)T_\beta$. This is a significant simplification but still does not a give us a method allowing construction of such points. In this section we investigate how the representation theorems form the previous section can be utilized for construction of common fixed points.

Let us first remind the reader the definition of Strong Opial property, [19, 21].

Definition 4.1. We say that L_ρ satisfies the ρ-a.e. strong Opial property if for every $\{x_n\} \in L_\rho$ which is ρ-a.e. convergent to 0 such that there exists a $\beta > 1$ for which

$$\sup_n \{\rho(\beta x_n)\} < \infty, \tag{4.1}$$

the following equality holds for any $y \in E_\rho$

$$\liminf_{n\to\infty} \rho(x_n + y) = \liminf_{n\to\infty} \rho(x_n) + \rho(y). \tag{4.2}$$

Remark 4.2. Also, note that, by virtue of Theorem 2.1 in [19], every convex, orthogonally additive function modular ρ has the ρ-a.e. strong Opial property. Let us recall that ρ is called orthogonally additive if $\rho(x1_{A\cup B}) = \rho(x1_A) + \rho(x1_B)$ *whenever* $A \cap B = \emptyset$. Therefore, all Orlicz and Musielak-Orlicz spaces must have the strong Opial property.

Note that the Opial property in the norm sense does not necessarily hold for several classical Banach function spaces. For instance the norm Opial property does not hold for L^p spaces for $1 \le p \ne 2$ while the modular strong Opial property holds in L^p for all $p \ge 1$.

Following [4], let us start with the definition of the generalized Mann iteration process.

Definition 4.3. Let $T)$ be a ρ-nonexpansive mappimng of $C \subset of L_\rho$. Let $\{t_k\} \subset (0,1)$ be bounded away from 0 and 1. The generalized Mann iteration process generated by the mapping T and the sequence $\{t_k\}$, denoted by $gM(T,\{t_k\})$ is defined by the following iterative formula:

$$x_{k+1} = t_k T^k(x_k) + (1-t_k)x_k, \text{ where } x_1 \in C \text{ is chosen arbitrarily.} \tag{4.3}$$

We will use the following convergence result for ρ-nonexpansive mappings.

Theorem 4.4 ([4])**.** *Let $\rho \in \Re$. Assume that*

(1) *ρ is (UUC),*
(2) *ρ has Strong Opial Property,*
(3) *ρ has Δ_2 property and is uniformly continuous.*

Let $C \subset L_\rho$ be a nonempty, sequentially compact with respect to the $\rho-a.e.$ convergence, convex, strongly ρ-bounded and ρ-closed. Let T be ρ-nonexpansive. Assume that a sequence $\{t_k\} \subset (0,1)$ is bounded away from 0 and 1. Let $gM(T,\{t_k\})$ be a generalized Mann iteration process. Then there exists $x \in F(T)$ such that $x_n \to x$ ρ-a.e.

Combining Theorem 3.7 with Theorem 4.4 one can easily obtain the following convergence result for semigroups of nonexpansive mappings.

Theorem 4.5. *Let C be a nonempty, sequentially compact with respect to the $\rho-a.e.$ convergence, convex, strongly ρ-bounded and ρ-closed subset of L_ρ, where $\rho \in \Re$ is uniformly continuous, (UUC) and has Δ_2 property. Let $\mathcal{F} = \{T_t : t \ge 0\}$ be a continuous nonexpansive semigroup on C. Assume that a sequence $\{t_k\} \subset (0,1)$ is bounded away from 0 and 1. Let $gM(T,\{t_k\})$ be a generalized Mann iteration process. Then there exists $x \in F(\mathcal{F})$ such that $x_n \to x$ ρ-a.e.*

The two-step Ishikawa iteration process is a generalization of the one-step Mann process. The Ishikawa iteration process provides more flexibility in defining the algorithm parameters which is important from the numerical implementation perspective.

Definition 4.6. Let T be ρ-nonexpansive mapping on C. Let $\{t_k\} \subset (0,1)$ be bounded away from 0 and 1, and $\{s_k\} \subset (0,1)$ be bounded away from 1. The generalized Ishikawa iteration process generated by the mapping T, the sequences $\{t_k\}$ and $\{s_k\}$, denoted by $gI(T,\{t_k\},\{s_k\})$, is defined by the following iterative formula:

(4.4)
$$x_{k+1} = t_kT^k(s_kT^k(x_k)+(1-s_k)x_k)+(1-t_k)x_k, \textit{ where } x_1 \in C \textit{ is chosen arbitrarily.}$$

Using Theorem 6.1 of [4] and Theorem 3.7 it is easy to obtain the following convergence result.

Theorem 4.7. *Let C be a nonempty, sequentially compact with respect to the $\rho-a.e.$ convergence, convex, strongly ρ-bounded and ρ-closed subset of L_ρ, where $\rho \in \Re$ is uniformly continuous, (UUC) and has Δ_2 property. Let $\mathcal{F} = \{T_t : t \geq 0\}$ be a continuous nonexpansive semigroup on C. Assume that a sequence $\{t_k\} \subset (0,1)$ is bounded away from 0 and 1, and that $\{s_k\} \subset (0,1)$ is bounded away from 1. Let $gI(T,\{t_k\},\{s_k\})$ be a generalized Ishikawa iteration process. Then there exists $x \in F(\mathcal{F})$ such that $x_n \to x$ ρ-a.e.*

In the theory of modular function spaces the $\rho-a.e.$ convergence plays role similar to that of weak convergence in Banach spaces. To obtain strong type convergence to a common fixed point we need to replace the assumption of compactness with respect to the $\rho - a.e.$ convergence by the strong compactness. First, let us recall the strong convergence theorem for nonexpansive mappings in modular function spaces.

Theorem 4.8. *[4] Let $\rho \in \Re$ satisfy conditions (UUC) and Δ_2. Let $C \subset L_\rho$ be a ρ-compact, ρ-bounded and convex set, and let T be ρ-nonexpansive in C. Let $\{t_k\} \subset (0,1)$ be bounded away from 0 and 1, and $\{s_k\} \subset (0,1)$ be bounded away from 1. Then there exists a fixed point $x \in F(T)$ such that then $\{x_k\}$ generated by $gM(T,\{t_k\})$ (resp. $gI(T,\{t_k\},\{s_k\})$) converges strongly to a fixed point of T, that is*

$$\lim_{k\to\infty} \rho(x_k - x) = 0. \tag{4.5}$$

Remark 4.9. Observe that in view of the Δ_2 assumption, the ρ-compactness of the set C assumed in Theorem 4.8 is equivalent to the compactness in the sense of the Luxemburg norm defined by ρ.

Finally we have the following strong convergence result.

Theorem 4.10. *Let $\rho \in \Re$ satisfy conditions (UUC) and Δ_2. Let $C \subset L_\rho$ be a ρ-compact, ρ-bounded and convex set. Let $\mathcal{F} = \{T_t : t \geq 0\}$ be a continuous nonexpansive semigroup on C. Let $\{t_k\} \subset (0,1)$ be bounded away from 0 and 1, and $\{s_k\} \subset (0,1)$ be bounded away from 1. Then there exists a fixed point $x \in F(\mathcal{F})$ such that then $\{x_k\}$ generated by $gM(T,\{t_k\})$ (resp. $gI(T,\{t_k\},\{s_k\})$) converges strongly* *to a common fixed point of $\mathcal{F}$, that is*

$$\lim_{k\to\infty} \rho(x_k - x) = 0. \tag{4.6}$$

Remark 4.11. Observe that in view of the Δ_2 assumption, the ρ-convergence in (4.6) is equivalent to the convergence in the sense of the Luxemburg norm defined by ρ.

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